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A REMARK ON THE STRONG LAW FOR B-VALUED ARRAYS OF RANDOM ELEMENTS

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Abstract

The conditions in the strong law of large numbers given by Li *et al.* ['A strong law for B-valued arrays', *Proc. Amer. Math. Soc.* **123** (1995), 3205–3212] for B-valued arrays are relaxed. Further, the compact logarithm rate law and the bounded logarithm rate law are discussed for the moving average process based on an array of random elements.

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1. Introduction and main results

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let B denote a real separable Banach space with norm $\|\cdot\|$ and topological dual space B^*. Let B_1^* denote the unit ball of B^*. A B-valued random element X is defined as a Borel measurable function from $(\Omega, \mathcal{F})$ into B. The expected value of a B-valued random element X is defined by a Bochner integral and is denoted by EX. Let H be the reproducing kernel Hilbert space associated with $\mu = \mathcal{L}(X)$, the law or distribution of X, and K the unit ball of H. For details of H and K, see Ledoux and Talagrand [4]. The symbol $\mathcal{C}(\{Y_n, n \ge 1\})$ stands for the cluster set of the sequence $\{Y_n, n \ge 1\}$ of random elements.

Let $\{X, X_n \ge 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables. Hartman and Wintner [2] established the following law of the iterated logarithm. If

$$EX = 0 \quad \text{and} \quad EX^2 = 1, \tag{1.1}$$

then

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n X_k}{\sqrt{2n \log \log n}} = 1 \text{ a.s.} \quad \text{and} \quad \liminf_{n\to\infty} \frac{\sum_{k=1}^n X_k}{\sqrt{2n \log \log n}} = -1 \text{ a.s.} \tag{1.2}$$

Strassen [8] proved that (1.1) is also necessary for (1.2) to hold.

Now let $\{X, X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of i.i.d. random variables. The almost sure convergence for arrays is quite different. For example, under the assumptions that $EX = 0$, $EX^2 = 1$, and $EX^4 < \infty$, Hu and Weber [3] proved that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n X_{nk}}{\sqrt{2n \log n}} = 1 \text{ a.s.} \quad \text{and} \quad \liminf_{n\to\infty} \frac{\sum_{k=1}^n X_{nk}}{\sqrt{2n \log n}} = -1 \text{ a.s.} \tag{1.3}$$

Hence the classical Hartman–Wintner law of the iterated logarithm does not hold for arrays. Qi [7] and Li *et al.* [6] independently proved that

$$EX = 0, \quad EX^2 = 1 \quad \text{and} \quad E\frac{|X|^4}{\log^2(e + |X|)} < \infty$$

are necessary and sufficient conditions for (1.3). Li and Huang [5] discussed strong invariance principles for arrays. For the case of arrays of B-valued random elements, Li *et al.* [6] obtained the following theorem.

THEOREM 1.1. *Let $\{X, X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of i.i.d. random elements. Suppose that*

$$\begin{cases} EX = 0, \\ E\dfrac{\|X\|^4}{\log^2(e + \|X\|)} < \infty, \\ \dfrac{\sum_{k=1}^n X_{nk}}{\sqrt{2n \log n}} \to 0 \quad \textit{in probability}. \end{cases} \tag{1.4}$$

Then

$$P\left(\left\{\frac{\sum_{k=1}^n X_{nk}}{\sqrt{2n \log n}}, n \ge 1\right\} \textit{ is conditionally compact in } B\right) = 1, \tag{1.5}$$

$$\mathcal{C}\left(\left\{\frac{\sum_{k=1}^n X_{nk}}{\sqrt{2n \log n}}, n \ge 1\right\}\right) = K \quad \textit{a.s.}, \tag{1.6}$$

$$\limsup_{n\to\infty} \frac{\|\sum_{k=1}^n X_{nk}\|}{\sqrt{2n \log n}} = \sup_{x\in K} \|x\| \quad \textit{a.s.} \tag{1.7}$$

Conversely, (1.5) implies (1.4).

