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COEFFICIENT INEQUALITIES AND YAMASHITA'S CONJECTURE FOR SOME CLASSES OF ANALYTIC FUNCTIONS

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Abstract

For any real number β with $\beta > 1$, let $\mathcal{M}(\beta)$ ($\mathcal{N}(\beta)$ respectively) denote the class of analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and satisfying $\operatorname{Re} P_f < \beta$ ($\operatorname{Re} Q_f < \beta$ respectively) in $\mathbb{D}$, where $P_f = zf'(z)/f(z)$ and $Q_f = 1 + zf''(z)/f'(z)$. Also, for $\beta > 1$, let $\mathcal{M}\Sigma(\beta)$ ($\mathcal{N}\Sigma(\beta)$ respectively) denote the class of analytic functions g of the form $g(z) = z(1 + \sum_{n=1}^{\infty} b_n z^{-n})$ and satisfying $\operatorname{Re} P_g < \beta$ ($\operatorname{Re} Q_g < \beta$ respectively) for $z \in \Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. In this paper, we shall determine the coefficient bounds, inverse coefficient bounds, the growth and distortion theorem and the upper bounds for the Fekete–Szegő functional $\Lambda_\lambda(f) = a_3 - \lambda a_2^2$ for functions f in the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$. Further, we shall solve the maximal area problem for functions of the type $z/f(z)$ when $f \in \mathcal{M}(\beta)$, which is Yamashita's conjecture for the class $\mathcal{M}(\beta)$. We shall obtain the radius of convexity for the class $\mathcal{N}(\beta)$. We shall also determine the coefficient bounds for functions g in the classes $\mathcal{M}\Sigma(\beta)$ and $\mathcal{N}\Sigma(\beta)$ and the inverse coefficient bounds for functions g in the class $\mathcal{M}\Sigma(\beta)$. All the results are sharp.

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Here we think of $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Let $\mathcal{A}$ denote the family of functions f in $\mathcal{H}$ normalized by $f(0) = 0 = f'(0) - 1$. If $f \in \mathcal{A}$, then $f(z)$ has the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

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A function f is said to be univalent in a domain $\Omega \subseteq \mathbb{C}$ if it is one-to-one in Ω. Let $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. A function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{S}^*(\alpha)$, called starlike functions of order α, if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D}$$

and is said to belong to the class $C(\alpha)$, called convex functions of order α, if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{D}.$$

The classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $C := C(0)$ are the familiar classes of starlike and convex functions, respectively. It is well known that $f \in C(\alpha)$ if and only if $zf' \in \mathcal{S}^*(\alpha)$.

Let f and g be analytic functions in the unit disk $\mathbb{D}$. The function f is said to be subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $f(z) = g(\omega(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. For a detailed study of differential subordination, we refer to the monograph of Miller and Mocanu [22].

For fixed $\beta > 1$, let the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ be defined by

$$\mathcal{M}(\beta) := \left\{f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \text{ for } z \in \mathbb{D}\right\}$$

and

$$\mathcal{N}(\beta) := \left\{f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \beta \text{ for } z \in \mathbb{D}\right\},$$

respectively. Then it is easy to see that $f \in \mathcal{N}(\beta)$ if and only if $zf' \in \mathcal{M}(\beta)$. In 1941, Ozaki [29] introduced the class $\mathcal{N}(\frac{3}{2})$ and proved that functions in the class $\mathcal{N}(\frac{3}{2})$ are univalent in the unit disk $\mathbb{D}$. Moreover, functions in the class $\mathcal{N}(\frac{3}{2})$ were proved to be starlike in the unit disk $\mathbb{D}$ (see [9, 33]). Thus, the class $\mathcal{N}(\beta)$ is included in the class $\mathcal{S}^*$ for $1 < \beta \leq \frac{3}{2}$. Also, we note that functions in the class $\mathcal{N}(\beta)$ need not be univalent in the unit disk $\mathbb{D}$ if $\beta > \frac{3}{2}$. For $1 < \beta \leq \frac{4}{3}$, the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ were introduced by Uralegaddi *et al.* [38]. Later, the full classes were investigated by Owa and Nishiwaki [23, 27] and also by Owa and Srivastava [28]. Recently, Obradović *et al.* [24] studied the class $\mathcal{N}(\beta)$ for $1 < \beta \leq \frac{3}{2}$.

Two more classes of our interest are $\mathcal{M}\Sigma(\beta)$ and $\mathcal{N}\Sigma(\beta)$, which are associated with the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$, respectively. For the sake of our computational purpose, here we use a slightly different notation for these classes. For $\beta > 1$, let $\mathcal{M}\Sigma(\beta)$ denote the class of analytic functions $g(z)$ in $\Delta := \{z \in \mathbb{C} : 1 < |z| < \infty\}$ of the form

$$g(z) = z\left(1 + \sum_{n=1}^{\infty} b_n z^{-n}\right) \quad \text{for } z \in \Delta, \tag{1.2}$$

which satisfy $\mathrm{Re}(zg'(z)/g(z)) < \beta$ in Δ. Similarly, for $\beta > 1$, let $\mathcal{N}\Sigma(\beta)$ denote the class of analytic functions $g(z)$ in Δ of the form (1.2) and satisfying $\mathrm{Re}(1 + zg''(z)/g'(z)) < \beta$

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in Δ. It is easy to see that if $g \in \mathcal{N}\Sigma(\beta)$, then $zg' \in \mathcal{M}\Sigma(\beta)$ but the converse may not be true.

In 1933, Fekete and Szegő [4] proved a remarkable result that if $f \in \mathcal{S}$, then

$$|a_3 - \lambda a_2^2| \le 1 + 2\exp(-2\lambda/(1-\lambda)) \quad \text{for } \lambda \in [0, 1).$$

This inequality is sharp. For a function $f \in \mathcal{A}$ of the form (1.1), the classical Fekete–Szegő functional, defined by

$$\Lambda_\lambda(f) = a_3 - \lambda a_2^2,$$

plays an important role in function theory. For example, the quantity $a_3 - a_2^2$ represents $S_f(0)/6$, where S_f denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of a locally univalent function f in $\mathbb{D}$. Moreover, $\Lambda_\lambda(f)$ behaves well with respect to rotation, namely $\Lambda_\lambda(e^{-i\theta} f(e^{i\theta} z)) = e^{2i\theta}\Lambda_\lambda(f), \theta \in \mathbb{R}$. The problem of maximizing the absolute value of the functional $\Lambda_\lambda(f)$ is called the Fekete–Szegő problem. In 1986, Pfluger [31] solved the Fekete–Szegő problem for the class $\mathcal{S}$ with complex parameter λ. In the literature, there are a large number of results available about the Fekete–Szegő problem (see for instance [1, 2, 12, 13, 19, 30]).

