Canad. Math. Bull. Vol. 57 (3), 2014 pp. 648–657
http://dx.doi.org/10.4153/CMB-2014-021-9

On the $\mathcal{F}\Phi$-Hypercentre of Finite Groups

Juping Tang and Long Miao

Abstract. Let G be a finite group and let $\mathcal{F}$ be a class of groups. Then $Z_{\mathcal{F}\Phi}(G)$ is the $\mathcal{F}\Phi$-hypercentre of G, which is the product of all normal subgroups of G whose non-Frattini G-chief factors are $\mathcal{F}$-central in G. A subgroup H is called $\mathcal{M}$-supplemented in a finite group G if there exists a subgroup B of G such that $G = HB$ and H_1B is a proper subgroup of G for any maximal subgroup H_1 of H. The main purpose of this paper is to prove the following: Let E be a normal subgroup of a group G. Suppose that every noncyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ is $\mathcal{M}$-supplemented in G, then $E \le Z_{\mathcal{U}\Phi}(G)$.

1 Introduction

All the groups in this paper are finite. Most of the notation is standard and can be found in [3,6,7]. In what follows, $\mathcal{U}$ denotes the formation of all supersoluble groups and $\mathcal{N}$ denotes the formation of all nilpotent groups.

Let $\mathcal{F}$ be a class of groups and let H/K be a chief factor of a group G. Then H/K is called *Frattini* provided $H/K \le \Phi(G/K)$. Moreover, H/K is called $\mathcal{F}$-*central* if the semidirect product $[H/K](G/C_G(H/K)) \in \mathcal{F}$. The symbol $Z_{\mathcal{F}}(G)$ denotes the $\mathcal{F}$-hypercentre of a group G, that is, the product of all normal subgroups H of G whose G-chief factors are $\mathcal{F}$-central. A subgroup H of G is said to be $\mathcal{F}$-*hypercentral* in G if $H \le Z_{\mathcal{F}}(G)$.

The $\mathcal{F}$-hypercentre essentially influences the structure of a group. Note that if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and $E \le Z_{\mathcal{F}}(G)$, then $G \in \mathcal{F}$ for any concrete classes $\mathcal{F}$.

Recently, L. A. Shemetkov and A. N. Skiba in [11] proposed the new concept of $\mathcal{F}\Phi$-hypercentre of G and investigated the structure of $Z_{\mathcal{F}\Phi}(G)$ by using weakly s-permutable primary subgroups. Then $Z_{\mathcal{F}\Phi}(G)$ denotes the $\mathcal{F}\Phi$-hypercentre of G, which is the product of all normal subgroups of G whose non-Frattini G-chief factors are $\mathcal{F}$-central in G. The subgroup $Z_{\mathcal{F}\Phi}(G)$ is characteristic in G and every non-Frattini G-chief factor of $Z_{\mathcal{F}\Phi}(G)$ is $\mathcal{F}$-central in G.

Recall that a subgroup H of G is said to be supplemented in G if there exists a subgroup K of G such that $G = HK$. The relationship between the property of primary subgroups and the supplements of some restricted conditions has been studied extensively by many scholars. For instance, in 1937 Hall [5] proved that a group G

Received by the editors February 28, 2013; revised March 17, 2014.
Published electronically April 28, 2014.
L. Miao is the corresponding author.
This research is supported by NSFC (Grant #11271016), the Postgraduate Innovation Project of Jiangsu Province (No. CXZZ13_0890), and the Natural Science Fund for Colleges and Universities in Anhui Province (Grant #KJ2013B138).
AMS subject classification: **20D10**, 20D20.
Keywords: $\mathcal{F}\Phi$-hypercentre, Sylow subgroups, $\mathcal{M}$-supplemented subgroups, formation.

is soluble if and only if every Sylow subgroup of G is complemented in G. In 1980, Srinivasan [14] stated that a group G is supersoluble if every maximal subgroup of the Sylow subgroups is normal in G. In 2000, A. Ballester-Bolinches, Y. Wang, and X. Guo ([2, 15]) introduced the concept of a c-supplemented subgroup and proved that G is soluble if and only if every Sylow subgroup of G is c-supplemented in G. In 2007, as an interesting application of these generalizations, A. N. Skiba [13] fixed in every noncyclic Sylow subgroup P of G a group D satisfying $1 < |D| < |P|$ and then investigated the structure of G under the assumption that all subgroups H with $|H| = |D|$ are weakly s-permutable in G. Recently, Miao and Lempken [9] considered $\mathcal{M}$-supplemented subgroups of finite groups and obtained some new characterization of saturated formations containing all supersoluble groups.

