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FLOPS

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§ 1. Introduction

The aim of this note is to study birational transformations of threefolds with nef canonical classes. One would like to write any such map as a composite of certain simple and basic transformations.

A special case was first considered by Kulikov [Ku] and later extensively studied by several authors. The next main conceptual step was Reid's study [R2] of small resolutions of terminal threefold singularities. He obtained the elementary birational transformations as follows: find a copy of such a small resolution inside the threefold and then replace it with another resolution. This approach was extended by Kawamata [K3] to canonical singularities.

If $g: X \dashrightarrow X'$ is a birational transformation between threefolds with nef canonical classes then using the above results one can start factoring g into the composite of such elementary transformations. I will call them flops. It is however not clear that the process will ever stop. The main contribution of this note is a simple proof of this fact. Besides extending the scope of applications it allows one to simplify considerably the proof of the existence of flops given in [K3].

Section two contains a proof of the existence of flops in the terminal case. This quick proof is based on an idea of Mori. I am grateful to him for allowing me to present it here. Section three is devoted to a special case of the termination of flops. In the next section this is used to solve the above mentioned factorisation problem of birational maps. Finally in sections five and six a simple proof of the main theorem of [K3] is given. The arguments show that going from terminal to canonical singularities is rather easy in all dimensions.

Most of the proofs work for algebraic and analytic threefolds as well.

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We will work simultaneously in both categories, unless projectivity is explicitly assumed. This is done in Sections 3.6 and 5.6.

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§ 2. Existence of flops; terminal case

The existence of flops for cDV points was established by Reid [R2, 2.12]. Here I present a different approach which was pointed out to me by Mori. This has the advantage of giving very precise geometric information.

Definition 2.1. (i) Let $f\colon Z \to Y$ be a proper surjective birational map with exceptional set $E \subset Z$. Assume that f is small (i.e. $\dim E \leq \dim Z - 2$). Let D be a divisor on Z such that D is f-ample.

Assume that f is indecomposable (i.e., if $f\colon Z \to U \to Y$ and U is normal then $Z \cong U$ or $Y \cong U$). Finally, assume that $K_Z = f^*K_Y$. A proper, surjective small map $f^+\colon Z^+ \to Y$ is called the D-flop of f if D^+, the proper transform of D on Z^+, is f^+ ample. It is easy to see that the D-flop of f is unique if it exists. Given Z and D there can be several maps $f_i\colon Z \to Y_i$ such that the above conditions are satisfied. Any of the $f_i^+\colon Z_i^+ \to Y_i$ will be called a D-flop.

(ii) Given Z and D we say that D-flops exist if they exist for every possible choice of f that satisfies the above conditions. Any resulting birational may $Z \dashrightarrow Z^+$ will be called a D-flop.

(iii) Given Z and D a sequence of varieties and maps (Z^i, D^i) and $g_i\colon Z^i \dashrightarrow Z^{i+1}$ is called a sequence of D-flops if $(Z^0, D^0) = (Z, D)$, each g_i is a D^i-flop and D^{i+1} is the proper transform of D^i under g_i.

(iv) All the above definitions make sense if D is a $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-divisor.

Proposition 2.2. *Let $f\colon Z \to Y$, D and E be as in* 2.1 (i). *Let Y' be a small* (*formal or analytic*) *neighborhood of $f(E) \subset Y$. Assume that there is a map $t\colon (f(E), Y') \to (f(E), Y')$ such that t induces* $-\mathrm{id}$ *on* $\mathrm{Pic}(Y' - f(E))$. *Then the D-flop $f^+\colon Z^+ \to Y$ exists and $f^{-1}(Y') \cong (f^+)^{-1}(Y')$.* (*The isomorphism is not the expected one.*)

Proof. We construct Z by attaching $f^{-1}(Y')$ and $(Y - f(E))$ via

$$f^{-1}(Y') - E \cong Y' - f(E) \xrightarrow{t} Y' - f(E) \subset Y - f(E).$$

f^+ is given as $t \circ f$ on $f^{-1}(Y')$ and as f on $Y - f(E)$.

By assumption $t^{-1}(D | Y' - f(E))$ is linearly equivalent to $-D | Y' - f(E)$ thus it extends to a Cartier divisor on $f^{-1}(Y')$. Therefore $t^{-1}(D)$ on $f^{-1}(Y')$ and D on $Y - E$ glue together to a Cartier divisor D^+ on Z^+. By construction D^+ is f^+ ample. This completes the construction of the D-flop and it also shows that f^+ is projective.

EXAMPLE 2.3. Let Y be a hypersurface singularity of multplicity two. In suitable coordinates its equation is of the form $x_1^2 + f(x_2, \cdots, x_n) = 0$. Let $t(x_1, x_2, \cdots, x_n) = (-x_1, x_2, \cdots, x_n)$. Then $Y/t = (C^{n-1}, 0)$. If $g\colon Y \to Y/t$ is the quotient map then for any Weil divisor D we have $D + t(D) = g^* g_*(D)$ which is Cartier. Thus for any t-invariant $F \subset Y$, t induces $-\mathrm{id}$ on $\mathrm{Pic}(Y - F)$.

THEOREM 2.4. *Let Z be an algebraic or analytic threefold with terminal singularities and let $D \subset Z$ be a Cartier divisor. Then D-flops exist. Moreover Z and Z^+ have the same analytic singularities.*

Proof. Let $f\colon Z \to Y$ be as in 2.1 (i). From the definition it is clear that Y has terminal singularities as well and so by [R1, 0.6] Y is locally the cyclic quotient of a hypersurface double point $X \subset C^4$.

By [K3, 3.2] this already implies that D-flops exist. We will however need more precise information.

By [R4, 6.9] we can assume that the coordinates on $C^4 \supset X$ are eigenfunctions of the cyclic group action G and that (x, X) is given by an equation of the form $x^2 + f(y, z, u) = 0$ or $xy + f(z, u) = 0$.

In the first case the involution $t(x, y, z, u) = (-x, y, z, u)$ commutes with G hence it gives an involution on Y. Thus by 2.2 and 2.3, Z and Z^+ have the same analytic singularities.

In the other case let $Z_X = Z \times_Y X$. Then $f_X\colon Z_X \to X$ is a small partial resolution. By Mori's classification [Mo1] the group G acts on the coordinates with weights $(1, -1, a, 0)$ for some a. $(u = 0)$ defines a DuVal singularity with equation $xy + z^n = 0$ and with a group action $(1, -1, a)$. The minimal resolution is obtained by repeatedly blowing up the origin, thus the group action on the minimal resolution is readily computable. One sees at once that if a partial resolution B is dominated by the minimal one then the following hold:

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(i) The cyclic group action lifts to an action on B.