For $g \in \mathcal{H}$, we denote the area of the image of $|z| < r$ under $w = g(z)$ by $\Delta(r, g)$, where $0 < r \le 1$. Thus, for $g(z) = \sum_{n=0}^{\infty} b_n z^n$,

$$\Delta(r, g) = \iint_{|z|<r} |g'(z)|^2 \, dx \, dy = \pi \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \quad (z = x + iy).$$

Computation of this area for an analytic function g is known as the area problem. We call g a Dirichlet-finite function whenever $\Delta(1, g)$ is finite. All polynomials and more generally all functions $f \in \mathcal{A}$ for which f' is bounded on $\mathbb{D}$ are Dirichlet-finite functions. In 1990, Yamashita [39] conjectured that

$$\max_{f \in C} \Delta\left(r, \frac{z}{f(z)}\right) = \pi r^2.$$

The maximum is attained only by the rotation of the function $f_0(z) = z/(1-z)$. In 2013, Yamashita's conjecture was settled by Obradović *et al.* [25] in a more general setting for the functions in $\mathcal{S}^*(\alpha)$. For more details, we refer to [26, 34].

The problem of determination of sharp coefficient estimates of inverse functions in various subclasses of univalent functions is interesting. If F is the inverse of a function $f \in \mathcal{S}$, then F has the following expansion near $w = 0$:

$$F(w) = w + \sum_{n=2}^{\infty} A_n w^n. \tag{1.3}$$

In 1923, Löwner [20] using a parametric method proved that for each $n \ge 2$, $|A_n| \le K_n$, where $K_n = (2n)!/(n!(n+1)!)$ and the inequality is sharp for the inverse of the Koebe function $K(z) = z/(1+z)^2$. An alternative proof of the inverse coefficient problem for the functions in $\mathcal{S}$ has been given by Schaeffer and Spencer [36] and FitzGerald [5].

Although the inverse coefficient problem for the class $\mathcal{S}$ was completely solved in 1923, only a few complete results are known on inverse coefficient estimates for most of the subclasses of $\mathcal{S}$ (see for instance [14, 16–18]). In some cases the inverse coefficient shows unexpected behavior. For example, it is known that if $f \in C$, then the coefficients of its inverse function satisfy $|A_n| \le 1$ for $n = 1, 2, \ldots, 8$ (see [15]), while $|A_{10}| > 1$ (see [11]) and the exact bounds $|A_9|$ and $|A_n|$ for $n > 10$ are still unknown. In 1979, Krzyż *et al.* [14] found the sharp inverse coefficient estimates of $|A_2|$ and $|A_3|$ for the class $\mathcal{S}^*(\alpha)$ and in 2007 Kapoor and Mishra [10] developed a new technique and extended these results (see also [37]).

Suppose that $f \in \mathcal{M}(\beta)$ (or $\mathcal{N}(\beta)$) is of the form (1.1). Since $f'(0) = 1 \neq 0$, $f(z)$ has an inverse $F(w)$ valid in some neighborhood of the origin and has an expansion of the form (1.3). Similarly, if $g \in \mathcal{M}\Sigma(\beta)$ is of the form (1.2), then it has an inverse $G(w)$ in some neighborhood of the point at infinity and has the following expansion:

$$G(w) = w\left(1 + \sum_{n=1}^{\infty} B_n w^{-n}\right).$$

In Section 2, we shall obtain the sharp coefficient bounds for the functions in the class $\mathcal{M}(\beta)$. As a consequence, we also find the sharp coefficient estimate for functions in the class $\mathcal{N}(\beta)$. Moreover, we shall obtain the growth and distortion theorem for functions in the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$. Finally, we prove that the radius of convexity for the class $\mathcal{N}(\beta)$ is $1/(2\beta - 1)$. In Section 3, we shall solve the Fekete–Szegő problem for both the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ with complex parameter λ.

In Section 4, first we prove Lemma 4.1 and as an application of the lemma we solve a Yamashita's conjecture for the class $\mathcal{M}(\beta)$, that is, we solve the maximal area problem for the functions of the type $z/f(z)$, where $f \in \mathcal{M}(\beta)$, and investigate the inverse coefficient problem for the class $\mathcal{M}(\beta)$. Also, we completely solve the inverse coefficient problem for the class $\mathcal{N}(\beta)$ for $\beta > 1$. Finally, using Lemma 4.1, we determine the sharp coefficient bounds and the sharp inverse coefficient bounds for functions in the class $\mathcal{M}\Sigma(\beta)$ and as a consequence we shall find the sharp coefficient bounds for functions in the class $\mathcal{N}\Sigma(\beta)$.

To prove our main results, we need the following lemma.

Lemma 1.1 [21]. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be an analytic function with* $\operatorname{Re} p(z) > 0$ *in $\mathbb{D}$ and μ be a complex number. Then*

$$|c_2 - \mu c_1^2| \le 2 \max\{1, |2\mu - 1|\}. \tag{1.4}$$

The result is sharp for the functions given by $p(z) = (1 + z^2)/(1 - z^2)$ and $p(z) = (1 + z)/(1 - z)$.

Lemma 1.2 [7]. *Let $h(z)$ be convex in $\mathbb{D}$ with $h(0) = a$. If $p(z)$ is analytic in $\mathbb{D}$, with $p(0) = a$ and $p(z) + zp'(z) \prec h(z)$, then*

$$p(z) \prec \frac{1}{z} \int_0^z h(t)\, dt.$$

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LEMMA 1.3 [8]. *Let $f \in \mathcal{A}$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then the inverse function $F(w)$ of the function $f(z)$ is analytic in $|w| < \rho$ for some $\rho > 0$. Also, suppose that*

$$\left(\frac{z}{f(z)}\right)^t = 1 + \sum_{n=1}^{\infty} a_n^{(-t)} z^n$$

and

$$\left(\frac{w}{F(w)}\right)^t = 1 + \sum_{n=1}^{\infty} A_n^{(-t)} z^n,$$

where $t = \pm 1, \pm 2, \pm 3, \ldots$. Then

$$A_n^{(t)} = \frac{t}{t+n} a_n^{(-(t+n))} \quad \text{for } t+n \neq 0 \text{ and } t = \pm 1, \pm 2, \pm 3, \ldots$$

and $A_{-t}^{(t)}$ is given by

$$\sum_{t=-\infty}^{\infty} A_{-t}^{(t)} z^{-t-1} = \frac{f'(z)}{f(z)}. \tag{1.5}$$

2. Coefficient and growth estimates

THEOREM 2.1. *Let $f(z)$ be of the form (1.1) and $f \in \mathcal{M}(\beta)$ for some $1 < \beta \le 2$. Then*

$$|a_n| \le \frac{2(\beta - 1)}{n-1} \quad \text{for } n \ge 2.$$

Equality is attained for the function $f_n(z) = z(1 - z^{n-1})^{(2(\beta-1))/(n-1)}$ for $n \ge 2$.