Theorem 1.1 gives the compact logarithm law for arrays under condition (1.4) which requires convergence in probability. We will relax this condition to require only that the normed sum is bounded in probability, that is, for every $\varepsilon > 0$, there exists a constant $A > 0$ such that

$$\sup_{n\ge1} P\left(\frac{\|\sum_{k=1}^n X_{nk}\|}{\sqrt{2n \log n}} > A\right) < \varepsilon.$$

Furthermore, we will extend Theorem 1.1 to the moving average process for arrays. We now state the main results.

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THEOREM 1.2. *Let* $\{X, X_{nk}, 1 \le k \le n, n \ge 1\}$ *be an array of i.i.d. random elements. Suppose that*

$$\begin{cases} EX = 0, \\ E\dfrac{\|X\|^4}{\log^2(e+\|X\|)} < \infty, \\ \dfrac{\sum_{k=1}^n X_{nk}}{\sqrt{2n\log n}} \quad \text{is bounded in probability.} \end{cases} \tag{1.8}$$

Then

$$\max\{\alpha, \beta\} \le \limsup_{n\to\infty} \frac{\|\sum_{k=1}^n X_{nk}\|}{\sqrt{2n\log n}} \le \alpha + \beta \quad \text{a.s.}, \tag{1.9}$$

where

$$\alpha = \sqrt{\sup\{Ef^2(X) : f \in B_1^*\}} \quad \text{and} \quad \beta = \limsup_{n\to\infty} \frac{E\|\sum_{i=1}^n X_{nk}\|}{\sqrt{2n\log n}}.$$

Conversely,

$$\limsup_{n\to\infty} \frac{\|\sum_{k=1}^n X_{nk}\|}{\sqrt{2n\log n}} < \infty \quad \text{a.s.} \tag{1.10}$$

implies that (1.8) holds.

REMARK 1.3. Theorem 1.2 gives the bounded logarithm law for arrays. Note that $\alpha = \sup_{x\in K} \|x\|$ and it is easy to show that $\sum_{k=1}^n X_{nk}/\sqrt{2n\log n} \to 0$ in probability implies $\beta = 0$. Hence, we can deduce that (1.4) implies (1.7) as an application of Theorem 1.2.

We will now present an example of an array where (1.8) holds but (1.4) does not.

EXAMPLE 1.4. We will adapt the construction in [4, Example 7.11]. Let $\{\xi_k\}$ be independent random variables with distribution

$$P(\xi_k = 1) = P(\xi_k = -1) = \tfrac{1}{2}(1 - P(\xi_k = 0)) = 1/(\log(k+1)).$$

Define $\beta_k = 1$, $k = 1, 2, 3$ and $\beta_k = \sqrt{(\log n)/n}$ whenever $2^n \le k < 2^{n+1}$, $k \ge 4$. Define the random element X in c_0, the separable Banach space of all real sequences tending to 0, equipped with the sup norm. X has coordinates $(\beta_k \xi_k)$ for $k \ge 1$. X is symmetric, a.s. bounded and has mean 0.

Denote by $\{\xi_{ki}\}$ independent copies of (ξ_k). Let

$$E_{nk} = \bigcap_{i=1}^n \{\xi_{ki} = 1\} \quad \text{and} \quad A_n = \bigcup_{k \le 2^n} E_{nk},$$

$$P(E_{nk}) = [\log(k+1)]^{-n},$$

$$P(A_n) = 1 - \prod_{k\le 2^n} P(E_{nk}^c) = 1 - \prod_{k \le 2^n}\left(1 - \frac{1}{[\log(k+1)]^n}\right),$$

so that $P(A_n) \to 1$.

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Let $\{X_{nk}\}$ be an array of i.i.d. random elements with the same distribution as X. Let $S_n = \sum_{k=1}^n X_{nk}$ and let (e_k) be the canonical basis for c_0. Use the superscript (n) to denote the terms in the nth row of the array, for example $A_n^{(n)}$ and $E_{nk}^{(n)}$, constructed as above. Thus

$$\frac{S_n}{\sqrt{n \log n}} = \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{\infty} \bigg(\sum_{i=1}^{n} \xi_{ki}^{(n)} \bigg) \beta_k e_k.$$

On $A_n^{(n)}$,

$$\frac{\|S_n\|}{\sqrt{n \log n}} \geq \max_{k \leq 2^n} \frac{1}{\sqrt{n \log n}} \beta_k n = 1,$$

so that, for every $\varepsilon \in (0, 1)$,

$$\liminf P\bigg(\frac{\|S_n\|}{\sqrt{n \log n}} > \varepsilon \bigg) > \liminf P(A_n^{(n)}) = 1,$$

so condition (1.4) of Theorem 1.1 does not hold.