(ii) If $z \in B$ is fixed by some non-identity element then it is fixed by the whole group and in suitable analytic coordinates B is given by $pq + r^k = 0$ with group action $(1, -1, b)$.

We apply this for $B = (f_X^{-1}(u) = 0)$. Since $f_X^{-1}(u)$ in invariant, this implies that at any fixed point Z_X is given by an equation $pq + g(r, s) = 0$ with group action $(1, -1, b, 0)$.

Now take $t(x, y, z, u) = (y, x, z, u)$. Then t acts as $-\mathrm{id}$ on $\mathrm{Pic}(X - 0)$. t acts only rationally on Z_X but tGt^{-1} acts regularly since it leaves the f_X-ample divisor tD invariant. One can easily see that $Z^+ = Z_X/tGt^{-1}$ is the D-flop. The tGt^{-1} action on Z_X is easy to compute. It has the same fixed points as the G-action and using the same local coordinates it acts with weights $(-1, 1, b, 0)$. Interchanging p and q shows that

$$(pq + g(r, s) = 0)/G \cong (pq + g(r, s) = 0)/tGt^{-1}.$$

Thus Z and Z^+ have the same analytic singularities. This completes the proof.

§ 3. Terminal flops: simple case

Definition 3.1. (i) Let X be a normal variety and $E \subset X$ a prime divisor. E defines a discrete rank one valuation, denoted by $v(E)$. We call a valuation algebraic if there is a proper birational map $X' \to X$ and a prime divisor $E' \subset X'$ such that $v(E') = v'$. We say that v' has small center on X if its center has codimension at least two.

(ii) If $K_X + \sum a_i F_i = K_X + F$ is a $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-divisor on X, and if $f\colon Y \to X$ is a birational map, then one can write $K_Y = f^*(K_X + F) + \sum a(F, E_i)E_i$, where the E_i are different prime divisors on Y. $a(F, E_i)$ will be called the F-discrepancy of E_i. If $v(E_i) = v_i$, then $a(F, v_i) = a(F, E_i)$ is the F-discrepancy of v_i. This is independent of the choice of E_i and Y.

(iii) If F is effective, then (X, F) is said to be terminal (resp. canonical, resp. log-terminal) if $a(F, v) > 0$ (resp. ≥ 0, resp. > -1) for every v which has small center on X and $a(F, F_i) > 0$ (resp. ≥ 0, resp. > -1) for every i. In particular, if (X, F) is terminal or canonical then $F = \emptyset$.

(iv) Now assume that X is smooth and that $\sum F_i$ is a divisor with simple normal crossings. Let $Z \subset X$ be an irreducible and reduced subvariety. Let $Y = B_Z X$ and E the exceptional divisor dominating Z.

Straightforward computation using adjunction formula for blow ups shows that

$$a(F, E) = \operatorname{codim}(Z, X) - 1 - \textstyle\sum_{F_i \supset Z} a_i .$$

Taking into account that $a(F, F_i) = -a_i$, this becomes the nicer formula

$$(*) \qquad 1 + a(F, E) = \textstyle\sum_{F_i \supset Z} [1 + a(F, F_i)] + [\operatorname{codim}(Z, X) - \#\{F_i \supset Z\}] .$$

COROLLARY 3.2. *Assume that* X *is smooth and* $\sum F_i$ *is a normal crossing divisor. Then*

(i) *If* $1 \geq c \geq -1$ *and* $a(F, F_i) \geq c$ *for every* i, *then* $a(F, v) \geq c$ *for every* v *which has small center on* X *and is algebraic.*

(ii) *If furthermore* $0 \geq c > -1/2$ *then there are only finitely many discrete rank one valuations* v_j *such that* $a(F, v_j) < 1 + c$ *and* v_j *has small center on* X.

(iii) *Assume furthermore that* $a(F, F_i) + a(F, F_j) \geq 0$ *if* F_i *and* F_j *intersect. Then any algebraic valuation* v *with small center on* X *such that* $a(F, v) < 1$ *is obtained as in* 3.1 (iv) *by blowing up a* $Z \subset X$ *such that* $\operatorname{codim}(Z, X) = 2$, *exactly one of the* F_i*'s contains* Z *and* $a(F, F_j) < 0$ *for* $Z \subset F_j$.

Proof. Let $Z \subset X$ be a subvariety and let $Y = B_Z X$, $f \colon Y \to X$ the natural projection. If Z is not smooth then Y can have very bad singularities. The simplest way to remedy this is to discard $\operatorname{Sing} Z$ from X.

Let $E = \sum -a(F, E_i)E_i$. Then $K_Y + E = f^*(K_X + F)$. Therefore if a statement involves the pull back of $K_X + F$ in a tower of blow ups then we are allowed to prove this for one blow up only.

In order to prove (i) we blow up the center of v on X. (*) implies the statement for one blow up. Now one can conclude by evoking a result of Zariski: by repeated blow ups the center of v eventually becomes a divisor (see e.g. [A2, 5.2]).

To prove (ii) we blow up $F_i \cap F_j$ such that $a(F, F_i) + a(F, F_j)$ is minimal. (*) easily implies that doing this repeatedly we reach a situation $g \colon X'' \to X$ such that $f^*(K_X + F) = K_{X''} + F''$ and F'' satisfies the condition of (iii). Therefore (iii) implies (ii).

To prove (iii) we again proceed one blow up at a time. Let v be the valuation obtained by blowing up $Z \subset X$. If $Z \not\subset \cup F_i$ then $a(F, v) \geq 1$. If Z is contained in at least two of the F_i's then $a(F, v) \geq \#\{F_i, F_i \supset Z\} - 1 + \sum_{F_i \supset Z} a(F, F_i)$. The last sum is non-negative by assumption, hence

$a(F, v) \geq 1$. If F_j is the only one that contains Z then $a(F, v) \geq a(F, F_j) + \operatorname{codim}(Z, X) - 1$. Thus $a(F, v) \geq 1 + c$ and $a(F, v) < 1$ only if $\operatorname{codim}(Z, X) = 2$. Since $c > -1/2$, the hypotheses of (iii) are satisfied by $B_Z X$ and we can use induction to complete the proof.

This result will be used in the following situation.

