AXIOMATIC TREATMENT OF RANK IN INFINITE SETS

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1. Summary. In many branches of mathematics the notion of rank plays an important part. H. Whitney [3] made a detailed axiomatic investigation of rank and several related ideas. All sets considered by Whitney are finite. In the present note the axiomatic treatment of rank is extended to sets of any cardinal. In the special case of algebraic dependence of elements of a field with respect to a sub-field, similar questions have already been considered by Steinitz [2].

Following Whitney, we investigate rank by means of a function $r(A)$ which associates a non-negative integer with every finite subset A of a given set M and which satisfies axioms (4)-(6) below. These axioms state that (i) the empty set is of rank zero, (ii) by adding one more element to A the rank is either unaltered or increased by 1, (iii) if neither the addition of the element x nor the addition of the element y increases the rank of A then the simultaneous addition of both, x and y, does not increase the rank of A. A subset L of M is called *independent*, if the rank $r(A)$ of every finite subset A of L is equal to the number of elements of A. A *base* of L is a maximal independent subset of L, i.e. an independent subset of L which is not a proper subset of another independent subset of L. It is easy to prove that every subset L of M possesses at least one base, more generally, that every independent subset of L is contained in some base of L. The main result of this note is that all bases of L have the same cardinal. Thus it is possible to define the rank $r(L)$ of any subset L of M as the common cardinal of all bases of L or, which is equivalent, as the largest cardinal of independent subsets of L.

Lemma 1 is a general combinatorial theorem which is capable of various applications. Lemma 2 generalizes an earlier theorem[1] on axiomatically defined independence of elements of a set.

2. A combinatorial lemma. We require a lemma which, in a certain sense, can be considered as a generalization of Cantor's diagonal process. Let there be given a system of finite, non-empty sets A_ν, where the index ν ranges over an arbitrary set, the index set. Corresponding to every finite set $N = \{\nu_1, \nu_2, \ldots, \nu_m\}$ of indices, select arbitrarily one element from each of the sets A_{ν_1}, $A_{\nu_2}, \ldots, A_{\nu_m}$. This means that the element selected, say, from A_{ν_1}, depends not only on ν_1 but also on the particular index set N of which ν_1 is a member. The assertion is that, under these circumstances, it is possible to make one

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[1][1], Theorem 3.

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further selection of elements, this time of one element x^*_ν from each A_ν, so that the following condition holds. However one chooses a finite index set N, there exists a finite index set N' containing N, such that, for every index ν of N, the element selected from A_ν in the given selection corresponding to the index set N', is x^*_ν.

Let M be an arbitrary set. Typical elements of M are denoted by the letters x and y, and typical finite subsets of M are denoted by A and B. Let M_1 be another set. Letters ν and N denote typical elements and finite subsets of M_1 respectively.

LEMMA 1. *Let* $A_\nu \subset M(\nu \epsilon M_1)$. *Suppose that the elements* $x(N, \nu)$ *satisfy*

$$x(N, \nu)\epsilon A_\nu \quad (\nu \epsilon N \subset M_1).$$

Then there are elements x^*_ν *such that, given any* N, *there exists* N' *satisfying* $N \subset N'$,

$$x^*_\nu = x(N', \nu) \quad (\nu \epsilon N).$$

Proof. We say that a system of sets $B_\nu(\nu\epsilon M_1)$ has the property R (possesses representatives) if, given any N, there exists N' satisfying $N \subset N'$,

$$x(N', \nu)\epsilon B_\nu \quad (\nu\epsilon N).$$

By hypothesis, the system A_ν possesses the property R, in fact, with $N' = N$. The assertion of the lemma is that there are elements x^*_ν such that the system $\{x^*_\nu\}$ has the property R.

We may assume that[2] $MM_1 = \theta$. Let M and M_1 be well-ordered.[3] The order relation is denoted by "$<$". Let $\bar{\nu}$ be the first element of M_1. Since $x(\{\nu\}, \nu)\epsilon A_\nu$, we have $A_\nu \neq \theta$. We shall define elements x_ν inductively. Let $\nu_0 \epsilon M_1$, and suppose that x_ν has already been defined for all $\nu < \nu_0$, and that $x_\nu \epsilon A_\nu (\nu < \nu_0)$. This includes the case $\nu_0 = \bar{\nu}$, when no assumption is made about the existence of elements x_ν. Consider the following systems of sets:

$$\text{system } S(\nu_0): \qquad B_\nu(\nu_0) = \begin{cases} \{x_\nu\} & (\nu < \nu_0) \\ A_\nu & (\nu \geqslant \nu_0) \end{cases}$$

$$\text{system } S'(\nu_0, x): \qquad B'_\nu(\nu_0, x) = \begin{cases} \{x_\nu\} & (\nu < \nu_0) \\ \{x\} & (\nu = \nu_0) \\ A_\nu & (\nu > \nu_0). \end{cases}$$

Case 1. Suppose that $S(\nu_0)$ does not possess the property R. Then define x_{ν_0} to be the first element of A_{ν_0}.