Next consider $P(\|S_n\|/\sqrt{n \log n} > C)$ for some constant C. Let $T_{nk} = \sum_{i=1}^n \xi_{ki}^{(n)}/\sqrt{n}$. Note that the T_{nk} are independent, zero mean random variables with $\mathrm{Var}(T_{nk}) = 2/(\log(k+1))$. Then

$$\begin{aligned} P\bigg(\frac{\|S_n\|}{\sqrt{n \log n}} > C \bigg) &= P\bigg(\sup_k \beta_k \frac{|T_{nk}|}{\sqrt{\log n}} > C \bigg) \\ &= 1 - P\Big(\sup_k \beta_k |T_{nk}| \leq C\sqrt{\log n} \Big) \\ &= 1 - \prod_k P(\beta_k |T_{nk}| \leq C\sqrt{\log n}) \\ &= 1 - \prod_k [1 - P(\beta_k |T_{nk}| > C\sqrt{\log n})]. \end{aligned}$$

Applying Bernstein's inequality,

$$\begin{aligned} P(\beta_k |T_{nk}| > C\sqrt{\log n}) &\leq P(|T_{nk}| > C\sqrt{\log n}) \quad \text{as } 0 < \beta_k \leq 1, \\ &\leq 2 \exp\bigg(- \frac{C^2 \log(k+1) \log(n)}{4 + C \log(k+1)\sqrt{(\log n)/n}} \bigg) = a_{nk}, \end{aligned}$$

say. Thus

$$P\bigg(\frac{\|S_n\|}{\sqrt{n \log n}} > C \bigg) \leq 1 - \prod_k (1 - a_{nk}). \tag{1.11}$$

Now $a_{nk} = O((k+1)^{-\frac{1}{4}C^2 \log n})$, so for fixed C and n large enough, a_{nk} is summable. Moreover, we can make the right-hand side of (1.11) arbitrarily small by selecting n large enough. Hence (1.8) holds.

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The following theorem extends Theorems 1.1 and 1.2 to the moving average process for arrays.

THEOREM 1.5. *Let* $\{X, X_{ni}, -\infty < i < \infty, n \geq 1\}$ *be an array of i.i.d. random elements and let* $\{a_i, -\infty < i < \infty\}$ *be a sequence of real constants with* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. *Set* $a = \sum_{i=-\infty}^{\infty} a_i$.

(i) *Condition (1.4) implies that*

$$P\left(\left\{\frac{\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_i X_{n,k-i}}{\sqrt{2n\log n}}, n \geq 1\right\} \text{ is conditionally compact in } B\right) = 1, \tag{1.12}$$

$$C\left(\left\{\frac{\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_i X_{n,k-i}}{\sqrt{2n\log n}}, n \geq 1\right\}\right) = |a|K \quad a.s., \tag{1.13}$$

$$\limsup_{n\to\infty} \frac{\|\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_i X_{n,k-i}\|}{\sqrt{2n\log n}} = |a| \sup_{x\in K} \|x\| \quad a.s. \tag{1.14}$$

(ii) *Condition (1.8) implies that*

$$|a| \max\{\alpha, \beta\} \leq \limsup_{n\to\infty} \frac{\|\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_i X_{n,k-i}\|}{\sqrt{2n\log n}} \leq |a|(\alpha+\beta) \quad a.s. \tag{1.15}$$

In the rest of this paper we let c denote a generic positive constant which may differ from one occurrence to the next.

2. Proofs

We begin with two lemmas needed in the proofs. The first lemma is a version of Theorem 3.1 in Einmahl and Li [1].