LEMMA 3.3. *Let Y be a normal variety with terminal singularities and let $D = \sum a_i D_i$ be an effective $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-divisor. Let $a = \max\{a_i\}$. Let $f\colon X \to Y$ be a resolution of singularities such that the proper transform of D and the exceptional divisors cross normally and the proper transforms of the D_i's are disjoint. For $\varepsilon > 0$ let*

$$K_X = f^*(K_Y + \varepsilon D) + \sum a(\varepsilon D, F_j)F_j .$$

If ε is sufficiently small, then all the conditions in 3.2 are satisfied by $(X, \sum a(\varepsilon D, F_j)F_j)$ and $a(\varepsilon D, F_j) > 0$ if F_j is f-exceptional. Thus for valuations with small centers on X we have the following two claims:

(i) *There are only finitely many v such that $a(\varepsilon D, v) < 1 - \varepsilon a$, and*

(ii) *if $a(\varepsilon D, v) < 1$, then either v is obtained by blowing up a codimension two subvariety B in Y such that D and Y are generically smooth along B, or v is one of finitely many other valuations.*

Finally we have to recall the following result of Shokurov.

PROPOSITION 3.4 [S, 2.13; K-M-M, 5-1-11]. *Let (Z, D) be a variety with log-terminal singularities and let $f\colon Z \to Y$ be a small contraction of a $K_X + D$ – extremal ray. Assume that the log-flip $f^+\colon Z^+ \to Y$ and D^+ exists. If v is an algebraic valuation, then $a(D^+, v) \geq a(D, v)$ and strict inequality holds if the center of v on Z is contained in the f-exceptional locus.*

Now we are ready to formulate the main result of this chapter.

THEOREM 3.5. *Let X be a proper threefold with terminal singularities and let $D \subset X$ be an effective $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-divisor. Then any sequence of D-flops is finite.*

The proof is a modification of an argument of Shokurov [S, 2.13]. It is especially simple in the case when $D = \sum D_i$, where the D_i are distinct prime divisors. The general case is postponed until chapter five.

Proof (special case). Let $D = \sum D_i$. For ε small 3.3 implies that there

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are only finitely many v with $a(\varepsilon D, v) < 1 - \varepsilon$. Now let (X^+, D^+) be a D-flop of (X, D). Let $E^+ \subset X^+$ be the exceptional set. X^+ is generically smooth along E^+ by 2.4. Let D^+ have generic multiplicity m along E^+. If v is one of the prime divisor obtained by blowing up E^+, then $a(\varepsilon D, v) < a(\varepsilon D^+, v) = 1 - m\varepsilon$. This implies that if we set

$$A(Z, \varepsilon D) = \textstyle\sum_1^\infty \sharp\{v \mid a(\varepsilon D, v) < 1 - j\varepsilon \text{ and } v \text{ has a small center on } Z\}$$

then $A(Z^+, \varepsilon D^+) \leq A(Z, \varepsilon D)$ and strict inequality holds if $m > 0$.

Since $A(Z, \varepsilon D)$ is a nonnegative integer, it cannot decrease indefinitely. Thus if $(X, D) = (X^0, D^0), (X^1, D^1), \cdots$ is an infinite sequence of D-flops, then for large i $A(X^i, \varepsilon D^i)$ is constant and $m = 0$.

If $E \subset X$ is the exceptional set, then $E \cdot D < 0$ by definition; hence $E \subset D$. If $m = 0$, then $E^+ \not\subset D^+$. Thus the normalization of D^+ is obtained from the normalization of D by contracting the preimage of E. Applying this to the sequence (X^i, D^i), we see that for large i the normalization of D^{i+1} is obtained from the normalization of D^i by contracting at least one curve of E. This procedure again must stop. Therefore the sequence of D-flops is finite.

An easy application is the following (cf. [Mo2]) 0.3.14.1).

PROPOSITION 3.6. *Let X be a projective threefold with $\boldsymbol{Q}$-factorial terminal singularities such that K_X is nef. Let $G \subset \operatorname{Bir} X$ be a finite subgroup. Then there is an $\bar{X}$ birational to X with terminal singularities such that $K_{\bar{X}}$ is nef and $G \subset \operatorname{Aut} \bar{X}$ (in the natural way). In general X is not $\boldsymbol{Q}$-factorial.*

Proof. Let H be ample on X and let $D = \sum_{g \in G} g[H]$. By 3.5 after finitely many D-flops, we get an (X^1, D^1) such that $nK_{X^1} + D^1$ is nef. Therefore some multiple of it gives a regular birational map $X^1 \to \bar{X}$ such that $nK_{\bar{X}} + \bar{D}$ is ample. The birational G-action on $\bar{X}$ maps the ample divisor $nK_{\bar{X}} + \bar{D}$ into itself; therefore the action is regular (this is quite easy, see e.g. [Ma-Mu]).

§ 4. Birational maps with K nef

As an application of the previous results we will study birational maps between threefolds with nef canonical classes. This generalises earlier results that were mostly restricted to the case of trivial canonical class (see e.g. Kulikov [Ku], Persson-Pinkham [P-P], Shepherd-Barron

[S-B], Reid [R2], Kawamata [K3]).

REDUCTION 4.1. If X' is a threefold with canonical singularities and K_X is nef then by Reid [R1, R2] there is a projective and birational map $f\colon X \to X'$ such that X has $\boldsymbol{Q}$-factorial terminal singularities and K_X is still nef. Therefore we will restrict ourselves to threefolds with $\boldsymbol{Q}$-factorial and terminal singularities. One should note that an algebraic variety with $\boldsymbol{Q}$-factorial singularities can well have singularities that are analytically not $\boldsymbol{Q}$-factorial.

If L is a line bundle on a complex space then we will say that L is nef if it has non negative degree on every compact curve contained in that space. Note that there might not be any compact curves.

The following lemma will be useful

LEMMA 4.2. *Let $(x, X) \subset \boldsymbol{C}^n$ be a three dimensional terminal singularity and let S be a small sphere around x. Then (x, X) is analytically $\boldsymbol{Q}$-factorial iff $H_i(X \cap S, \boldsymbol{Q}) = 0$ for $0 < i < 5 = \dim X \cap S$.*

Proof. The fundamental group of $X \cap S$ is finite cyclic by [R2, 0.6; G, X 3.4] hence $H_1 = 0$. By Flenner [F, 6.1] $H^2(X \cap S, \boldsymbol{Z}) = \operatorname{Pic}(X - x)$ thus (x, X) is $\boldsymbol{Q}$-factorial iff $H^2(X \cap S, \boldsymbol{Q}) = 0$. The rest follows from Poincaré duality.