As a continuation of this work, we shall investigate extensively the properties of $Z_{\mathcal{F}\Phi}(G)$ in which some primary subgroups are $\mathcal{M}$-supplemented.

Definition 1.1 A subgroup H is called $\mathcal{M}$-*supplemented* in a finite group G, if there exists a subgroup B of G such that $G = HB$ and H_1B is a proper subgroup of G for any maximal subgroup H_1 of H.

Recall that a subgroup H is called *weakly s-permutable in G* [11], if there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_{sG}$. In fact, the following example indicates that the $\mathcal{M}$-supplementation of subgroups cannot be deduced from weakly s-permutable subgroups.

Example 1.2 Let $G = S_4$ and let $H = \langle (1234) \rangle$ be a cyclic subgroup of order 4. Then $G = HA_4$ where A_4 is the alternating group of degree 4. Clearly, since $A_4 \trianglelefteq G$, A_4 permutes every maximal subgroup of H, and hence H is $\mathcal{M}$-supplemented in G. On the other hand, we have $H_{sG} = 1$. To see this, suppose first that H is s-permutable in G, then H is normal in G, a contradiction. If $H_{sG} = \langle (13)(24) \rangle$ is s-permutable in G, then $\langle (13)(24) \rangle$ is normal in G, which is also a contradiction. Therefore H is not weakly s-permutable in G.

2 Preliminaries

For the sake of convenience, we first list some results that will be used in the sequel.

Lemma 2.1 ([9, Lemmas 2.1 and 2.2]) *Let G be a finite group. Then the following hold:*

(i) *If $H \leq M \leq G$ and H is $\mathcal{M}$-supplemented in G, then H is also $\mathcal{M}$-supplemented in M.*

(ii) *Let $N \trianglelefteq G$ and $N \leq H \leq G$. If H is $\mathcal{M}$-supplemented in G, then H/N is $\mathcal{M}$-supplemented in G/N.*

(iii) *Let K be a normal π'-subgroup and H be a π-subgroup of G for a set π of primes. Then H is $\mathcal{M}$-supplemented in G if and only if HK/K is $\mathcal{M}$-supplemented in G/K.*

(iv) *If P is a p-subgroup of G where $p \in \pi(G)$ and P is $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $P \cap B = P_1 \cap B = \Phi(P) \cap B$ and $|G{:}P_1B| = p$ for any maximal subgroup P_1 of P.*

https://doi.org/10.4153/CMB-2014-021-9 Published online by Cambridge University Press

Lemma 2.2 ([4, Theorem 1.8.17]) *Let N be a nontrivial soluble normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G that are contained in N.*

Lemma 2.3 ([11, Lemma 2.3]) *Let $Z = Z_{\mathcal{F}\Phi}(G)$ and N and T be normal subgroups of G.*

(i) *Every non-Frattini G-chief factor of Z is $\mathcal{F}$-central in G.*
(ii) *$ZN/N \leq Z_{\mathcal{F}\Phi}(G/N)$.*
(iii) *If $TN/N \leq Z_{\mathcal{F}\Phi}(G/N)$ and $(|T|, |N|) = 1$, then $T \leq Z$.*

Lemma 2.4 ([1, Lemma 3.5]) *Let P be a normal p-subgroup of G where p is a prime divisor of $|G|$. If every subgroup of P of order p is complemented in G, then $P \leq Z_{\mathcal{U}}(G)$.*

Lemma 2.5 ([16, Lemma 2.8]) *Let M be a maximal subgroup of G and let P be a normal p-subgroup of G such that $G = PM$ where p is a prime of $|G|$.*

(i) *$P \cap M$ is a normal subgroup of G.*
(ii) *If $p > 2$ and all minimal subgroups of P are normal in G, then M has index p in G.*

Lemma 2.6 ([12, Theorem 9.15]) *Let $\mathcal{F}$ be one of the classes $\mathcal{N}$ or $\mathcal{U}$. Then*

$$G/C_G(Z_{\mathcal{F}}(G)) \in \mathcal{F}.$$

Lemma 2.7 ([8, Lemma 2.7]) *Let P be an elementary abelian p-group of order p^d, $d \geq 2$, let p be a prime, and let $\mathcal{M}_d(P) = \{M_1, \dots, M_d\}$.*

(i) *$X_i = \bigcap_{i \neq j} M_j$ is cyclic of order p.*
(ii) *$P = \langle X_1, \dots, X_d \rangle$.*

Lemma 2.8 ([7]) *Let G be a group and N a subgroup of G. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G.*