PROOF. Let $f \in \mathcal{M}(\beta)$. By definition of the class $\mathcal{M}(\beta)$,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \quad \text{for } z \in \mathbb{D}$$

and, equivalently,

$$\mathrm{Re}\left(\frac{\beta - \frac{zf'(z)}{f(z)}}{\beta - 1}\right) > 0 \quad \text{for } z \in \mathbb{D}. \tag{2.1}$$

Then there exists an analytic function $\omega : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$\frac{\beta - \frac{zf'(z)}{f(z)}}{\beta - 1} = \frac{1 + z\omega(z)}{1 - z\omega(z)}.$$

By a simple computation, this can be written as

$$zf'(z) - f(z) = \omega(z)(z^2 f'(z) + (1 - 2\beta) z f(z)). \tag{2.2}$$

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Then, applying Clunie's method to the equation (2.2), which can be found in [3, 32, 35], we obtain the following inequality:

$$\sum_{k=2}^{n}(k-1)^2|a_k|^2 \le \sum_{k=1}^{n-1}(k+1-2\beta)^2|a_k|^2 \tag{2.3}$$

for each $n \ge 2$ with $a_1 = 1$.

Therefore, (2.3) can be written as

$$\begin{aligned}(n-1)^2|a_n|^2 &\le 4(\beta-1)^2 + \sum_{k=2}^{n-1}(((k-1)-2(\beta-1))^2-(k-1)^2)|a_k|^2\\ &\le 4(\beta-1)^2\end{aligned}$$

and, consequently,

$$|a_n| \le \frac{2(\beta-1)}{n-1} \quad \text{for } n \ge 2. \qquad \square$$

Theorem 2.2. *Let $f \in \mathcal{N}(\beta)$ be of the form (1.1) for some $1 < \beta \le 2$. Then*

$$|a_n| \le \frac{2(\beta-1)}{n(n-1)} \quad \textit{for } n \ge 2.$$

Equality is attained for the function $f_n(z)$ given by $f_n'(z) = (1-z^{n-1})^{(2(\beta-1))/(n-1)}$, $n \ge 2$.

Proof. If $f(z)$ is in $\mathcal{N}(\beta)$, then clearly $zf'(z) = z + \sum_{n=2}^{\infty} na_nz^n$ is in $\mathcal{M}(\beta)$. Therefore, by Theorem 2.1, it immediately follows that

$$|a_n| \le \frac{2(\beta-1)}{n(n-1)} \quad \text{for } n \ge 2. \qquad \square$$

Theorem 2.3.

(i) *If $f \in \mathcal{M}(\beta)$ for some $\beta > 1$, then*

$$\frac{z}{f(z)} \prec (1-z)^{-2(\beta-1)}. \tag{2.4}$$

(ii) *If $f \in \mathcal{N}(\beta)$ for some $\beta > 1$, then*

$$f'(z) \prec (1-z)^{2(\beta-1)} \tag{2.5}$$

and

$$\frac{f(z)}{z} \prec \frac{1-(1-z)^{2\beta-1}}{(2\beta-1)z}. \tag{2.6}$$

Proof. (i) Let $f \in \mathcal{M}(\beta)$ for some $\beta > 1$. In view of (2.1), $f \in \mathcal{M}(\beta)$ if and only if

$$\frac{1}{\beta-1}\Big(\beta - \frac{zf'(z)}{f(z)}\Big) \prec \frac{1+z}{1-z} \quad \text{for } z \in \mathbb{D}.$$

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A simple computation yields the following subordination relation:

$$\frac{zf'(z)}{f(z)} \prec \beta - (\beta - 1)\frac{1+z}{1-z} = \frac{1+(1-2\beta)z}{1-z}. \tag{2.7}$$

Let $g(z) = z/f(z)$. Then, from the subordination relation (2.7),

$$\frac{zg'(z)}{g(z)} = 1 - \frac{zf'(z)}{f(z)} \prec 1 - \frac{1+(1-2\beta)z}{1-z} = \frac{2(\beta-1)z}{1-z} =: \phi(z).$$

Since $\phi(z)$ is convex in $\mathbb{D}$ and $\phi(0) = 0$, it follows that (see, for example, [22, Corollary 3.1d.1, page 76])

$$\frac{z}{f(z)} = g(z) \prec \exp\Big(\int_0^z \frac{2(\beta-1)}{1-t}\,dt\Big) = (1-z)^{-2(\beta-1)}.$$

(ii) Let $f \in \mathcal{N}(\beta)$ for some $\beta > 1$. Then, by the definition of the class $\mathcal{N}(\beta)$,

$$\mathrm{Re}\bigg(\frac{\beta - 1 - \frac{zf''(z)}{f'(z)}}{\beta - 1}\bigg) > 0 \quad \text{for } z \in \mathbb{D}. \tag{2.8}$$

Therefore,

$$\frac{1}{\beta-1}\Big(\beta - 1 - \frac{zf''(z)}{f'(z)}\Big) \prec \frac{1+z}{1-z} \quad \text{for } z \in \mathbb{D},$$

which is equivalent to

$$\frac{zf''(z)}{f'(z)} \prec (\beta-1)\Big(1 - \frac{1+z}{1-z}\Big) = -\frac{2(\beta-1)z}{1-z} =: \psi(z) \quad \text{for } z \in \mathbb{D}. \tag{2.9}$$

Since $\psi(z)$ is convex in $\mathbb{D}$ and $\psi(0) = 0$, it follows that (see, for example, [22, Corollary 3.1d.1, page 76])

$$f'(z) \prec \exp\Big[-\int_0^z \frac{2(\beta-1)}{1-t}\,dt\Big] = (1-z)^{2(\beta-1)}. \tag{2.10}$$