LEMMA 4.3 (Hanamura [Ha, 3.4]). *Let X and X' be compact n-folds with terminal singularities such that K_X and $K_{X'}$ are both nef. Let $g\colon X \dashrightarrow X'$ be a bimeromorphic map. Then g is an isomorphism in codimension one.*

Proof. Let Y be a desingularisation of the graph of g and let $p\colon Y \to X$ and $q\colon Y \to X'$ be the projections. By E_i (resp. F_i resp. G_i) we mean some non negative linear combination of divisors on Y that are both p and q-exceptional (resp. p but not q-exceptional, resp. q but not p-exceptional). Then

$$K_Y = p^*K_X + E_1 + F_1 \quad \text{and} \quad K_Y = q^*K_{X'} + E_2 + G_2 .$$

Since $p_*(q^*K_{X'} + G_2) = K_X$ we can write

$$p^*K_X = q^*K_{X'} + G_2 + (p\text{-exceptional components}).$$

I claim that the p-exceptional component is effective. It is sufficient to prove this after some further blow ups, thus we may assume that p is

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projective. Now the question is local on X and by repeated hyperplane section we reduce the problem to the following:

Let $f: U \to V$ be a resolution of a surface singularity. Let D be a Cartier divisor on V and assume that $f^*D = D' + G +$ (f-exceptional components) where D' is f-nef and G is effective containing no f-exceptional curves. Then the f-exceptional part is effective. This can be proved easily; see e.g. [K-S-B, 3.5].

Thus $q^*K_{X'} + G_2 = p^*K_X - E_3 - F_3$; hence $E_1 + F_1 = E_2 - E_3 - F_3$. In particular $F_1 = F_3 = 0$. Since X has terminal singularities this implies that any p-exceptional divisor is q-exceptional as well. If $E \subset Y$ is the union of all p-exceptional divisors then $p(E)$ and $q(E)$ have codimension at least two and g defines an isomorphism between $X - p(E)$ and $X' - q(E)$. This was to be proved.

LEMMA 4.4 (Notation as in 4.3). *Assume in addition that* $\dim X = 3$ *and X and X' are $\mathbf{Q}$-factorial. Let D be a divisor on X and let D' be its proper transform on X'. If both D and D' are nef then they are both numerically trivial along the exceptional loci (i.e. subsets where g (resp. g^{-1}) is not regular).*

Proof. Since $D = p_*q^*D'$ the previous considerations show that $q^*D' = p^*D -$ (effective exceptional divisor). Reversing the roles gives that $p^*D = q^*D'$. If $C \subset q(E)$ is any curve then we can find a $\tilde{C} \subset Y$ such that $q(\tilde{C}) = mC$ for some $m > 0$ and $p(\tilde{C}) =$ point. Thus $mCD' = \tilde{C}q^*D' = \tilde{C}p^*D = p(\tilde{C})D = 0$.

COROLLARY 4.5. *With the above notation both K_X and $K_{X'}$ are numerically trivial along the g exceptional loci.*

PROPOSITION 4.6. *Notation as in* 4.4. *Then the exceptional loci are unions of rational curves.*

Proof. Let $C' \subset X'$ be a component of the exceptional locus and let $E \subset Y$ be a divisor such that $q(E) = C'$ and $p(E) = C$ is a curve too. Let D be a small two dimensional disc intersecting C transversally at a general point. Then $p^{-1}(D) \cap E$ is a union of $p^{-1}(D) \to D$ exceptional curves and all these are rational. Since the above intersection dominates C', it is rational as well.

Remark 4.7. If is likely that the exceptional loci are covered by

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rational curves for $\dim X$ arbitrary.

EXAMPLE 4.8. Let S be a surface and embed it into a threefold X such that its normal bundle is $N_{S|X} \cong K_S$. (We might not be able to do this for a compact X. If S is a Del Pezzo surface then one can choose X to be even projective and of general type.) Let $E = \{E_i\}$ and $F = \{F_j\}$ be two sets of -1 curves in S such that within one set the curves do not intersect. These curves have $\mathcal{O}(-1) + \mathcal{O}(-1)$ normal bundle in X and therefore they can be flopped (cf. [Ku]). By flopping all the curves in the set E one obtains a threefold X_E. The proper transform of S is a surface S_E which is obtained from S by contracting all the curves in E. For each i there is a curve $E'_i = \boldsymbol{CP}^1$ contained in X_E which intersects S_E in the point corresponding to E_i. Similarly one can get X_F etc. The exceptional locus of the natural map $X_E \dashrightarrow X_F$ is $\{E'_i\} \cup \{$the images of F_j in $S_E\}$. Now let us see some concrete examples.

(i) Let S be $\boldsymbol{CP}^2$ blown up in three general points. Let E be the set of three exceptional curves and let F be the set of proper transforms of the three lines connecting our three points. Then $S_E = \boldsymbol{CP}^2$ and the three lines in it are part of the exceptional locus. These lines can move in $\boldsymbol{CP}^2$ thus the exceptional locus can not be contracted to a point.

(ii) Pick a line and a conic in $\boldsymbol{CP}^2$. Blow up two points on the line and five on the conic to get S. Let E be the proper transform of the line and F be the proper transform of the conic. The image of F in S_E will be a nodal or cuspidal rational curve. Thus the components of the exceptional locus need not be smooth.

In both of the above examples X can be chosen to be projective and K_X nef.

THEOREM 4.9. *Let X and X' be projective (resp. compact analytic) threefolds with $\boldsymbol{Q}$-factorial terminal singularities. Assume that K_X and $K_{X'}$ are both nef. Let $g\colon X \dashrightarrow X'$ be a bimeromorphic map. Then g can be written as the composite of algebraic (resp. analytic) flops.*

Proof. Let $\{C_i\}$ resp. $\{C'_i\}$ be the components of the exceptional loci on X resp. X'. If X' is projective then let D' be an ample divisor on X'. Otherwise let $D' = \cup D'_i$ where D'_i is a small two dimensional disc intersecting C'_i at a general point. Let $D \subset X$ be the proper transform of D'. By 4.4 $DC_j < 0$ for some j.

In the projective case one can use the cone theorem (cf. [K-M-M, 4.2]) for $K_X + \varepsilon D$ (ε sufficiently small) to contract some $K_X + \varepsilon D$-extremal curves. Since D is ample on $X - \cup C_i$ the contracted extremal curves are among the C_i's. By 2.4 D-flops exist and repeated application gives an X'' and D'' such that D'' is nef. Since D'' is the proper transform of D on X'' we can use 4.4 to conclude that D is numerically trivial along the $X \dashrightarrow X''$ exceptional locus. D is ample on X therefore the exceptional locus must be empty and X and X'' are isomorphic.