(i) *If N is normal in G, then $F(N) = N \cap F(G)$ and $F^*(N) = N \cap F^*(G)$.*
(ii) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \operatorname{Soc}(F(G)C_G(F(G)))/F(G)$.*
(iii) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*
(iv) *$C_G(F^*(G)) \leq F(G)$.*
(v) *Let $P \trianglelefteq G$ and $P \leq O_p(G)$. Then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.*
(vi) *If K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

Lemma 2.9 ([9, Lemma 2.7]) *Let H and L be normal subgroups of G and let $p \in \pi(G)$. Then the following hold:*

(i) *$\Phi(H) \leq \Phi(G)$;*
(ii) *if $L \leq \Phi(G)$, then $F(G/L) = F(G)/L$;*
(iii) *if $L \leq H \cap \Phi(G)$, then $F(H/L) = F(H)/L$;*
(iv) *if H is a p-group and $L \leq \Phi(H)$, then $F^*(H/L) = F^*(H)/L$.*

Lemma 2.10 ([9, Lemma 2.12]) *Let p be the smallest prime divisor of $|G|$ and let $P \in \mathrm{Syl}_p(G)$. Then G is p-nilpotent if and only if P has a nontrivial proper subgroup D such that every subgroup E of P with $|E| = |D|$ has a p-nilpotent supplement or an $\mathcal{M}$-supplement in G.*

Lemma 2.11 ([9, Theorem 3.2]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Suppose that every noncyclic Sylow subgroup P of H has a nontrivial proper subgroup D such that every subgroup $E \leq P$ of order $|D|$ has a supersoluble supplement or an $\mathcal{M}$-supplement in G. Then $G \in \mathcal{F}$.*

Lemma 2.12 ([9, Theorem 3.6]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Suppose that every noncyclic Sylow subgroup P of $F^*(H)$ has a nontrivial proper subgroup D such that every subgroup $E \leq P$ of order $|D|$ has a supersoluble supplement or an $\mathcal{M}$-supplement in G. Then $G \in \mathcal{F}$.*

Lemma 2.13 ([10, Corollary 2.1]) *Suppose that G is a group and*

$$\pi(G) = \{p_1, p_2 = p, p_3, \dots, p_n\}, \quad p_1 < p_2 = p < p_3 < \cdots < p_n.$$

If a Sylow p-subgroup is $\mathcal{M}$-supplemented in G, then G is p-supersoluble.

3 Main Results

Theorem 3.1 *Let E be a normal subgroup of G and let P be a Sylow p-subgroup of E where p is the smallest prime dividing $|E|$. Suppose that P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|D| = |H|$ having no p-nilpotent supplement in G is $\mathcal{M}$-supplemented in G. Then $E/O_{p'}(E) \leq Z_{\mathcal{U}\Phi}(G/O_{p'}(E))$.*

Proof Suppose that this theorem is false and consider a counterexample (G, E) for which $|G||E|$ is minimal.

(1) $O_{p'}(E) = 1$:

Suppose that $O_{p'}(E) \neq 1$. By Lemma 2.1(iii) the hypothesis also holds for $(G/O_{p'}(E), E/O_{p'}(E))$ and hence for (G, E), a contradiction.

(2) $E = P$:

If $E = G$, by Lemma 2.10, G is p-nilpotent and hence $E \leq Z_{\mathcal{U}\Phi}(G)$, a contradiction. Suppose that $E \neq G$. By Lemma 2.1(i), the hypothesis is still true for (E, E), so E is p-nilpotent by Lemma 2.10. Since $O_{p'}(E) = 1$, we have $E = P$.

(3) $|D| > p$:

Suppose that $|D| = p$. Then every minimal subgroup of P having no p-nilpotent supplement is $\mathcal{M}$-supplemented in G. Indeed, every minimal subgroup of P is complemented in G, by Lemma 2.4, $E \leq Z_{\mathcal{U}}(G) \leq Z_{\mathcal{U}\Phi}(G)$, a contradiction.

(4) Suppose that $|P{:}D| > p$. Then every subgroup H of P with $|D| = |H|$ has a p-nilpotent supplement in G:

https://doi.org/10.4153/CMB-2014-021-9 Published online by Cambridge University Press

Otherwise, if there exists a subgroup H with $|D| = |H|$ that is $\mathcal{M}$-supplemented in G, then there exists a subgroup B such that $G = HB$ and $H_1B < G$ for every maximal subgroup H_1 of H. By Lemma 2.1(iv), $|G{:}H_1B| = p$ and $G = P(H_1B)$. Clearly, $P \cap H_1B \trianglelefteq G$ by Lemma 2.5, so the hypothesis holds for $(G, P \cap H_1B)$. Hence $P \cap H_1B \le Z_{\mathcal{U}\Phi}(G)$. On the other hand, it follows from $|P/P \cap H_1B| = p$ that the chief factor $P/P \cap H_1B$ is $\mathcal{U}$-central in G. Hence the theorem is true for (G, E), a contradiction.