Next, suppose that $h(z) = f(z)/z$ and so $zh'(z) + h(z) = f'(z)$. Therefore, (2.10) becomes

$$zh'(z) + h(z) = f'(z) \prec (1-z)^{2(\beta-1)}.$$

Again, by applying Lemma 1.2 to the previous relation,

$$\frac{f(z)}{z} = h(z) \prec \frac{1}{z}\int_0^z (1-t)^{2(\beta-1)}\,dt = \frac{1-(1-z)^{2\beta-1}}{(2\beta-1)z}. \qquad \square$$

COROLLARY 2.4. *For $f \in \mathcal{M}(\beta)$ for some $\beta > 1$, the following hold.*

(i) $$r(1-r)^{2(\beta-1)} \le |f(z)| \le r(1+r)^{2(\beta-1)}, \quad |z| = r < 1.$$

Equality holds for the function $f(z) = z(1-z)^{2(\beta-1)}$ or its rotation.

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(ii) $$\left|\frac{zf'(z)}{f(z)} - \frac{1-(2\beta-1)r^2}{1-r^2}\right| \leq \frac{2\beta r}{1-r^2}, \quad |z| = r < 1.$$

Equality holds in the above inequality for the function $f(z) = z(1-z)^{2(\beta-1)}$ or its rotation.

Proof. To prove (i), we observe that $(1-z)^{-2(\beta-1)} \neq 0$ for $z \in \mathbb{D}$. Then (2.4) is equivalent to

$$\frac{f(z)}{z} \prec (1-z)^{2(\beta-1)} \quad \text{for } z \in \mathbb{D},$$

from which the result follows.

To prove (ii), consider the relation (2.7) and note that the function $w = (1+(1-2\beta)z)/(1-z)$ maps the disk $|z| \leq r < 1$ onto the disk $|w - (1-(2\beta-1)r^2)/(1-r^2)| \leq 2\beta r/(1-r^2)$. □

Corollary 2.5.

(i) *If $f \in \mathcal{N}(\beta)$ for some $\beta > 1$, then, for each $z = re^{i\theta}$ in $\mathbb{D}$,*

$$(1-r)^{2(\beta-1)} \leq |f'(z)| \leq (1+r)^{2(\beta-1)}. \tag{2.11}$$

Equality holds for the function $f(z)$ given by $f'(z) = (1-z)^{2(\beta-1)}$ or its rotation.

(ii) *For each $f \in \mathcal{N}(\beta)$ $(\beta > 1)$,*

$$|\arg f'(z)| \leq 2(\beta-1)\sin^{-1} r, \quad |z| = r < 1.$$

Equality holds for the function $f(z)$ given by $f'(z) = (1-z)^{2(\beta-1)}$ or its rotation.

(iii) *If $f \in \mathcal{N}(\beta)$ and $\beta > 1$, then, for each $z = re^{i\theta}$ in $\mathbb{D}$,*

$$|f(z)| \leq \frac{(1+r)^{2\beta-1}-1}{2\beta-1}.$$

And, if $f \in \mathcal{N}(\beta)$ with $1 < \beta \leq \frac{3}{2}$, then

$$\frac{1-(1-r)^{2\beta-1}}{2\beta-1} \leq |f(z)| \leq \frac{(1+r)^{2\beta-1}-1}{2\beta-1} \quad \textit{for } |z| = r < 1.$$

Equality holds in the above inequalities for the function $f(z)$ given by $f'(z) = (1-z)^{2(\beta-1)}$ or its rotation.

Proof. Proofs of (i) and (ii) easily follow from the relation (2.5). And, the proof of the first part of (iii) follows from the relation (2.6). To prove the second part of (iii), just observe that if $f \in \mathcal{N}(\beta)$ with $1 < \beta \leq \frac{3}{2}$, then $f(z)$ is starlike univalent in $\mathbb{D}$ (see [9, 33]) and so the desired result follows on integration of (2.11). □

Theorem 2.6. *Let $f \in \mathcal{N}(\beta)$ for some $\beta > 1$. Then, for every positive number $r \leq 1/(2\beta-1)$, the function f maps the disk $|z| < r$ onto a convex domain. The result is best possible, that is, the radius of convexity for the class $\mathcal{N}(\beta)$ is $1/(2\beta-1)$.*

PROOF. Let $f \in \mathcal{N}(\beta)$. Then, by the subordination relation (2.9),

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{2(\beta-1)r}{1-r}$$

and, consequently,

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &\geq 1-\frac{2(\beta-1)r}{1-r}\\ &= \frac{1-(2\beta-1)r}{1-r} \quad \text{for } |z|=r<1.\end{aligned}$$

Since $1-(2\beta-1)r>0$ for $r<1/(2\beta-1)$, $f(z)$ must map such a disk $|z|<r$ onto a convex domain.

For the function $f \in \mathcal{N}(\beta)$ given by $f'(z)=(1-z)^{2(\beta-1)}$, a simple computation gives

$$1+\frac{zf''(z)}{f'(z)} = \frac{1-(2\beta-1)z}{1-z}.$$

This shows that the bound $1/(2\beta-1)$ is sharp. □

REMARK 2.7. If $f \in \mathcal{M}(\beta)$ for some $\beta>1$, then, by the relation (2.7),

$$\left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+(2\beta-1)r}{1-r}$$

and so

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{-1-(2\beta-1)r}{1-r} \quad \text{for } |z|=r<1.$$

But $-1-(2\beta-1)r<0$ for any $r \in (0,1)$. Hence, we can say that there exists no $r \in (0,1)$ such that each $f \in \mathcal{M}(\beta)$ maps $|z|<r$ onto a starlike domain. In other words, the radius of starlikeness of the class $\mathcal{M}(\beta)$ is zero.

3. Fekete–Szegő problem

THEOREM 3.1. *Let $f(z)$ be of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $f \in \mathcal{M}(\beta)$ for some $\beta>1$. Then, for any $\lambda \in \mathbb{C}$,*

$$|a_3-\lambda a_2^2| \leq \begin{cases} \beta-1 & \text{for } \left|\lambda-\dfrac{2\beta-3}{4(\beta-1)}\right| \leq \dfrac{1}{4(\beta-1)},\\[2ex] 4(\beta-1)^2\left|\lambda-\dfrac{2\beta-3}{4(\beta-1)}\right| & \text{for } \left|\lambda-\dfrac{2\beta-3}{4(\beta-1)}\right| \geq \dfrac{1}{4(\beta-1)}.\end{cases}$$

For each $\lambda \in \mathbb{C}$, there are functions in $\mathcal{M}(\beta)$ such that equality holds for both of the cases.