In the analytic case let $E' = \cup E'_i$ be another union of two dimensional discs with the same properties as D' but disjoint from D'. Clearly D' and E' are numerically equivalent along $\cup C'_i$ thus so are their proper transforms D and E. Therefore there is a j such that $DC_j = EC_j < 0$. By construction $D \cap E \subset \cup C_i$ thus C_j is a component of $D \cap E$. We will shortly see that C_j is analytically contractible. Assuming this for a moment one can flop C_i and conclude the proof of the theorem as in the projective case. The contractibility is given by

LEMMA 4.10. *Let X be an analytic threefold and let D and E be Cartier divisors. Let $C \subset X$ be a compact irreducible curve. Assume that C is a component of $D \cap E$ and that $DC < 0$, $EC < 0$. Then C is analytically contractible.*

Proof. Let I_C be the biggest ideal sheaf on X that agrees with $(\mathcal{O}(-D), \mathcal{O}(-E)) \subset \mathcal{O}_X$ generically along C. It is easy to check that I_C and the map $\operatorname{Spec} \mathcal{O}_X/I_C \to \operatorname{Spec} \boldsymbol{C}$ satisfy the contractibility criterion of Artin [A1.6.2] (see [Bi, 6.1] for the analytic case).

Here are some nice corollaries of 4.9.

COROLLARY 4.11. *Notation as in* 4.9. *Then X and X' have the same analytic singularities.*

Proof. By 2.5 the singularities are unchanged under a flop.

COROLLARY 4.12. *Notation as in* 4.9. *Then g induces an isomorphism between the intersection homology groups $IH_i(X, \boldsymbol{Q})$ and $IH_i(X', \boldsymbol{Q})$. If X and X' are projective then this map* (*tensored with $\boldsymbol{C}$*) *is an isomorphism of the corresponding pure Hodge structures as well.*

Proof. We refer to [G-M] for the necessary definitions. Let $f: X \to Z$ be a small map between threefolds with terminal singularities. Let $h: Y \to X$ be a small map such that Y has only analytically $\boldsymbol{Q}$-factorial terminal singularities. By 4.2 Y is a rational homology manifold hence h and fh are rational homologically small [G-M, 6.2]. Thus by [G-M, 6.2, Theorem] $Rh_*IC^{\cdot}_Y(\boldsymbol{Q}) = IC^{\cdot}_X(\boldsymbol{Q})$ and $R(fh)_*IC^{\cdot}_Y(\boldsymbol{Q}) = IC^{\cdot}_Z(\boldsymbol{Q})$. Therefore $Rf_*IC^{\cdot}_X(\boldsymbol{Q}) = IC^{\cdot}_Z(\boldsymbol{Q})$ and this yields that $IH_i(Y, \boldsymbol{Q}) = IH_i(Z, \boldsymbol{Q})$.

Each flop gives a pair of maps $X \to Z \leftarrow X^+$ and so a pair of natural isomorphisms $IH_i(X, \boldsymbol{Q}) = IH_i(Z, \boldsymbol{Q}) = IH_i(X^+, \boldsymbol{Q})$. If X and X' are projective then each of these steps gives an isomorphism of the natural Hodge structures [Sa, Corollaire 3]. By 4.9 this completes the proof.

Remark 4.13. (i) The above isomorphism is for unpolarised Hodge structures only. However in the most important case of IH_3, if $h^1(X, \mathcal{O}_X) = 0$ then the polarisation is given by Poincaré duality thus the isomorphism preserves polarisation.

(ii) If X'' is any smooth projective variety birationally equivalent to the above X then $IH_i(X, \boldsymbol{C})$ is a direct summand of $H_i(X'', \boldsymbol{C})$, Hodge structures included. Therefore $IH_i(X, \boldsymbol{C})$ can be viewed as the invariant part of the Hodge structure on $H_i(X'', \boldsymbol{C})$ as X'' varies in its birational equivalence class.

Remark 4.14. Example 4.8 shows that exceptional loci of birational maps can be quite complicated even when K is nef. It turns out however that the exceptional locus of a single flop is quite simple. Since these are the same as exceptional loci of small partial resolutions of terminal threefold singularities we study the latter instead.

Let f: $X \to Y$ be a small map between threefolds with terminal singularities. Let $E = E_i$ be the exceptional curves. $R^1f_*\mathcal{O}_Y = 0$ implies that $H^1(E, \mathcal{O}_E) = 0$ thus all the E_i are smooth and they intersect transversally. Next we will describe the possible configurations of the exceptional curves and the indices of the singularities along E. For more detailed study of a special case see [M].

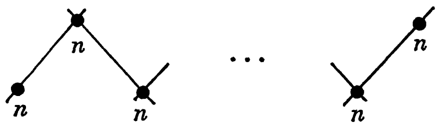
First we recall results of Reid [R2, 1.14] about the case when X has index one.

Case 1. X has index one. Let $H \subset X$ be a hyperplane section of X which has a Du Val singularity at the origin. Then $f^{-1}(H) \subset Y$ is normal and is dominated by the minimal resolution of H. Thus the configuration of curves is one of the Dynkin diagrams and every point of Y has index one. All possible Dynkin diagrams do occur.

Case 2. X has index larger than one. Let $g: X' \to X$ be the index one cyclic cover of X and let $f': Y' \to X'$ be the pull back of f to X'. Mori [Mo1] gave a complete list of the possible covers $g: X' \to X$ and his results show that there is always an $H' \subset X'$ such that $H' = g^{-1}(H)$ for some Weil divisor $H \subset X$ and H' has a Du Val singularity at the origin. Thus $f^{-1}(H)$ is normal and is dominated by a suitable quotient of the minimal resolution of H'. This quotient can be readily identified using Mori's explicit equations.

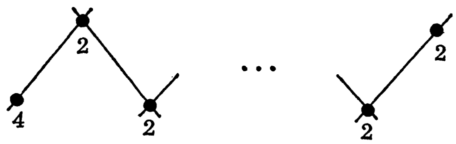
Next we list all possible configurations. A line segment represents an irreducible component of E, intersecting segments correspond to intersecting components. A $\bullet$ represents a singularity of index larger than one, we write the index under the $\bullet$. We use Mori's convention [Mo1] for the equations and for the group actions.

(i) $xy + f(z, u^n)/Z_n(1, -1, 0, a)$



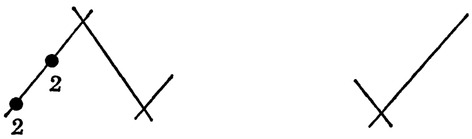
example: $xy + z^{nk} + u^{nk}$; E has $k - 1$ components.