(5) $|N| \le |D|$ for any minimal normal subgroup N of G contained in P:

Assume that $|D| < |N|$. If some subgroup H of N with order $|D| = |H|$ has a p-nilpotent supplement T in G, then $G = HT = NT$. Clearly, $N \cap T = 1$, otherwise, $G = T$ is p-nilpotent, a contradiction. So $H = N$, also is a contradiction. Hence H is $\mathcal{M}$-supplemented in G, there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for every maximal subgroup H_1 of H. Clearly, $G = HB = NB$, and $N \cap B \trianglelefteq G$. If $N \cap B = N$, then $G = B$, a contradiction. If $N \cap B = 1$, then $H = N$, also is a contradiction. Thus we prove (5).

(6) If N is a minimal normal subgroup of G contained in E, then the hypothesis is still true for $(G/N, E/N)$:

If $|P{:}D| = p$, by (5), $|N| \le |D|$. If $|D| = |N|$, then $|P/N| = p$ and hence $P/N \le Z_{\mathcal{U}\Phi}(G/N)$. If $|N| < |D|$, then by Lemma 2.1(ii), the theorem is true for $(G/N, E/N)$.

So we may assume $|P{:}D| > p$. By (4), every subgroup H of P with $|D| = |H|$ has a p-nilpotent supplement in G. If $|N| < |D|$, then every subgroup H/N of P/N has a p-nilpotent supplement in G/N. It follows that the hypothesis is still true for $(G/N, E/N)$. If $|D| = |N|$, then we consider every subgroup M/N of P/N with $|M/N| = p$. Clearly, M is noncyclic. Otherwise, $|N| = p$; this contradicts (3). Hence there exists a subgroup H of M such that $|H| = |N| = |D|$ and $M = HN$. By (4), H has a p-nilpotent supplement in G. Hence M/N also has a p-nilpotent supplement in G/N. It follows that the hypothesis is still true for $(G/N, E/N)$.

(7) The final contradiction:

Let N be any minimal normal subgroup of G contained in P. Then by (6), the hypothesis holds for $(G/N, E/N)$. Hence $E/N \le Z_{\mathcal{U}\Phi}(G/N)$, $N \nleq \Phi(G)$, and $|N| > p$. Therefore, $\Phi(G) \cap E = 1$. Then by Lemma 2.2, P is the direct product of some minimal normal subgroups of G. In view of (5), $N < P$. Hence for some minimal normal subgroup R of G contained in P, $R \ne N$. Then by [3, Lemma A.9.11], $NR/N \nleq \Phi(G/N)$. Therefore $|R| = |NR/N| = p$, which implies that the theorem is true for (G, E), a contradiction.

The final contradiction completes our proof. ■

Corollary 3.2 *Let E be a normal subgroup of G and let P be a Sylow p-subgroup of E, where p is the smallest prime dividing $|G|$. Suppose that P has a subgroup $|D|$ such that $1 < |D| < |P|$ and every subgroup H of P with order $|D| = |H|$ having no p-nilpotent supplement in G is $\mathcal{M}$-supplemented in G. Then $E/O_{p'}(E) \le Z_{\Phi}(G/O_{p'}(E))$.*

Theorem 3.3 *Let E be a p-soluble normal subgroup of G and let P be a Sylow p-subgroup of E where p is a prime dividing $|E|$. Suppose that P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|D| = |H|$ is $\mathcal{M}$-supplemented in G. Then $E/O_{p'}(E) \le Z_{\mathcal{F}\Phi}(G/O_{p'}(E))$, where $\mathcal{F}$ is the class of all p-supersoluble groups.*

Proof Suppose that this theorem is false and consider a counterexample (G, E) for which $|G||E|$ is minimal.

(1) $O_{p'}(E) = 1$:

Suppose that $O_{p'}(E) \neq 1$. By Lemma 2.1(iii) the hypothesis also holds for $(G/O_{p'}(E), E/O_{p'}(E))$, and hence for (G, E), a contradiction.

(2) $O_p(E) \neq 1$:

Since E is p-soluble and $O_{p'}(E) = 1$, we have that the minimal normal subgroup of G contained in E is an elementary abelian p-group, and hence $O_p(E) \neq 1$.