LEMMA 2.1. *Let* $Z_1, Z_2, \ldots, Z_n$ *be independent random elements with mean zero such that for some* $s > 2$, $E\|Z_k\|^s < \infty$, $1 \leq k \leq n$. *Then, for* $0 < \eta \leq 1$, $\delta > 0$ *and any* $t > 0$,

$$P\left(\left\|\sum_{k=1}^{n} Z_k\right\| > (1+\eta)E\left\|\sum_{k=1}^{n} Z_k\right\| + t\right) \leq \exp\left(-\frac{t^2}{2(1+\delta)\Lambda_n}\right) + ct^{-s}\sum_{k=1}^{n} E\|Z_k\|^s,$$

where

$$\Lambda_n = \sup\left\{Ef^2\left(\sum_{k=1}^{n} Z_k\right) : f \in B_1^*\right\}$$

and c *is a positive constant depending on* η, δ *and* s.

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LEMMA 2.2. *Let $\{X, X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of i.i.d. random elements. Then (1.8) implies that*

$$E\left(\sup_{n\ge 2} \frac{\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}}\right) < \infty.$$

PROOF. Let $\alpha^2 = \sup\{Ef^2(X) : f \in B_1^*\}$ and

$$\beta = \limsup_{n\to\infty} \frac{E\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}}.$$

It is easy to show that $\alpha + \beta < \infty$. We have

$$\begin{aligned}
E\sup_{n\ge 2} \frac{\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} &= \int_0^\infty P\left(\sup_{n\ge 2} \frac{\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} > x\right) dx \\
&\le 2(\alpha+\beta) + \int_{2(\alpha+\beta)}^\infty P\left(\sup_{n\ge 2} \frac{\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} > x\right) dx \\
&\le 2(\alpha+\beta) + \int_{2(\alpha+\beta)}^\infty \sum_{n=2}^{\infty} P\left(\left\|\sum_{k=1}^{n} X_{nk}\right\| > x\sqrt{2n\log n}\right) dx.
\end{aligned}$$

Let $x = 2(\alpha+\beta)y/\sqrt{2n\log n}$. Then

$$\begin{aligned}
&E\sup_{n\ge 2} \frac{\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} \\
&\quad \le 2(\alpha+\beta) + \sqrt{2}(\alpha+\beta)\sum_{n=2}^{\infty} \frac{1}{\sqrt{n\log n}} \int_{\sqrt{2n\log n}}^\infty P\left(\left\|\sum_{k=1}^{n} X_{nk}\right\| > 2(\alpha+\beta)y\right) dy \\
&\quad \le 2(\alpha+\beta) + \sqrt{2}(\alpha+\beta)\sum_{n=2}^{\infty} \frac{1}{\sqrt{n\log n}} \int_{\sqrt{2n\log n}}^\infty nP(\|X\| > y)\, dy \\
&\qquad + \sqrt{2}(\alpha+\beta)\sum_{n=2}^{\infty} \frac{1}{\sqrt{n\log n}} \\
&\qquad \times \int_{\sqrt{2n\log n}}^\infty P\left(\left\|\sum_{k=1}^{n} X_{nk} I(\|X_{nk}\| \le y)\right\| > 2(\alpha+\beta)y\right) dy \\
&\quad = 2(\alpha+\beta) + \sqrt{2}(\alpha+\beta)(I_1 + I_2).
\end{aligned}$$

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For I_1,

$$I_1 = \int_{\sqrt{4\log 2}}^{\infty} \sum_{\sqrt{2n\log n}\le y} \sqrt{\frac{n}{\log n}}\, P(\|X\| > y)\, dy$$
$$\le c\int_{\sqrt{4\log 2}}^{\infty} \frac{y^3}{\log^2 y} P(\|X\| > y)\, dy$$
$$\le cE\frac{\|X\|^4}{\log^2(e+\|X\|)} < \infty.$$