Case 2. Suppose that $S(\nu_0)$ possesses the property R. Then x_{ν_0} is defined as the first element x of A_{ν_0} which satisfies the condition that $S'(\nu_0, x)$ possesses the property R. It must now be shown that there exists such an element x. Let us therefore assume that for no x of A_{ν_0} the system $S'(\nu_0, x)$ has the prop-

[2] $\{X, Y, \ldots\}$ denotes the set whose elements are $X, Y, \ldots$. $S + T$ and ST are the union and meet of sets S and T respectively, and $S - ST$ is the set of elements of S not belonging to T. The empty set is denoted by θ, and the cardinal of S is $|S|$.

[3] The well-ordering of M can be dispensed with.

erty R. Then, given any $x \epsilon A_{\nu_0}$, there is $N(x)$ satisfying the following condition. For no N' containing $N(x)$ is

(1) $$x(N', \nu) \epsilon B'_\nu(\nu_0, x) \qquad (\nu \epsilon N(x)).$$

Put $$N^* = \sum_{x \epsilon A_{\nu_0}} N(x).$$

Then N^* is finite. Since $S(\nu_0)$ has the property R, there exists N' satisfying $N^* \subset N'$,

(2) $$x(N', \nu) \epsilon B_\nu(\nu_0) \qquad (\nu \epsilon N^*).$$

Let $x \epsilon A_{\nu_0}$. Then $N(x) \subset N^* \subset N'$. Hence, by the assumption which we want to prove false, (1) does not hold. By (2),

(3) $$x(N', \nu) \epsilon B_\nu(\nu_0) \qquad (\nu \epsilon N(x)).$$

By comparing (1) and (3), one concludes that, for some $\nu \epsilon N(x)$,

$$B'_\nu(\nu_0, x) \neq B_\nu(\nu_0).$$

Therefore $$\nu_0 \epsilon N(x) \subset N^* \; (x \epsilon A_{\nu_0}).$$

Put $x(N', \nu_0) = x'$. Then

$$x' \epsilon A_{\nu_0}; \; x(N', \nu) \epsilon B'_\nu(\nu_0, x') \quad (\nu \epsilon N(x')),$$

in contradiction to the assumption that (1) is false for every x of A_{ν_0}. Hence this assumption is false, and it is possible to define an element x_{ν_0} in the way indicated. Thus, by transfinite construction, one obtains elements $x_\nu \epsilon A_\nu$ which satisfy the following condition C: if, for some ν_0, the system $S(\nu_0)$ has the property R, then the system $S'(\nu_0, x_{\nu_0})$ has the property R.

Put $$S'(\nu_0, x_{\nu_0}) = S^*(\nu_0),$$
$$B'_\nu(\nu_0, x_{\nu_0}) = B^*_\nu(\nu_0).$$

By the hypothesis of the lemma, $S(\bar{\nu})$ has the property R. Hence, by condition C, $S^*(\bar{\nu})$ has the property R. Let $\nu_0 > \bar{\nu}$, and assume that, for every $\nu_1 < \nu_0$, $S^*(\nu_1)$ has the property R. Consider any set N. Since N is finite, there is an element $\nu_1 < \nu_0$ satisfying

$$\nu \leq \nu_1 \quad (\nu \epsilon N; \; \nu < \nu_0).$$

Since $S^*(\nu_1)$ has the property R, there exists N' satisfying

$$N \subset N'; \; x(N', \nu) \epsilon B^*_\nu(\nu_1) \qquad (\nu \epsilon N).$$

Then $$x(N', \nu) \epsilon B^*_\nu(\nu_1) \subset B_\nu(\nu_0) \qquad (\nu \epsilon N).$$

Since N is arbitrary, this shows that $S(\nu_0)$ has the property R. Hence, by condition C, $S^*(\nu_0)$ has the property R. Thus it follows by induction that every system $S^*(\nu)$ has the property R.

Let $N \neq \theta$. Let ν' be the last element of N. Since $S^*(\nu')$ has the property R, there exists N' such that

$$N \subset N'; x(N', \nu) \epsilon B^*_\nu(\nu') = \{x_\nu\} \; (\nu \epsilon N),$$
$$x_\nu = x(N', \nu) \; (\nu \epsilon N).$$

Hence the assertion of the lemma is true for $x^*_\nu = x_\nu$.