(3) $O_p(E) \cap \Phi(G) = 1$:

Otherwise, if $O_p(E) \cap \Phi(G) \neq 1$, then we may choose a minimal normal subgroup L of G with $L \le O_p(E) \cap \Phi(G)$. If $|D| \le |L|$, then we may choose $S \le L$ with $|S| = |D|$. By hypothesis, S is $\mathcal{M}$-supplemented in G. Thus there exists a subgroup B of G such that $G = SB$ and $S_iB < G$ for any maximal subgroup S_i of S. Since $S \le L \le \Phi(G)$, we get $G = SB = B$, a contradiction.

Assume that $|D| > |L|$ and fix a subgroup H of P with $L < H$ and $|H| = |D|$. If H is $\mathcal{M}$-supplemented in G, then Lemma 2.1(ii) shows that H/L is $\mathcal{M}$-supplemented in G/L. Now we easily verify that $(G/L, E/L)$ satisfies the hypothesis of the theorem and $E/L \le Z_{\mathcal{F}\Phi}(G/L)$ by the induction. It follows from $L \le O_p(E) \cap \Phi(G)$ that $E \le Z_{\mathcal{F}\Phi}(G)$, a contradiction. So we may assume that $O_p(E) \cap \Phi(G) = 1$.

(4) The final contradiction:

By Lemma 2.2 and (3), $O_p(E) = R_1 \times \cdots \times R_t$ with minimal normal subgroups $R_1, \dots, R_t$ of G. Let L be any minimal normal subgroup of G contained in $O_p(E)$. Assume that $|D| < |L|$ for some $L \in \{R_1, \dots, R_t\}$ and let $H < L$ with $|H| = |D|$. By hypothesis, H is $\mathcal{M}$-supplemented in G, *i.e.,* there exists a subgroup B of G such that $G = HB$ and $H_iB < G$ for any maximal subgroup H_i of H. Now we have $G = HB = LB$ and thus $1 \neq L \cap B \trianglelefteq G$. Since L is minimal normal in G, we get $L \le B$ and hence $G = LB = B$, a contradiction.

Now let $L \le H \le P$ with $|H| = |D|$. Assume that H is $\mathcal{M}$-supplemented in G; *i.e.,* there exists $B \le G$ such that $G = HB$ and $H_iB < G$ for any maximal subgroup H_i of H. Since $|G : H_iB| = p$ by Lemma 2.1(iv) and $O_p(E) \cap \Phi(G) = 1$, there exists maximal subgroup H_i of H with $L \nleq H_i$ and hence $H = LH_i$ as well as $G = HB = LH_iB$ and $L \cap H_iB \trianglelefteq G$. As L is minimal normal in G, we get $L \nleq H_iB$ and thus $|L| = |G : H_iB| = p$; otherwise, if $L \le H_iB$, then $H_iB = LH_iB = HB = G$, a contradiction.

Thus $O_p(E)$ is the direct product of some minimal normal subgroup of order p of G. Since $C_E(O_p(E)) = O_p(E)$ and $O_p(E) \le Z(P)$, we have $O_p(E) = P$. Therefore, $E \le Z_{\mathcal{F}\Phi}(G)$, a final contradiction. ■

https://doi.org/10.4153/CMB-2014-021-9 Published online by Cambridge University Press

Corollary 3.4 *Let E be a normal subgroup of G where*

$$\pi(E) = \{p_1, p_2 = p, p_3, \dots, p_n, p_1 < p_2 = p < p_3 < \cdots < p_n\}$$

and P be a Sylow p-subgroup of E. Suppose that P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ is $\mathcal{M}$-supplemented in G. Then $E/O_{p'}(E) \le Z_{\mathcal{F}\Phi}(G/O_{p'}(E))$, where $\mathcal{F}$ is the class of all p-supersoluble groups.

Proof By Theorem 3.3, we only need to show that E is p-soluble. Now we induct on the order of E. Since every subgroup H of P with order $|H| = |D|$ is $\mathcal{M}$-supplemented in G, by Lemma 2.1(i), H is $\mathcal{M}$-supplemented in E. Let T be a subgroup of P with $|T| = |D|$. By hypothesis, there exists a subgroup B of E such that $E = TB$ and $T_iB < E$ for every maximal subgroup T_i of T. According to Lemma 2.1(iv), we get that $|E{:}T_iB| = p$ and hence $E/(T_iB)_E$ is isomorphic to a subgroup of the symmetric group S_p of degree p. Obviously, $E/(T_iB)_E$ is p-supersoluble. If $(T_iB)_E = 1$, then E is p-soluble. So we may assume that $(T_iB)_E \ne 1$ and then $T_iB \ne 1$. Let L be a Sylow p-subgroup of T_iB. Actually, L is a maximal subgroup of P. If $|L| = |D|$, then T_iB is p-supersoluble by Lemma 2.13 and hence E is p-soluble. So we may assert that $|L| > |D|$. Therefore, T_iB is also p-soluble and then E is p-soluble. ■