PROOF. Let $f \in \mathcal{M}(\beta)$. Then the relation (2.1) holds. Hence, there exists an analytic function $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$ in the unit disk $\mathbb{D}$ with $\operatorname{Re} p(z)>0$ for $z \in \mathbb{D}$ such that

$$\frac{1}{\beta-1}\left(\beta-\frac{zf'(z)}{f(z)}\right)=p(z),$$

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which is equivalent to

$$zf'(z) - \beta f(z) = (1-\beta)p(z)f(z). \tag{3.1}$$

Equating the coefficients of z^2 and z^3 on both sides of (3.1),

$$a_2 = (1-\beta)c_1 \tag{3.2}$$

and

$$2a_3 = (1-\beta)(a_2c_1 + c_2). \tag{3.3}$$

By using (3.2) in (3.3),

$$a_3 = \frac{1-\beta}{2}c_2 + \frac{(\beta-1)^2}{2}c_1^2.$$

Therefore,

$$|a_3 - \lambda a_2^2| = \left|\frac{1-\beta}{2}c_2 + \left(\frac{(\beta-1)^2}{2} - \lambda(1-\beta)^2\right)c_1^2\right| = \frac{\beta-1}{2}|c_2 - \mu c_1^2|, \tag{3.4}$$

where

$$\mu = (\beta-1)(1-2\lambda).$$

A simple application of Lemma 1.1 in (3.4) yields

$$\begin{aligned}|a_3 - \lambda a_2^2| &\le (\beta-1)\max\{1, |2\mu - 1|\}\\ &= (\beta-1)\max\{1, |4\lambda(\beta-1) - (2\beta-3)|\}.\end{aligned}$$

Since the inequality (1.4) is sharp, we obtain the following sharp inequality for any $\lambda \in \mathbb{C}$:

$$|a_3 - \lambda a_2^2| \le \beta - 1 \quad \text{for} \left|\lambda - \frac{2\beta-3}{4(\beta-1)}\right| \le \frac{1}{4(\beta-1)}$$

and the equality is attained for the functions $g(z) = z(1-z^2)^{\beta-1}$ or any rotation of $g(z)$.

Also, for the other case,

$$|a_3 - \lambda a_2^2| \le 4(\beta-1)^2\left|\lambda - \frac{2\beta-3}{4(\beta-1)}\right| \quad \text{for} \left|\lambda - \frac{2\beta-3}{4(\beta-1)}\right| \ge \frac{1}{4(\beta-1)}$$

and the equality is attained for the functions $h(z) = z(1-z)^{2(\beta-1)}$ or any rotation of $h(z)$. □

THEOREM 3.2. *Let $f(z)$ be of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $f \in \mathcal{M}(\beta)$ for some $\beta > 1$. Then, for any $\lambda \in \mathbb{C}$,*

$$|a_3 - \lambda a_2^2| \le \begin{cases} \dfrac{\beta-1}{3} & \textit{for} \left|\lambda - \dfrac{2\beta-3}{3(\beta-1)}\right| \le \dfrac{1}{3(\beta-1)},\\[2ex] (\beta-1)^2\left|\lambda - \dfrac{2\beta-3}{3(\beta-1)}\right| & \textit{for} \left|\lambda - \dfrac{2\beta-3}{3(\beta-1)}\right| \ge \dfrac{1}{3(\beta-1)}.\end{cases}$$

For each $\lambda \in \mathbb{C}$, there are functions in $\mathcal{N}(\beta)$ such that equality holds for both of the cases.

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Proof. Here we follow the same method as used in Theorem 3.1. Let $f \in \mathcal{N}(\beta)$. In view of the relation (2.8), there exists an analytic function $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ in the unit disk $\mathbb{D}$ with $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$ such that

$$\frac{1}{\beta - 1}\left(\beta - 1 - \frac{zf''(z)}{f'(z)}\right) = p(z). \tag{3.5}$$

Equating the coefficients of z^2 and z^3 on both sides of (3.5),

$$2a_2 = (1-\beta)c_1 \tag{3.6}$$

and

$$6a_3 = (1-\beta)(2a_2c_1 + c_2). \tag{3.7}$$

By using (3.6) in (3.7),

$$a_3 = \frac{1-\beta}{6}c_2 + \frac{(\beta-1)^2}{6}c_1^2.$$

Therefore,

$$|a_3 - \lambda a_2^2| = \left|\frac{1-\beta}{6}c_2 + \left(\frac{(\beta-1)^2}{6} - \lambda\left(\frac{1-\beta}{2}\right)^2\right)c_1^2\right| = \frac{\beta-1}{6}|c_2 - \mu c_1^2|, \tag{3.8}$$

where

$$\mu = (\beta-1)\left(1 - \frac{3\lambda}{2}\right).$$

Applying Lemma 1.1 in (3.8),

$$\begin{aligned}|a_3 - \lambda a_2^2| &\le \frac{\beta-1}{3}\max\{1, |2\mu - 1|\}\\ &= \frac{\beta-1}{3}\max\{1, |3\lambda(\beta-1) - (2\beta-3)|\}.\end{aligned}$$

Since the inequality (1.4) is sharp, we obtain the following sharp inequality for any $\lambda \in \mathbb{C}$:

$$|a_3 - \lambda a_2^2| \le \frac{\beta-1}{3} \quad \text{for } \left|\lambda - \frac{2\beta-3}{3(\beta-1)}\right| \le \frac{1}{3(\beta-1)}$$

and the equality is attained for the functions $g(z)$ given by $g'(z) = (1 - z^2e^{i\theta})^{\beta-1}, \theta \in [0, 2\pi)$.