For I_2, since $EX = 0$, note that

$$\sup_{y\ge\sqrt{2n\log n}} y^{-1}\left\|\sum_{k=1}^{n} EX_{nk}I(\|X_{nk}\|\le y)\right\|$$
$$\le \sup_{y\ge\sqrt{2n\log n}} y^{-1} nE\|X\|I(\|X\| > y)$$
$$= \frac{n}{\sqrt{2n\log n}} E\|X\|I(\|X\| > \sqrt{2n\log n})$$
$$\le \frac{n}{\sqrt{2n\log n}}\cdot\frac{1}{\sqrt{2n\log n}} E\|X\|^2 I(\|X\| > \sqrt{2n\log n})$$
$$\le \frac{E\|X\|^2}{2\log n} \to 0,$$

$$\limsup_{n\to\infty} \sup_{y\ge\sqrt{2n\log n}} y^{-1}E\left\|\sum_{k=1}^{n}(X_{nk}I(\|X_{nk}\|\le y) - EX_{nk}I(\|X_{nk}\|\le y))\right\|$$
$$\le \limsup_{n\to\infty} \frac{1}{\sqrt{2n\log n}} E\left\|\sum_{k=1}^{n} X_{nk}\right\| = \beta$$

and

$$\sup\left\{Ef^2\left(\sum_{k=1}^{n}(X_{nk}I(\|X_{nk}\|\le y) - EX_{nk}I(\|X_{nk}\|\le y))\right) : f\in B_1^*\right\} \le n\alpha^2.$$

Hence there exists η, $0 < \eta < 1$, such that

$$\int_{\sqrt{2n\log n}}^{\infty} P\left(\left\|\sum_{k=1}^{n} X_{nk}I(\|X_{nk}\|\le y)\right\| > 2(\alpha+\beta)y\right) dy$$
$$\le \int_{\sqrt{2n\log n}}^{\infty} P\left(\left\|\sum_{k=1}^{n}(X_{nk}I(\|X_{nk}\|\le y) - EX_{nk}I(\|X_{nk}\|\le y))\right\| > \right.$$
$$\left.(1+\eta)E\left\|\sum_{k=1}^{n}(X_{nk}I(\|X_{nk}\|\le y) - EX_{nk}I(\|X_{nk}\|\le y))\right\| + (1+\eta)\alpha y\right) dy$$

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when n is large enough. So by Lemma 2.1 for $s > 4$,

$$\begin{aligned} I_2 &\le \sum_{n=2}^{\infty} \frac{1}{\sqrt{n \log n}} \int_{\sqrt{2n \log n}}^{\infty} \exp\left(-\frac{(1+\eta)^2\alpha^2 y^2}{2(1+\eta)n\alpha^2}\right) dy \\ &\quad + c\sum_{n=2}^{\infty} \frac{n}{\sqrt{n \log n}} \int_{\sqrt{2n \log n}}^{\infty} y^{-s} E\|X\|^s I(\|X\| \le y)\, dy \\ &= I_3 + I_4. \end{aligned}$$

For I_3, substituting $y = t\sqrt{2n \log n}$,

$$\begin{aligned} I_3 &= \sum_{n=2}^{\infty} \frac{1}{\sqrt{n \log n}} \int_{\sqrt{2n \log n}}^{\infty} \exp\left(-\frac{(1+\eta)y^2}{2n}\right) dy \\ &= \sqrt{2}\sum_{n=2}^{\infty} \int_1^{\infty} \exp\left(-\frac{(1+\eta)(2n\log n)t^2}{2n}\right) dt \\ &= \sqrt{2}\int_1^{\infty} \sum_{n=2}^{\infty} n^{-(1+\eta)t^2}\, dt \\ &\le c\int_1^{\infty} 2^{-\eta t^2}\, dt < \infty. \end{aligned}$$

For I_4, standard computation gives

$$\begin{aligned} I_4 &= c\sum_{n=2}^{\infty} \frac{n}{\sqrt{n \log n}} \int_{\sqrt{2n \log n}}^{\infty} y^{-s} E\|X\|^s I(\|X\| \le y)\, dy \\ &\le cE\frac{\|X\|^4}{\log^2(e + \|X\|)} < \infty, \end{aligned}$$

which completes the proof. □

PROOF OF THEOREM 1.2. First we establish the upper bound, that is,

$$\limsup_{n\to\infty} \frac{\|\sum_{k=1}^n X_{nk}\|}{\sqrt{2n \log n}} \le \alpha + \beta \quad \text{a.s.} \tag{2.1}$$

By the Borel–Cantelli lemma, it is enough to prove that for every $\varepsilon > 0$,

$$\sum_{n=2}^{\infty} P\left(\left\|\sum_{k=1}^n X_{nk}\right\| > (1+\varepsilon)(\alpha+\beta)\sqrt{2n \log n}\right) < \infty. \tag{2.2}$$