Corollary 3.5 *Let E be a normal subgroup of a group G. Suppose that every noncyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ is $\mathcal{M}$-supplemented in G, then $E \le Z_{\mathcal{U}\Phi}(G)$.*

Proof By Theorem 3.1, E is soluble. Assume that H/K is a non-Frattini G-chief factor of E. Then for some $p \in \pi(E)$, $HO_{p'}(E)/KO_{p'}(E) \cong H/K$ and by Theorem 3.3, $E/O_{p'}(E) \le Z_{\mathcal{F}\Phi}(G/O_{p'}(E))$, where $\mathcal{F}$ is the class of all p-supersoluble groups. Clearly, H/K is complemented in G/K. There exists a maximal subgroup M of G such that $G/K = (H/K)(M/K)$ and $(H/K) \cap (M/K) = 1$. So $|G{:}M|$ is p-number and $O_{p'}(E) \le M$. $G/KO_{p'}(E) = HO_{p'}(E)/(KO_{p'}(E))(M/KO_{p'}(E))$. If $G \ne O_{p'}(E)(H)$, then $HO_{p'}(E)/KO_{p'}(E)$ is complemented in $G/KO_{p'}(E)$ and hence $HO_{p'}(E)/KO_{p'}(E)$ is $\mathcal{F}$-central in G. Therefore, $|HO_{p'}(E)/KO_{p'}(E)| = p$. On the other hand, if $G = O_{p'}(E)H$, then $M = KO_{p'}(E) \trianglelefteq G$ and $|G/M| = p$. This completes our proof. ■

Theorem 3.6 *Let E be a normal subgroup of a group G. Suppose that every noncyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ is $\mathcal{M}$-supplemented in G. Then $E \le Z_{\mathcal{U}\Phi}(G)$.*

Proof Suppose that in this case the theorem is false and let (G, E) be a counterexample with $|G||E|$ minimal. Let $F = F(E)$ and $F^* = F^*(E)$. We use p to denote the smallest prime divisor of $|F|$ and let P be a Sylow p-subgroup of F.

(1) $F^* = F \ne E$:

By hypothesis and Lemma 2.11, F^* is supersoluble, and hence $F^* = F \ne E$ by Lemma 2.8(iii) and Corollary 3.5.

(2) $P \le Z_{\mathcal{U}\Phi}(G)$ and $E/P \not\le Z_{\mathcal{U}\Phi}(G/P)$:

https://doi.org/10.4153/CMB-2014-021-9 Published online by Cambridge University Press

Since P char $F = F^*$ char $E \trianglelefteq G$, $P \trianglelefteq G$. Hence by hypothesis, $P \leq Z_{\mathcal{U}\Phi}(G)$. Therefore, $E/P \nleq Z_{\mathcal{U}\Phi}(G/P)$. Otherwise, $E \leq Z_{\mathcal{U}\Phi}(G)$, which is a contradiction.

(3) If $E \neq G$, then E is supersoluble by Lemma 2.12.

(4) $|D| > p$ and P is noncyclic:

Suppose $|D| = p$. We show that every minimal subgroup L of P is normal in G. But we first claim that $\Phi(P) = 1$. If not, we pick a subgroup S of $\Phi(P)$ with order p. By the hypothesis S is $\mathcal{M}$-supplemented in G, then S is complemented in G. That is, there exits a subgroup K of G such that $G = SK$ and $S \cap K = 1$. Since $S \leq \Phi(P)$, $G = SK = K$, a contradiction. Therefore $\Phi(P) = 1$, and hence P is an elementary abelian normal subgroup of G.