Also, for the other case,

$$|a_3 - \lambda a_2^2| \le (\beta-1)^2\left|\lambda - \frac{2\beta-3}{3(\beta-1)}\right| \quad \text{for } \left|\lambda - \frac{2\beta-3}{3(\beta-1)}\right| \ge \frac{1}{3(\beta-1)}$$

and the equality is attained for the functions $h(z)$ given by $h'(z) = (1 - ze^{i\theta})^{2(\beta-1)}, \theta \in [0, 2\pi)$. □

4. Inverse coefficient estimates and maximal area problem

Our main tool in this section is an estimate for the Taylor coefficients of the function $(z/f(z))^n$ for functions $f(z)$ in $\mathcal{M}(\beta)$. Let $g(z)$ and $h(z)$ be given by the power series

$$g(z)=\sum_{n=0}^{\infty}a_nz^n \quad \text{and} \quad h(z)=\sum_{n=0}^{\infty}b_nz^n,$$

which converge in some disk $|z|<R$. We say that $g(z)$ is dominated by $h(z)$ (or $h(z)$ dominates $g(z)$), written as $g(z)\ll h(z)$, if $|a_n|\le b_n$ for each integer $n\ge 0$. For more details of the technique of dominant power series, we refer to [6, Vol. I, page 81].

Lemma 4.1. *Let $f(z)$ be of the form (1.1) and $f\in\mathcal{M}(\beta)$ for some $\beta>1$. Also, for a fixed $n\in\mathbb{N}$, let $(f(z)/z)^{-n}$ have an expansion of the form*

$$\left(\frac{f(z)}{z}\right)^{-n}=1+\sum_{k=1}^{\infty}a_k^{(-n)}z^k.$$

Then, for each $k\ge 1$,

$$|a_k^{(-n)}|\le\left|\binom{-2n(\beta-1)}{k}\right|.$$

Equality holds in the above inequality for the function $f(z)=z(1-z)^{2(\beta-1)}$ or its rotation.

Proof. The main idea of the proof of this lemma is the technique of dominant power series. If $f\in\mathcal{M}(\beta)$, then

$$\mathrm{Re}\left(\frac{\beta-\frac{zf'(z)}{f(z)}}{\beta-1}\right)>0 \quad \text{for } z\in\mathbb{D}.$$

Therefore,

$$\frac{\beta-\frac{zf'(z)}{f(z)}}{\beta-1}\ll\frac{1+z}{1-z},$$

where $\ll$ denotes coefficient domination.

The previous relation yields

$$1-\frac{zf'(z)}{f(z)}\ll(\beta-1)\frac{1+z}{1-z}-(\beta-1)=\frac{2(\beta-1)z}{1-z}$$

and hence

$$\frac{1}{z}-\frac{f'(z)}{f(z)}\ll\frac{2(\beta-1)}{1-z}. \tag{4.1}$$

On integrating (4.1) from 0 to z,

$$\ln\left(\frac{z}{f(z)}\right)\ll-2(\beta-1)\ln(1-z)$$

and, consequently,

$$\frac{z}{f(z)} \ll (1-z)^{-2(\beta-1)}. \tag{4.2}$$

Finally, from (4.2), it follows that for each fixed $n \in \mathbb{N}$,

$$\left(\frac{z}{f(z)}\right)^n \ll (1-z)^{-2n(\beta-1)}. \tag{4.3}$$

Since

$$(1-z)^{-2n(\beta-1)} = 1 + \sum_{k=1}^{\infty} c_k z^k \quad \text{where } c_k = (-1)^k \binom{-2n(\beta-1)}{k},$$

by (4.3), the required result follows immediately. □

To state our next theorem, we need some preparation. For a, b and c complex numbers with $c \neq 0, -1, -2, -3, \ldots$, the function

$$F(a,b;c;z) := {}_2F_1(a,b;c;z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

is called the Gaussian hypergeometric function which is analytic in $\mathbb{D}$. Here $(a)_0 = 1$ for $a \neq 0$ and $(a)_n$ denotes the Pochhammer symbol $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1) \cdots (a+n-1)$ for $n \in \mathbb{N}$.

THEOREM 4.2. *Let $f \in \mathcal{M}(\beta)$ for some fixed $\beta > 1$. Then, for $0 < r \leq 1$,*

$$\max_{f \in \mathcal{M}(\beta)} \Delta\left(r, \frac{z}{f(z)}\right) = 4\pi r^2 (\beta-1)^2 \, F(2\beta-1, 2\beta-1; 2; r^2).$$

The maximum is attained for the function $f_0(z) = z(1-z)^{2(\beta-1)}$.

PROOF. Since $f \in \mathcal{M}(\beta)$, $f(z) \neq 0$ for $0 < |z| < 1$, otherwise the function $zf'(z)/f(z)$ has a pole at that point. Suppose that

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}.$$

Also, it can be easily verified that $f_0 \in \mathcal{M}(\beta)$ and $z/f_0(z)$ has the following representation:

$$\frac{z}{f_0(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \quad \text{for } z \in \mathbb{D},$$

where

$$c_n = (-1)^n \binom{-2(\beta-1)}{n} \quad \text{for } n \geq 1.$$

Then, by the relation (4.2),

$$|b_n| \leq c_n \quad \text{for all } n \geq 1.$$

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Therefore,

$$\begin{aligned}\Delta\Big(r, \frac{z}{f(z)}\Big) &= \iint_{|z|<r} \Big|\Big(\frac{z}{f(z)}\Big)'\Big|^2 \,dx\,dy \quad (z = x+iy)\\ &= \pi \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}\\ &\le \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}\\ &= \pi \sum_{n=1}^{\infty} n\Big(\frac{(\alpha)_n}{(1)_n}\Big)^2 r^{2n} \quad \text{where } \alpha = 2(\beta-1)\\ &= \pi r^2\alpha^2 \sum_{n=0}^{\infty} \frac{(\alpha+1)_n(\alpha+1)_n}{(2)_n(1)_n} r^{2n}\\ &= \pi r^2\alpha^2 F(\alpha+1, \alpha+1; 2; r^2)\\ &= 4\pi r^2(\beta-1)^2 F(2\beta-1, 2\beta-1; 2; r^2).\end{aligned}$$ □

THEOREM 4.3. *Let $f \in \mathcal{M}(\beta)$ for some $\beta > 1$ and $F(w)$ be the inverse function of $f(z)$ having the expansion $F(w) = w + \sum_{n=2}^{\infty} A_n w^n$, which is valid in some neighborhood of the origin. Then*

$$|A_n| \le \frac{1}{n}\left|\binom{-2n(\beta-1)}{n-1}\right| \quad \textit{for } n \ge 2.$$

Equality holds for the function $f(z) = z(1-z)^{2(\beta-1)}$.