Note that

$$\begin{aligned} &\sum_{n=2}^{\infty} P\left(\left\|\sum_{k=1}^n X_{nk}\right\| > (1+\varepsilon)(\alpha+\beta)\sqrt{2n \log n}\right) \\ &\quad \le \sum_{n=2}^{\infty} nP(\|X\| > \sqrt{n \log n}) \end{aligned}$$

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$$+\sum_{n=2}^{\infty} P\left(\left\|\sum_{k=1}^{n} X_{nk} I(\|X_{nk}\| \le \sqrt{n\log n})\right\| > (1+\varepsilon)(\alpha+\beta)\sqrt{2n\log n}\right)$$
$$= I_1 + I_2.$$

It is easy to show that $E(\|X\|^4/\log^2(e+\|X\|)) < \infty$ implies $I_1 < \infty$. For I_2, note that

$$\frac{1}{\sqrt{2n\log n}}\left\|\sum_{k=1}^{n} EX_{nk} I(\|X_{nk}\| \le \sqrt{n\log n})\right\|$$
$$\le \frac{n}{\sqrt{2n\log n}} E\|X\| I(\|X\| > \sqrt{n\log n})$$
$$\le \frac{E\|X\|^2}{\sqrt{2}\log n} \to 0,$$
$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n\log n}} E\left\|\sum_{k=1}^{n}(X_{nk} I(\|X_{nk}\| \le \sqrt{n\log n}) - EX_{nk} I(\|X_{nk}\| \le \sqrt{n\log n}))\right\| \le \beta,$$

and

$$\sup\left\{Ef^2\left(\sum_{k=1}^{n}(X_{nk} I(\|X_{nk}\| \le \sqrt{n\log n}) - EX_{nk} I(\|X_{nk}\| \le \sqrt{n\log n}))\right)\right\} \le n\alpha^2.$$

Hence there exists η, $0 < \eta < \varepsilon$, when n is large enough, such that

$$P\left(\left\|\sum_{k=1}^{n} X_{nk} I(\|X_{nk}\| \le \sqrt{n\log n})\right\| > (1+\varepsilon)(\alpha+\beta)\sqrt{2n\log n}\right)$$
$$\le P\left(\left\|\sum_{k=1}^{n}(X_{nk} I(\|X_{nk}\| \le \sqrt{n\log n}) - EX_{nk} I(\|X_{nk}\| \le \sqrt{n\log n}))\right\| > (1+\eta)E\left\|\sum_{k=1}^{n}(X_{nk} I(\|X_{nk}\| \le \sqrt{n\log n}) - EX_{nk} I(\|X_{nk}\| \le \sqrt{n\log n}))\right\| + (1+\eta)\alpha\sqrt{2n\log n}\right).$$

So by Lemma 2.1 for $s > 4$,

$$I_2 \le \sum_{n=2}^{\infty} \exp\left(-\frac{(1+\eta)^2\alpha^2 \cdot 2n\log n}{2(1+\eta)n\alpha^2}\right) + c\sum_{n=2}^{\infty}(n\log n)^{-s/2} nE\|X\|^s I(\|X\| \le \sqrt{n\log n})$$
$$= I_3 + I_4.$$

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For I_3,

$$I_3 = \sum_{n=2}^{\infty} \exp\{-(1+\eta)\log n\} = \sum_{n=2}^{\infty} n^{-(1+\eta)} < \infty.$$

For I_4,

$$\begin{aligned} I_4 &= c\sum_{n=1}^{\infty}(n\log n)^{-s/2} n E\|X\|^s I(\|X\| \le \sqrt{n\log n}) \\ &\le cE\frac{\|X\|^4}{\log^2(e+\|X\|)} < \infty. \end{aligned}$$

So (2.2) follows, and hence (2.1) holds.