Therefore, every minimal subgroup of P is $\mathcal{M}$-supplemented in G, and is also complemented in G. Let L be a subgroup of P with order p. By hypothesis, L is complemented in G and there exists a subgroup K such that $G = LK$ and $L \cap K = 1$. By Lemma 2.5, $P \cap K \trianglelefteq G$. Since $P = L(P \cap K)$, we have every maximal subgroup of P is normal in G. Then by Lemma 2.7, every minimal subgroup of P is normal in G, and hence $P \leq Z(F)$. Next we show that the hypothesis is still true for $(G/P, C_G(P) \cap E/P)$. Indeed, $F^* = F \leq F^*(C_G(P) \cap E)$, and by Lemma 2.8(iii), $F^*(C_G(P) \cap E) \leq F^*$. Hence $F^*(C_G(P) \cap E) = F^*$, and so by Lemma 2.8(i), $F^*(C_G(P) \cap E/P) = F^*/P$, since $P \leq Z(C_E(P))$. Now by Lemma 2.1(iii) and Lemma 2.8(vi), we know that $(G/P, C_G(P) \cap E/P)$ satisfies the condition of the theorem, and hence $(C_G(P) \cap E)/P \leq Z_{\mathcal{U}\Phi}(G/P)$, by the choice of (G, E). On the other hand, by Lemma 2.6, $G/C_G(P)$ is supersoluble, and every G-chief factor between E and $E \cap C_G(P)$ has prime order. Hence $E \leq Z_{\mathcal{U}\Phi}(G)$, a contradiction.

(5) If L is a minimal normal subgroup of G contained in P, then $|L| > p$:

Assume that $|L| = p$. Let $C = C_E(L)$. Then the hypothesis is true for $(G/L, C/L)$. Indeed, since $F = F^* \leq C$ and $L \leq Z(F)$, we have $F^*(C/L) = F^*/L$ by Lemma 2.8(vi). On the other hand, if H/L is a subgroup of G/L such that $|H| = |D|$, we have $1 < |H/L| < |P/L|$ by (4). Besides, H/L is $\mathcal{M}$-supplemented in G/L by Lemma 2.1(ii). Now by Lemmas 2.1(iii) and 2.8(vi), the hypothesis still holds for $(G/L, C/L)$. Hence $C/L \leq Z_{\mathcal{U}\Phi}(G/L)$, which implies that $E \leq Z_{\mathcal{U}\Phi}(G)$, a contradiction.

(6) $\Phi(G) \cap P \neq 1$:

If $\Phi(G) \cap P = 1$, then P is the direct product of some minimal normal subgroups of G contained in P by Lemma 2.2. Let S be a subgroup of P with $|D| = |S|$. By hypothesis, S is $\mathcal{M}$-supplemented in G, and then there exists a subgroup B such that $G = SB$ and $S_1B < G$ for every maximal subgroup S_1 of S. By Lemma 2.1(iv), $|G{:}S_1B| = p$. Clearly, there exists at least a minimal normal subgroup L of G contained in P such that $L \nleq S_1B$. Therefore $|L| = p$, contrary to (5).

(7) $E = G$ is not soluble:

First, we show that $\Phi(P) = 1$. If not, there exists a minimal normal subgroup N of G contained in $\Phi(P)$. If $|D| \leq |N|$, then we choose a subgroup S of N with order $|D|$. By the hypothesis, S is $\mathcal{M}$-supplemented in G. So there exists a subgroup K of G such that $G = SK$ and and $S_1K < G$ for every maximal subgroup S_1 of S. Clearly, since $S \leq \Phi(P) \leq \Phi(G)$, $G = K$, a contradiction. So we may assume that $|D| > |N|$.

https://doi.org/10.4153/CMB-2014-021-9 Published online by Cambridge University Press

Then the hypothesis still holds for $(G/N, E/N)$. Hence $E/N \leq Z_{\mathfrak{U}\Phi}(G/N)$, which implies that $E \leq Z_{\mathfrak{U}\Phi}(G)$, a contradiction. Therefore $\Phi(P) = 1$ and hence P is an elementary abelian p-group.

Next we will prove $E = G$. By (3), E is soluble if $E < G$. Let L be a minimal normal subgroup of G contained in $\Phi(G) \cap P$. By Lemma 2.9(ii), $F/L = F(E/L) = F^*(E/L)$. Hence by (1), $F^*(E/L) = F(E/L) = F^*/L$. On the other hand, if $|D| < |L|$, then we may choose a subgroup S of L with $|D| = |S|$. By hypothesis, S is $\mathcal{M}$-supplemented in G, so there exists a subgroup B such that $G = SB$ and $S_1B < G$ for every maximal subgroup S_1 of S. Clearly, since $S \leq \Phi(G)$, $G = B$, a contradiction. If $|D| = |L|$, then let L_1 be a maximal subgroup of L, and then $P = L \times \langle x_1 \rangle \times \cdots \times \langle x_t \rangle$. Let $T = L_1\langle x_1 \rangle$, where L_1 is the maximal subgroup of L. Clearly, $L \neq T$. By hypothesis, T is $\mathcal{M}$-supplemented in G. There exists a subgroup B such that $G = TB$ and $T_1B < G$. Let $T_1 = \langle x_1 \rangle L_2$ where L_2 is the maximal subgroup of L_1. Clearly, $|G{:}T_1B| = p$ and $L \leq T_1B$. It follows that $T_1B = LT_1B = G$, a contradiction. So we have $|D| > |L|$ and the hypothesis still holds for $(G/L, E/L)$. Hence $E/L \leq Z_{\mathfrak{U}\Phi}(G/L)$, which implies that $E \leq Z_{\mathfrak{U}\Phi}(G)$, a contradiction.