(ii) $x^2 + y^2 + f(z, u^2)/Z_4(1, 3, 2, 1)$



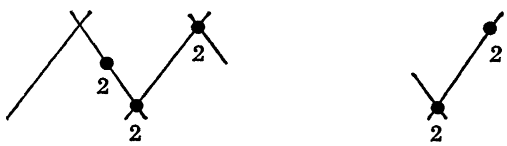
example: $x^2 + y^2 + (z + 2u^2)(z^{4k} + u^{4k})$; E has k components.

(iii) $x^2 + y^2 + f(z, u)/Z_2(1, 0, 1, 1)$



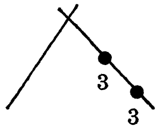
example: $x^2 + y^2 + z^{4k} + u^{2k}$; E has $k - 1$ components.

(iv) $u^2 + y^2z + z^n + xg(x, y, z, u)/Z_2(1, 1, 0, 1)$

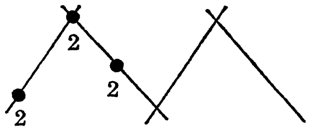


example: $u^2 + y^2z + z^k + x^{2k}$; E has k components.

(v) $u^2 + x^3 + y^3 + zg(x, y, z, u)/Z_3(1, 2, 2, 0)$



(vi) $u^2 + x^3 + y^4 + zg(x, y, z, u)/Z_2(0, 1, 1, 1)$



The above are the configurations with the maximal number of curves for the given class of singularities. Of course we can contract any subset of these configurations. In cases (iii)–(vi) this yields further possibilities. I do not have examples for most of these cases.

§ 5. Terminal flops, general case

The proof of 3.5 in the general case relies on the same ideas as the proof of the special case. However, it requires a little more careful bookkeeping. We start with a few simple observations.

Remark 5.1. (i) Let (x, X) be an isolated cDV point. Then $X - x$ is simply connected by [G, *X*. 3.4], and hence every $\boldsymbol{Q}$-Cartier divisor is Cartier.

(ii) Let Y be a threefold with terminal singularities and $D = \sum a_j D_j$ an effective $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-divisor. Let N_1 be the l.c.m. of indices of the singularities of Y. Assume furthermore that $N_2 a_j$ is an integer for every j. Let $N = N_1 N_2$. Then by (i) $N \cdot D$ is a Cartier divisor.

(iii) With the above notation, if (Y^+, D^+) is a D-flop of (Y, D), then $N \cdot D^+$ is Cartier again. This follows from 2.4.

Definition 5.2. Let φ: $(Y, D) \dashrightarrow (Y^+, D^+)$ be a D-flop and let $E \subset Y$ be the exceptional set. We define

$$a(D, \varphi) = \min\{a(D, v) \,|\, \text{center of } v \text{ is contained in } E\}.$$

If N is chosen as in 5.1 (ii), then $Na(D, v)$ is an integer where $a(D, v)$ is as in 3.1. Hence the above minimum is achived by some v if $a(D, v)$ is bounded from below (c.f. 3.2 (ii)).

LEMMA 5.3. *Let* $\varphi\colon (Y, D) \dashrightarrow (Y^+. D^+)$ *be a D-flop. Let* $E \subset Y$ *and* $E^+ \subset Y^+$ *be the exceptional sets. Let* $\bar{?}$ *denote the normalization of* ?. *Then*

(i) *If* $C \subset D_j$ *is a curve and if* v *is the valuation given by the divisor that we obtain from blowing up* C, *then* $a(D, v) \leq 1 - a_j$. *If* C *is not contained in* D, *then* $a(D, v) = 1$.

(ii) $a(D, \varphi) < 1$.

(iii) *If* $1 - a_j < a(D, \varphi)$, *then* $E \not\subset D_j$ *and* $E^+ \not\subset D_j^+$. *Hence* $\bar{D}_j$ *and* $\bar{D}_j^+$ *are isomorphic.*

(iv) *If* $1 - a_j = a(D, \varphi)$ *and* $E \subset D_j$, *then* $E^+ \not\subset D_j^+$. *Hence* $\bar{D}_j^+$ *is obtained from* $\bar{D}_j$ *by contracting the components of* E.

Proof. (i) is an easy computation. To see (ii) let v be obtained from blowing up E^+. Then $a(D, \varphi) \leq a(D, v) < a(D^+, v) \leq 1$. (iii) and (iv) follow from (i) and 8.4.

5.4. *Proof of* 3.5. Let $h\colon X' \to X$ be a resolution of singularities of X such that h^*D is a divisor with simple normal crossings. For our purpose we can always replace D by εD where ε is a small positive rational number. Thus we may and will assume that X' and $F = h^*D$ satisfy the conditions of 3.2 (ii).

Let $(X, D) = (X^0, D^0)$ and let $\phi_i\colon (X^i, D^i) \dashrightarrow (X^{i+1}, D^{i+1})$ be an infinite sequence of D-flops. I want to see that this cannot exist. If we define N as in 5.1 (ii), then $\min\{a(D^i, \varphi_i)\} = k/N$ for some k. We prove the non-existence by descending induction on k.

By 5.3 (ii) necessarily $k < N$. Let F^i be the union of those components D_j^i of D^i such that $1 - a_j = k/N$.

We consider those valuations v such that $a(D, v) = k/N$. By 3.2 (iii) these come from curves $C \subset F^0$ along which D^0 is generically smooth, (we denote these by v_C) and finitely many others $v_1, \cdots, v_m$.

By 3.2 (iii) if $a(D^i, v) = k/N$ for some i and v, then either $v = v_C$ or $v = v_j$ for some C or j. Furthermore, if the center of v is contained in $E_s \subset Y^s$, then $a(D^i, v) > k/N$ for $i > s$ (3.4). Thus there can be only finitely many flops φ_i such that E_i contains the center of some of the v_j's. We can assume that this will not happen if $i > i_0$.

If there is an i_1 such that for $i > i_1$ we have $a(D^i, \varphi_i) > k/N$, then by induction the sequence $(X^{i_1}, D^{i_1}), \cdots$ is finite and we are done. Otherwise let us look at the surfaces $\bar{F}^i$ for $i > i_0$. If $a(D^i, \varphi_i) > k/N$, then $\bar{F}^i$ and

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$\bar{F}^{i+1}$ are isomorphic by 5.3 (iii). If $a(D^i, \varphi_i) = k/N$ then $a(D^i, v_C) = k/N$ for some $C \subset E_i \subset \bar{F}^i$ by 5.3 (i). Hence by (5.3) (iv) $\bar{F}^{i+1}$ is obtained from $\bar{F}^i$ by contracting E_i. We cannot contract infinitely many times. Thus $a(D^i, \phi_i) > k/N$ for $i > i_1$ and this completes the proof.