PROOF. It is well known that (see for example [6, Vol. I, page 54])

$$A_n = \frac{1}{2\pi i n}\int_{|z|=r} \frac{1}{[f(z)]^n}\,dz = \frac{1}{n}a_{n-1}^{(-n)}$$

for $n \ge 2$, where $a_{n-1}^{(-n)}$ is defined in Lemma 4.1. Then, by Lemma 4.1, it follows that for each $n \ge 2$,

$$|A_n| = \frac{1}{n}|a_{n-1}^{(-n)}| \le \frac{1}{n}\left|\binom{-2n(\beta-1)}{n-1}\right|.$$ □

THEOREM 4.4. *Let the function $f(z)$ be in $\mathcal{N}(\beta)$ for some $\beta > 1$ and $F(w)$ be the inverse function of $f(z)$, with the following expansion:*

$$F(w) = w + \sum_{n=2}^{\infty} A_n w^n, \tag{4.4}$$

which is valid in some neighborhood of the origin. Then

$$|A_n| \le (-1)^{n+1}(2\beta-1)^n \binom{\frac{1}{2\beta-1}}{n} \quad \textit{for } n \ge 2.$$

Equality holds for the function $f_0(z)$ given by $f_0'(z) = (1-z)^{2(\beta-1)}$.

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PROOF. Let $F_0(w)$ be the inverse function of $f_0(z)$. Since $f_0(z) = 1/(2\beta - 1)(1 - (1 - z)^{2\beta-1})$,

$$F_0(w) = 1 - (1 - (2\beta - 1)w)^{1/(2\beta-1)} = w + \sum_{n=2}^{\infty} \gamma_n z^n$$

is valid in some neighborhood of the origin. It is a simple exercise to see that

$$\gamma_n = (-1)^{n+1}(2\beta - 1)^n \binom{\frac{1}{2\beta-1}}{n} > 0 \quad \text{for } n \geq 2.$$

Since $f \in \mathcal{N}(\beta)$, in view of (2.8), there exists an analytic function $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ in the unit disk $\mathbb{D}$ with Re $p(z) > 0$ for $z \in \mathbb{D}$ such that

$$\frac{\beta - 1 - \frac{zf''(z)}{f'(z)}}{\beta - 1} = p(z).$$

This implies that

$$-\frac{zf''(z)}{f'(z)} = (\beta - 1)(p(z) - 1). \tag{4.5}$$

Now, using the following relations:

$$f(F(w)) = w, f'(F(w))F'(w) = 1 \quad \text{and} \quad f''(F(w))(F'(w))^2 + f'(F(w))F''(w) = 0,$$

equation (4.5) can be written as

$$\frac{F(w)F''(w)}{(F'(w))^2} = (\beta - 1)(p(F(w)) - 1).$$

A simplification yields

$$F''(w) = (\beta - 1)(F'(w))^2 \sum_{n=1}^{\infty} c_n (F(w))^{n-1}. \tag{4.6}$$

Again, using the series expansion (4.4) of $F(w)$ and

$$F'(w) = 1 + \sum_{n=2}^{\infty} nA_n w^{n-1}, \quad F''(w) = \sum_{n=2}^{\infty} n(n-1)A_n w^{n-2}$$

in (4.6) and on simplification of it,

$$\sum_{n=2}^{\infty} n(n-1)A_n w^{n-2} = (\beta - 1)\Big(\sum_{n=0}^{\infty} B_n w^n\Big)\Big(\sum_{n=0}^{\infty} D_n w^n\Big), \tag{4.7}$$

where

$$B_0 = 1, B_n = \sum_{r=0}^{n} (n + 1 - r)(r + 1)A_{n+1-r}A_{r+1} \quad \text{for } n \geq 1$$

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and

$$D_0 = c_1, D_n = \sum_{r=2}^{n+1} c_r \{F^{r-1}(w)\}_n \quad \text{for } n \geq 1$$

with $A_1 = 1$. Here $\{F^{r-1}(w)\}_n$ denotes the coefficient of w^n in $(F(w))^{r-1}$.

We observe that $\{F^{r-1}(w)\}_n = X_r(A_1, A_2, \dots, A_n)$ are polynomials in A_1, A_2, …, A_n with nonnegative coefficients. And, hence, the D_n are polynomials in A_1, A_2, …, A_n, c_1, c_2, …, c_{n+1} with nonnegative coefficients. Also, we note that the B_n are polynomials in $A_1, A_2, \dots, A_{n+1}$ with nonnegative coefficients.

By equating the coefficient of w^{n-2} in (4.7),

$$n(n-1)A_n = (\beta - 1)\, Q(A_1, A_2, \dots, A_{n-1}, c_1, c_2, \dots, c_{n-1}) \quad \text{for } n \geq 2, \tag{4.8}$$

where

$$Q(A_1, A_2, \dots, A_{n-1}, c_1, c_2, \dots, c_{n-1}) = \sum_{r=0}^{n-2} B_{n-r-2} D_r$$

is a polynomial in $A_1, A_2, \dots, A_{n-1}, c_1, c_2, \dots, c_{n-1}$ with nonnegative coefficients.

As $A_1 = 1$ and $|c_n| \leq 2$ for each $n \geq 2$, we see from (4.8) that

$$n(n-1)|A_n| \leq (\beta - 1)\, Q^*(|A_2|, |A_3|, \dots, |A_{n-1}|) \tag{4.9}$$

holds for each $n > 2$, where $Q^*(|A_2|, |A_3|, \dots, |A_{n-1}|)$ is obtained from $Q(|A_1|, |A_2|, \dots, |A_{n-1}|, |c_1|, |c_2|, \dots, |c_{n-1}|)$ by replacing A_1 with 1 and $|c_1|, |c_2|, \dots, |c_{n-1}|$ with 2.