(8) The final contradiction:

By (7), $F^* = F = F^*(G)$, G is supersoluble by Lemma 2.12. This contradiction completes our proof. ■

Corollary 3.7 *Let E be a soluble normal subgroup of a group G. Suppose that every noncyclic Sylow subgroup P of $F(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ is $\mathcal{M}$-supplemented in G. Then $E \leq Z_{\mathfrak{U}\Phi}(G)$.*

Acknowledgment The authors are grateful for the valuable discussion with Prof. Adolfo Ballester-Bolinches and Prof. Ramon Esteban-Romero.

References

[1] M. Asaad, *Finite groups with certain subgroups of Sylow subgroups complemented.* J. Algebra **323**(2010), no. 7, 1958–1965. http://dx.doi.org/10.1016/j.jalgebra.2010.02.006
[2] A. Ballester-Bolinches, Y. Wang, and X. Guo, *C-supplemented subgroups of finite groups.* Glasg. Math. J. **42**(2000), no. 3, 383–389. http://dx.doi.org/10.1017/S001708950003007X
[3] K. Doerk and T. Hawkes, *Finite soluble groups.* de Gruyter Expositions in Mathematics, 4, Walter de Gruyter, Berlin, 1992.
[4] W. Guo, *The theory of classes of groups.* Mathematics and its Applications, 505, Kluwer Academic Publishers Group, Dordrecht; Science Press, Beijing, 2000.
[5] P. Hall, *A characteristic property of soluble groups.* J. London Math. Soc. **12**(1937), 189–200. http://dx.doi.org/10.1112/jlms/s1-12.2.198
[6] B. Huppert, *Endliche Gruppen. I* Die Grundlehren der Mathematischen Wissenschaften, 134, Springer-Verlag, Berlin-New York, 1967.
[7] B. Huppert and N. Blackburn, *Finite groups. III.* Grundlehren der Mathematischen Wissenschaften, 243, Springer-Verlag, Berlin-New York, 1982.
[8] S. Li and X. He, *On normally embedded subgroups of prime power order in finite groups.* Comm. Algebra **36**(2008), no. 6, 2333–2340. http://dx.doi.org/10.1080/00927870701509370
[9] L. Miao and W. Lempken, *On $\mathcal{M}$-supplemented subgroups of finite groups.* J. Group Theory **12**(2009), no. 2, 271–289.
[10] V. S. Monakhov and A. V. Shnyparkov, *On the p-supersolubility of a finite group with a μ-complemented Sylow p-subgroup.* Sib. Math. J. **50**(2009), no. 4, 681–686.
[11] L. A. Shemetkov and A. N. Skiba, *On the $\mathfrak{X}\Phi$-hypercentre of finite groups.* J. Algebra **322**(2009), no. 6, 2106–2117. http://dx.doi.org/10.1016/j.jalgebra.2009.03.029

[12] L. A. Shemetkov, *Formations of finite groups.* (Russian) Nauka, Moscow, 1978.
[13] A. N. Skiba, *On weakly s-permutable subgroups of finite groups.* J. Algebra **315**(2007), no. 1, 192–209. http://dx.doi.org/10.1016/j.jalgebra.2007.04.025
[14] S. Srinivasan, *Two sufficient conditions for supersolvability of finite groups.* Israel J. Math. **35**(1980), no. 3, 210–214. http://dx.doi.org/10.1007/BF02761191
[15] Y. Wang, *Finite groups with some subgroups of Sylow subgroups c-supplemented.* J. Algebra **224**(2000), no. 2, 467–478. http://dx.doi.org/10.1006/jabr.1999.8079
[16] Y. Wang, H. Wei, and Y. Li, *A generalization of Kramer's theorem and its applications.* Bull. Aust. Math. Soc. **65**(2002), no. 3, 467–475. http://dx.doi.org/10.1017/S0004972700020517

School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, People's Republic of China
e-mail: lmiao@yzu.edu.cn tangjuping_205@126.com