As an application of the general form of 3.5, we give a short proof of a recent result of Kawamata-Matsuki [K-M].

Definition 5.5. Let X be a variety with canonical singularities. A proper birational morphism $f: Y \to X$ is called a crepant partial resolution if $K_Y = f^*K_X$ and Y is normal.

Corollary 5.6 [K-M]. *Let X be a threefold with canonical singularities. Then X has only finitely many projective crepant partial resolutions.*

Proof. Let $f: Y \to X$ be a crepant partial resolution. Then Y has canonical singularities as well. By [R1, 2.11] there is a crepant partial resolution $g: Z \to Y$ such that Z has only terminal singularities. By [R2, 8.2] there is a small resolution $h: U \to Z$ such that U has only $\boldsymbol{Q}$-factorial terminal singularities.

Let $p: U \to X$ be the composition of the above maps. Let $D \subset U$ be an effective divisor such that $-D$ is p-ample. The Cone Theorem [K2, R3, Ko] for $K_U + \varepsilon D$ gives that there are only finitely many p-extremal rays. $g \circ h: U \to Y$ is obtained by contracting some of these extremal rays. Thus U dominates only finitely many crepant partial resolutions. Therefore it is sufficient to prove that the number of crepant partial resolutions with $\boldsymbol{Q}$-factorial terminal singularities is finite. Let these be $p_i: U_i \to X$.

Let $H \approx -cD$ be an effective p-ample divisor on U and let $H_i \subset U_i$ be its proper transform on U_i. As in 3.6 we get that U is obtained from U_i by performing finitely many H-flops. Thus doing everything backwards we see that U_i is obtained from U by performing finitely many D-flops.

If there are infinitely many U_i's, then there are infinitely many finite sequences of D-flops. As we saw above, the number of p_j-extremal rays is finite. Hence the number of D_j-flops is finite for any j. Thus by König's theorem we would have an infinite sequence of D-flops. This contradicts 3.5, and hence the corollary is proved.

§ 6. Canonical flops

In this chapter we prove the existence and termination of flops for threefolds with $\boldsymbol{Q}$-factorial canonical singularities. This gives a generalisation of a result of Kawamata [K3, § 6].

It is worthwhile to note that the result remains true if X is a threefold with canonical singularities and D is a $\boldsymbol{Q}$-Cartier divisor. The existence of D-flops in this slightly more general situation is implied by 6.6. The same result can be used to reduce the termination to the $\boldsymbol{Q}$-factorial case.

Following Kawamata [K3, § 6] the proof proceeds by induction on the following measure of non-terminality:

DEFINITION 6.1. If X is a threefold with canonical singularities, then let $e(X)$ be the number of algebraic valuations v with small center on X such that $a(\emptyset, v) = 0$ (i.e. is crepant).

THEOREM 6.2. *Let X be a threefold with $\boldsymbol{Q}$-factorial canonical singularities. Let $D \subset X$ be an effective $\boldsymbol{Q}$-divisor. Then D-flops exist and any sequence of them is finite.*

The proof will be given in 6.8.

COROLLARY 6.3. *With the above assumptions assume that $K_X | D \equiv 0$. Then after finitely many D-flops the proper transform of D is either nef or some component of it is contractible.*

Proof. Any sequence of D-flops must stop. Call this (X^+, D^+). We still have $K_{X^+} | D^+ \equiv 0$ by 4.4. By the Contraction Theorem [K-M-M, 3.2.1] D^+ is either nef or some component of it is contractible.

COROLLARY 6.4. *Let X be a threefold with canonical singularities and let $v_1, \cdots, v_k$ be the crepant valuations centered on X. Then there is a sequence of maps $f_i \colon X_i \to X_{i-1}$ and a $g \colon X_0 \to X$ such that*

(i) *$X_0, \cdots, X_k$ are all $\boldsymbol{Q}$-factorial;*

(ii) *g is a small morphism;*

(iii) *f_i contracts a divisor corresponding to v_i and* $\dim N^1(X_i/X_{i-1}) = 1$

(iv) *All the X_i are projective over X.*

Proof. Let $h \colon Y \to X$ be a maximal projective crepant partial resolution [R2, 2.11]. As above we may assume that Y is $\boldsymbol{Q}$-factorial. Let $E \subset Y$ be any irreducible exceptional divisor. We apply 6.3 for (Y, E). Since E is exceptional, it will never become nef; hence it can be contracted after some E-flops. Continuing in this way, we can contract all exceptional divisors and this gives $X_0 \to X$.

If h_0: $Y_0 \to X_0$ is a maximal crepant $\boldsymbol{Q}$-factorial resolution, then we contract the divisors corresponding to $v_2, \cdots, v_k$. With the divisor corresponding to v_1 we flop until it becomes contractible. This gives X_1. The others are obtained similarly.

COROLLARY 6.5. *Any three dimensional canonical singularity has a $\boldsymbol{Q}$-factorial small partial resolution.*

COROLLARY 6.6 (Kawamata [K3, 6.1]). *Let X be a three dimensional canonical singularity and $D \subset X$ a Weil divisor. Then there is a small resolution f: $X^1 \to X$ such that the proper transform D^1 of D is $\boldsymbol{Q}$-Cartier and f-ample.*

Proof. Let g: $X_0 \to X$ be a $\boldsymbol{Q}$-factorial resolution; let D_0 be the proper transform of D. After some D_0-flops we get Y^+ and D^+ and D^+ is f^+-nef (no contraction is possible). Some multiple of D^+ is base-point free and maps Y^+ onto the required X^1.

COROLLARY 6.7 (Resolution of the DuVal locus). *Let X be a threefold with canonical singularities. There exists a p: $\overline{X} \to X$ such that p has at most one dimensional fibres and p (Sing $\overline{X}$) is finite. (I do not claim that $\overline{X}$ has isolated singularities).*

Proof. In 6.4 we arrange that $v_1, \cdots, v_i$ dominate the curves of Sing X and we take $\overline{X} = X_i$.

6.8. *Proof of* 6.2. We use induction on $e(X)$. The case of terminal singularities is $e(X) = 0$, this is already settled. Assume that 6.2 is already proved for $e(X) = n$. This implies 6.4 for $e(X) = n + 1$. In particular since X is $\boldsymbol{Q}$-factorial, $X_0 = X$ and f: $X_1 \to X$ is the contraction of a single extremal ray. We denote this map by f: $X' \to X$. We will use f to lift back the flopping problem from X to X' where we already solved it by induction. The proof of the existence part is the same as in [K3, § 6].