To obtain the lower bound it is enough to show that

$$\limsup_{n\to\infty} \frac{\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} \ge \alpha \quad \text{a.s.} \tag{2.3}$$

and

$$\limsup_{n\to\infty} \frac{\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} \ge \beta \quad \text{a.s.} \tag{2.4}$$

For every $f \in B_1^*$, $\{f(X), f(X_{nk}), 1 \le k \le n, n \ge 1\}$ is an array of i.i.d. random variables with $Ef(X) = 0$ and $E(f^4(X)/\log^2(e+|f(X)|)) < \infty$. From Qi [7] or Li *et al.* [6],

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n} f(X_{nk})|}{\sqrt{2n\log n}} = \sqrt{Ef^2(X)} \quad \text{a.s.}$$

Hence (2.3) holds. By the upper bound and the Kolmogorov 0–1 law,

$$\limsup_{n\to\infty} \frac{\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}}$$

is a constant in $[0, \alpha+\beta]$ almost surely, and by Lemma 2.2 and Fatou's lemma,

$$E\limsup_{n\to\infty} \frac{\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} \ge \limsup_{n\to\infty} \frac{E\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} = \beta.$$

Hence (2.4) holds.

It is obvious that (1.10) implies that $(\sum_{k=1}^{n} X_{nk})/\sqrt{2n\log n}$ is bounded in probability. And by the same argument as in Li *et al.* [6], (1.10) implies the moment conditions in (1.8). The proof is complete. □

PROOF OF THEOREM 1.5. By Theorems 1.1 and 1.2, it is enough to prove that

$$\lim_{n\to\infty} \frac{\|\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_i X_{n,k-i} - \sum_{i=-\infty}^{\infty} a_i \sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} = 0 \quad \text{a.s.} \tag{2.5}$$

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Given $m > 0$, set

$$Y_{nm} = \sum_{k=1}^{n} \sum_{i=-m}^{m} a_i X_{n,k-i},$$

$$\widetilde{a}_m = 0, \quad \widetilde{a}_i = \sum_{j=i+1}^{m} a_j, \quad i = 0, \ldots, m-1,$$

$$\widetilde{\widetilde{a}}_{-m} = 0, \quad \widetilde{\widetilde{a}}_i = \sum_{j=-m}^{i-1} a_j, \quad i = -m+1, -m+2, \ldots, 0,$$

$$\widetilde{X}_{nk} = \sum_{i=0}^{m} \widetilde{a}_i X_{n,k-i}, \quad \widetilde{\widetilde{X}}_{nk} = \sum_{i=-m}^{0} \widetilde{\widetilde{a}}_i X_{n,k-i}.$$

Then

$$Y_{nm} = \left(\sum_{i=-m}^{m} a_i\right) \sum_{k=1}^{n} X_{nk} + (\widetilde{X}_{n0} - \widetilde{X}_{nn} + \widetilde{\widetilde{X}}_{n,n+1} - \widetilde{\widetilde{X}}_{n1}) \tag{2.6}$$

and

$$\sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i X_{n,k-i} = Y_{nm} + \sum_{k=1}^{n} \sum_{|i|>m} a_i X_{n,k-i}. \tag{2.7}$$

For every i, $E(\|X\|^4/\log^2(e + \|X\|)) < \infty$ implies, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\{\|X_{n,n-i}\| > \varepsilon\sqrt{2n \log n}\} < \infty.$$

Hence, by the Borel–Cantelli lemma,

$$\lim_{n\to\infty} \frac{\|X_{n,n-i}\|}{\sqrt{2n \log n}} = 0 \quad \text{a.s.}$$

So

$$\lim_{n\to\infty} \frac{\|\widetilde{X}_{nn}\|}{\sqrt{2n \log n}} = 0 \quad \text{a.s.}$$

and

$$\lim_{n\to\infty} \frac{\|\widetilde{\widetilde{X}}_{n,n+1}\|}{\sqrt{2n \log n}} = 0 \quad \text{a.s.}$$

Furthermore,

$$\lim_{n\to\infty} \frac{\|\widetilde{X}_{n0}\|}{\sqrt{2n \log n}} = \lim_{n\to\infty} \frac{\|\widetilde{\widetilde{X}}_{n1}\|}{\sqrt{2n \log n}} = 0 \quad \text{a.s.}$$