From the relation (4.8), it is not difficult to conclude that Theorem 4.4 is true for $n = 2$. Our aim is to prove that $|A_n| \leq \gamma_n$ for $n \geq 2$ by using mathematical induction. Suppose that the theorem is true for $k = 2, 3, \dots, n-1$. Then, from (4.9),

$$n(n-1)|A_n| \leq (\beta - 1)\, Q^*(\gamma_2, \gamma_3, \dots, \gamma_{n-1}). \tag{4.10}$$

Now, for the function $f_0(z)$,

$$\frac{1}{\beta - 1}\left(\beta - 1 - \frac{z f_0''(z)}{f_0'(z)}\right) = \frac{1+z}{1-z}.$$

By proceeding as above, the value of $Q(A_1, A_2, \dots, A_{n-1}, c_1, c_2, \dots, c_{n-1})$ for the function $f_0(z)$ is equal to $Q^*(\gamma_2, \gamma_3, \dots, \gamma_{n-1})$. Consequently, the relation (4.8) for the function $f_0(z)$ becomes

$$n(n-1)\gamma_n = (\beta - 1)\, Q^*(\gamma_2, \gamma_3, \dots, \gamma_{n-1}). \tag{4.11}$$

Finally, (4.10) and (4.11) together imply that

$$|A_n| \leq \gamma_n.$$

Thus, by mathematical induction, the theorem is true for all $n \geq 2$. □

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Theorem 4.5. *Let $g \in \mathcal{M}\Sigma(\beta)\,(\beta > 1)$ be of the form $g(z) = z(1 + \sum_{n=1}^{\infty} b_n z^{-n})$ for $z \in \Delta$. Then, for each $n \geq 1$,*

$$|b_n| \leq \left|\binom{-2(\beta-1)}{n}\right| \tag{4.12}$$

and the estimation (4.12) is sharp.

Proof. The mapping $f(z) \mapsto g(z) := 1/f(1/z)$ establishes a one-to-one correspondence between the functions in the classes $\mathcal{M}(\beta)$ and $\mathcal{M}\Sigma(\beta)$ because

$$\frac{zg'(z)}{g(z)} = \frac{z\left(\frac{1}{f(1/z)}\right)'}{\left(\frac{1}{f(1/z)}\right)} = \frac{1}{z}\frac{f'(1/z)}{f(1/z)}.$$

A careful observation shows that the coefficient $a_n^{(-1)}$ $(n \geq 1)$ in the expansion of $1/f(1/z)$ in $\mathbb{D}\backslash\{0\}$ is equal to the corresponding coefficient b_n $(n \geq 1)$ in the expansion of $g(z)$, where $a_n^{(-1)}$ are given as in Lemma 4.1.

Therefore,

$$\max_{g \in \mathcal{M}\Sigma(\beta)} |b_n| = \max_{f \in \mathcal{M}(\beta)} |a_n^{(-1)}| \quad \text{for } n \geq 1.$$

Thus, by Lemma 4.1, we find that for each $n \geq 1$,

$$|b_n| \leq \left|\binom{-2(\beta-1)}{n}\right|.$$

The inequality is sharp. One can easily see that the function

$$g_0(z) = z\left(1 - \frac{1}{z}\right)^{-2(\beta-1)}$$

belongs to the class $\mathcal{M}\Sigma(\beta)$ and the equality holds in (4.12) for the function $g_0(z)$. □

Theorem 4.6. *Let $g \in \mathcal{N}\Sigma(\beta)\,(\beta > 1)$ be given by $g(z) = z(1 + \sum_{n=1}^{\infty} b_n z^{-n})$ for $z \in \Delta$. Then, for $n \geq 2$,*

$$|b_n| \leq \frac{1}{n-1}\left|\binom{-2(\beta-1)}{n}\right|. \tag{4.13}$$

The estimation in (4.13) is best possible.

Proof. If $g \in \mathcal{N}\Sigma(\beta)$, then clearly $zg'(z) = z(1 + \sum_{n=1}^{\infty}(1-n)b_n z^{-n})$ is in $\mathcal{M}\Sigma(\beta)$. Hence, by Theorem 4.5,

$$|(1-n)b_n| \leq \left|\binom{-2(\beta-1)}{n}\right|,$$

from which (4.13) follows.

It can be easily verified that the function $g_0(z)$ given by

$$g_0'(z) = \left(1 - \frac{1}{z}\right)^{-2(\beta-1)}$$

belongs to the class $\mathcal{N}\Sigma(\beta)$. Equality holds in (4.13) for the function $g_0(z)$. □

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REMARK 4.7. If the function $g(z)$ is in the class $\mathcal{N}\Sigma(\beta)$, then, for any complex number c, the function $g(z)+c$ is also in the class $\mathcal{N}\Sigma(\beta)$. It follows that $|b_1(g)|$ has no upper bound in the class $\mathcal{N}\Sigma(\beta)$.

THEOREM 4.8. *Let $g \in \mathcal{M}\Sigma(\beta)$ $(\beta > 1)$ and $G(w)$ be the inverse of $g(z)$ and suppose that $G(w)$ has the following expansion:*

$$G(w) = w\left(1 + \sum_{n=1}^{\infty} B_n w^{-n}\right)$$

in some neighborhood of the point at infinity. Then:

(i) $|B_1| \le 2(\beta-1)$;

(ii) *for $n \ge 2$,*

$$|B_n| \le \frac{1}{n-1}\left|\binom{-2(n-1)(\beta-1)}{n}\right|.$$

Both of the inequalities (i) and (ii) are sharp.

PROOF. For any $g \in \mathcal{M}\Sigma(\beta)$, there exists $f \in \mathcal{M}(\beta)$ such that $g(z) = 1/f(1/z)$. Also, it can be easily verified that $G(w) = 1/F(1/w)$, where $F(w)$ is the inverse of $f(z)$. Therefore,

$$B_n = A_n^{(-1)} \quad \text{for } n \ge 1, \tag{4.14}$$

where the $A_n^{(-1)}$ are defined as in Lemma 1.3.

Our first aim is to estimate B_1. Since $f \in \mathcal{M}(\beta)$ and $f(z)$ is of the form (1.1),

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + a_2 + (2a_3 - a_2^2)z + \cdots. \tag{4.15}$$

By comparing the coefficients in (1.5) and (4.15), we find that $A_1^{(-1)} = a_2$ and, therefore, by Theorem 2.1,

$$|B_1| = |A_1^{(-1)}| = |a_2| \le 2(\beta-1).$$

Our next aim is to estimate $A_n^{(-1)}$ for $n \ge 2$. From Lemma 1.3,

$$A_n^{(-1)} = -\frac{1}{n-1} a_n^{(-(n-1))} \quad \text{for } n \ge 2. \tag{4.16}$$

Then, from (4.14) and (4.16) and applying Lemma 4.1,

$$|B_n| = \frac{1}{n-1}|a_n^{(-(n-1))}| \le \frac{1}{n-1}\left|\binom{-2(n-1)(\beta-1)}{n}\right| \quad \text{for } n \ge 2.$$

Both of the inequalities (i) and (ii) are sharp and equality holds for the function

$$g_0(z) = z\left(1 - \frac{1}{z}\right)^{-2(\beta-1)}.$$

□

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