6.9. *Existence of flops.* Let p: $X \to Z$ be a small contraction such that K_X is p-trivial and D is p-negative. Let f: $X' \to X$ be the map obtained above. Let $E \subset X'$ be the exceptional surface. By induction on $e(X)$ the f^*D-flops exist and terminate after finitely many steps with $(X'^+, (f^*D)^+)$.

Assume first that $(f^*D)^+$ is nef on X'^+/Z. Then it maps onto some

X^+ and there is a divisor $D^+ \subset X^+$ such that $(f^*D)^+ = f^{+*}(D^+)$ where $f^+ \colon X'^+ \to X^+$ is the natural map. Assume that f^+ contracts only finitely many curves. Then $C^+ \cdot (f^*D)^+ > 0$ unless $f^+(C^+) = \text{point}$ and therefore $C \cdot f^*D > 0$ for all but finitely many curves $C \subset X'$, a contradiction. Therefore f^+ contracts a prime divisor E^+, $p^+ \colon X^+ \to Z$ is small, and (X^+, D^+) is the D-flop.

Otherwise $(f^*D)^+$ is not nef and it defines a divisorial contraction $f^+ \colon X'^+ \to X^+$ and $(f^*D)^+ = f^{+*}(D^+) + aE^+$ for some $D^+ \subset X^+$ and $a > 0$. Again I claim that (X^+, D^+) is the D-flop of (X, D). This follows once we establish that D^+ is X^+/Z-ample. It cannot be X^+/Z trivial because then $p(D) = p^+(D^+)$ were $\boldsymbol{Q}$-Cartier. If D^+ is X^+/Z negative, then $X^+ \cong X$ and so $(f^*D)^+ = f^{+*}(D) = f^{+*}(D^+)$, which contradicts the previous formula. This proves the existence of the D-flops.

6.10. *Termination of flops.* This is again done by induction on $e(X)$. Assume to the contrary that there is an infinite sequence (X_i, D_i). Let $f_0 \colon X'_0 \to X_0$ be the map given in 6.8. If $f_i \colon X'_i \to X_i$ is already constructed, then using the notation of 6.9 let $X'_{i+1} = X'^+_i$ and $f_{i+1} = f^+_i$. Recall that we had two different constructions for X^+_1. We shall say that $X_i \dashrightarrow X_{i+1}$ is an N-flop if $(f^*_i D_i)$ is nef on X'^+_i/Z_i and a C-flop if it is not nef.

If $X_i \dashrightarrow X_{i+1}$ is an N-flop, then $(X'_{i+1}, f^*_{i+1} D_{i+1})$ is obtained from $(X_i, f^*_i D_i)$ by finitely many flops. Therefore an infinite sequence of N-flops induces an infinite sequence of flops of the pair $(X'_j, f^*_j D_j)$. This is impossible by induction.

Thus in the infinite sequence (X_i, D_i) there must be infinitely many C-flops. If $X_i \dashrightarrow X_{i+1}$ is a C-flop, then $(f^*_i D_i)^+ = f^*_{i+1}(D_{i+1}) + a_i E_{i+1}$. If $?^\vee$ denotes the proper transform of a divisor ? on X'_0 and if $j_1, \cdots, j_k$ are the first k C-flops in the sequence, then

$$f^*_0 D_0 = a_{j_1} E^\vee_{j_1} + \cdots + a_{j_k} E^\vee_{j_k} + (f^*_{j_k} D_{j_k})^\vee .$$

Since all the divisors in the above equality are effective, this is possible only if $a_{j_k} \to 0$ as $k \to \infty$.

If there is a natural member M such that $M \cdot D_i$ is Cartier for every i then a_i is a multiple of $1/M$ for every i, contradicting $a_{j_k} \to 0$. In order to find M we use the following result

PROPOSITION 6.11 (Kawamata [K3]). *Let X be a threefold with canonical singularities. Let $r(X)$ be the index of X. Let D be a $\boldsymbol{Q}$-Cartier Weil* *divisor. Then $N \cdot D$ is Cartier for some $N \leq b(r(X), e(X))$.*

Remark 6.12. (i) The above proposition is the content of Step 3 of § 6 of [K3]. See also 6.15.

(ii) One can choose $b(r, e) = r \cdot 3^{2e}$ (see 6.15).

6.13. *End of proof.* If $X \to Z \leftarrow X^+$ is a flop, then $r(X) = r(Z) = r(X^+)$. Hence the indices of the X_i's are the same. If $D_0 = \sum b_j D_0^j$ is its expression as the sum of irreducible Weil divisors, then $D_i = \sum b_j D_i^j$. Then if the $K \cdot b_j$ are all integers, then by 6.11 $M = K \cdot b(r(X_0), e(X_0))!$ satisfies the requirements. This completes the proof.

Remark 6.14. Shephered-Barron proved [S-B] that if (x, X) is a three dimensional canonical singualrity, then the algebraic fundamental group $\hat{\pi}_1(X - x)$ is finite. Using 6.12 (ii), one can show that its order divides $r(X) \cdot (3^{2e(X)})!$.

Remark 6.15. The proof of 6.11 shows a principle which might be quite useful in other contexts as well.

Let X be an n-dimensional projective variety with log-terminal singularities and let $f: X \to Y$ be a divisorial contraction with exceptional divisor E. Then [Mi-Mo, Cor. 3] implies that E is covered by rational curves C satisfying $0 < -(E + K_X) \cdot C \leq 2n - 2$.

Assume that X is $\boldsymbol{Q}$-factorial and let $D \subset Y$ be a Weil divisor, $D' \subset X$ its proper transform. For some m_1 and m_2 the divisors $m_1 D'$ and $m_2 E$ are Cartier. $-E \cdot C \leq 2n - 2$ implies that $0 < -m_2 EC \leq m_2(2n - 2)$. The divisor $(m_2 E \cdot C) m_1 D' - (m_1 D' \cdot C) m_2 E$ is Cartier and has zero intersection with C. Therefore it descends to Y and we see that $(m_2 E \cdot C) m_1 D$ is Cartier. This implies the following.

PROPOSITION 6.16. *With the above notation assume that for any Weil divisor F on X mF is Cartier for some $0 < m \leq M$. Then for any Weil divisor D on Y mD is Cartier for some $0 < m \leq (2n - 2)M^2$.*

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