Hence

$$\lim_{n\to\infty} \frac{\|\widetilde{X}_{n0} - \widetilde{X}_{nn} + \widetilde{\widetilde{X}}_{n,n+1} - \widetilde{\widetilde{X}}_{n1}\|}{\sqrt{2n \log n}} = 0 \quad \text{a.s.} \tag{2.8}$$

By (2.6)–(2.8) and Theorem 1.2,

$$\begin{aligned}
\limsup_{n\to\infty} & \frac{\|\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_i X_{n,k-i} - \sum_{i=-\infty}^{\infty} a_i \sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} \\
&= \limsup_{n\to\infty} \frac{\|\sum_{|i|>m} a_i \sum_{k=1}^{n} X_{nk} + \sum_{|i|>m} a_i \sum_{k=1}^{n} X_{n,k-i}\|}{\sqrt{2n\log n}} \\
&\le \limsup_{n\to\infty} \frac{|\sum_{|i|>m} a_i|\,\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} \\
&\quad + \limsup_{n\to\infty} \frac{\sum_{|i|>m} |a_i|\,\|\sum_{k=1}^{n} X_{n,k-i}\|}{\sqrt{2n\log n}} \\
&\le \Big|\sum_{|i|>m} a_i\Big|(\alpha+\beta) + \sum_{|i|>m} |a_i| \sup_{n\ge 1} \frac{\|\sum_{k=1}^{n} X_{n,k-i}\|}{\sqrt{2n\log n}} \quad \text{a.s.} \qquad (2.9)
\end{aligned}$$

By the stationarity of $\{X_{ni},\ -\infty < i < \infty,\ n \ge 1\}$ and Lemma 2.2,

$$\begin{aligned}
E\sum_{i=-\infty}^{\infty} |a_i| \sup_{n\ge 1} \frac{\|\sum_{k=1}^{n} X_{n,k-i}\|}{\sqrt{2n\log n}} &\le \sum_{i=-\infty}^{\infty} |a_i| E \sup_{n\ge 1} \frac{\|\sum_{k=1}^{n} X_{n,k-i}\|}{\sqrt{2n\log n}} \\
&= \sum_{i=-\infty}^{\infty} |a_i| E \sup_{n\ge 1} \frac{\|\sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} \\
&< \infty.
\end{aligned}$$

Hence

$$\sum_{i=-\infty}^{\infty} |a_i| \sup_{n\ge 1} \frac{\|\sum_{k=1}^{n} X_{n,k-i}\|}{\sqrt{2n\log n}} < \infty \quad \text{a.s.}$$

Letting $m \to \infty$ in (2.9),

$$\limsup_{n\to\infty} \frac{\|\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_i X_{n,k-i} - \sum_{i=-\infty}^{\infty} a_i \sum_{k=1}^{n} X_{nk}\|}{\sqrt{2n\log n}} = 0 \quad \text{a.s.},$$

which completes the proof. □

References

[1] U. Einmahl and D. L. Li, 'Characterization of the LIL behavior in Banach space', *Trans. Amer. Math. Soc.* **360** (2008), 6677–6693.

[2] P. Hartman and P. Wintner, 'On the law of the iterated logarithm', *Amer. J. Math.* **63** (1941), 169–176.

[3] T.-C. Hu and N. C. Weber, 'On the rate of convergence in the strong law of large numbers for arrays', *Bull. Aust. Math. Soc.* **45** (1992), 279–282.

[4] M. Ledoux and M. Talagrand, *Probability in Banach Spaces* (Springer, Berlin, 1991).

[5] D. L. Li and M. L. Huang, 'Strong invariance principles for arrays', *Bull. Inst. Math. Acad. Sin.* **28** (2000), 167–181.

https://doi.org/10.1017/S0004972710000158 Published online by Cambridge University Press

[6] D. L. Li, M. B. Rao and K. J. Tomkins, 'A strong law for B-valued arrays', *Proc. Amer. Math. Soc.* **123** (1995), 3205–3212.

[7] Y. C. Qi, 'On the strong convergence of arrays', *Bull. Aust. Math. Soc.* **50** (1994), 219–223.

[8] V. Strassen, 'A converse to the law of the iterated logarithm', *Z. Wahrscheinlichkeitstheor. Verwandte Geb.* **4** (1960), 265–268.

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