I should like to point out that the lemma is no longer valid if in both, hypothesis and conclusion, the condition that A and N are finite sets, is replaced by the condition that A and N are at most denumerable sets. For let M be the set of all positive integers and M_1 be a non-denumerable set, $A_\nu = M$ $(\nu \in M_1)$. If N is at most denumerable, say[4] $N = \{\nu_1, \nu_2, \ldots\}_{\neq}$, define $x(N, \nu)$, for $\nu \in N$, by putting $x(N, \nu_\lambda) = \lambda$ $(\nu_\lambda \in N)$. Then the modified hypothesis holds, but the modified conclusion of the lemma does not hold.

3. A lemma on rank functions. We use the notation of sec. 2. In addition, we denote by L any subset of M. By definition, a *rank function* in M is a function r which associates with every A an integer $r(A)$ and satisfies the following axioms[5]:

$$r(\theta) = 0, \tag{4}$$

$$r(A) \leq r(A + \{x\}) \leq r(A) + 1, \tag{5}$$

$$\begin{cases} \text{if} & r(A) = r(A + \{x\}) = r(A + \{y\}), \\ \text{then} & r(A) = r(A + \{x, y\}). \end{cases} \tag{6}$$

Put $f(L) = 1$, if $r(A) = |A|$ $(A \subset L)$, and $f(L) = 0$ otherwise. In particular, $f(\theta) = 1$. The equation $f(L) = 1$ expresses the fact that the set L is independent, in the sense of this term as defined in the summary of this note.

Let A_ν be a system of finite subsets of M which has the property that, given any finite number k of distinct indices ν_κ, the union of the corresponding k sets A_{ν_κ} is at least of rank k. Then Lemma 2 below states that it is possible to select one element from each A_ν in such a way that the selected elements are mutually distinct and the set of all selected elements is independent.

LEMMA 2. *Let $r(A)$ be a rank function in M. Let*

$$A_\nu \subset M \quad (\nu \in M_1).$$

Suppose that

$$r(\sum_{\nu \in N} A_\nu) \geqslant |N| \quad (N \subset M_1). \tag{7}$$

Then there are elements $x^*_\nu \in A_\nu$ satisfying

$$x^*_{\nu_1} \neq x^*_{\nu_2} \quad (\nu_1 \neq \nu_2), \tag{8}$$

$$f(\sum_{\nu \in M_1} \{x^*_\nu\}) = 1. \tag{9}$$

Clearly, (7) is necessary for the existence of such x^*_ν.

Proof. Whitney[6] has shown that the definition of a rank function r in M satisfying (4), (5) and (6) is equivalent to a classification of all finite subsets of M into dependent and independent sets, this classification satisfying certain axioms.[7] Given the definition of rank, Whitney's definition of independence is identical with the definition given in the present note. Actually in Whitney's

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[4]The symbol $\{X, Y, \ldots\}_{\neq}$ denotes the set $\{X, Y, \ldots\}$ and, at the same time, expresses the fact that the objects $X, Y, \ldots$ are different from each other.

[5][3], 510, $(R_1)-(R_3)$.

[6][3], 6.

[7][3], 513, (I_1) and (I_2).

case, M is finite, but in this particular instance no use is made of this fact. In view of Whitney's result, the case of Lemma 2 in which M_1 is finite is contained in a known theorem.[8] Now let M_1 be of any cardinal. Then, by applying this last result to a finite subset N of M_1, one can find elements $x(N, \nu)$ satisfying

$$x(N, \nu) \epsilon A_\nu \ (\nu \epsilon N \subset M_1),$$

(10) $$r(\sum_{\nu \epsilon N} \{x(N, \nu)\}) = |N| \quad (N \subset M_1).$$

By Lemma 1, there are elements x^*_ν satisfying

$$x^*_\nu = x(N', \nu) \quad (\nu \epsilon N \subset M_1),$$

where $N' = N'(N)$ is a certain set containing N. Put

$$A = A(N) = \sum_{\nu \epsilon N} \{x(N', \nu)\} \ ; \ B = B(N) = \sum_{\nu \epsilon N' - N} \{x(N', \nu)\} .$$

Then, by using (10) and two simple results of (1),[9] one deduces that

$$|N'| = r(A + B) \leq r(A) + r(B) \leq |A| + |B| \leq |N| + |N' - N| = |N'| ,$$

$$r(A) = |A| = |N| ,$$

$$r(\sum_{\nu \epsilon N} \{x^*_\nu\}) = r(\sum_{\nu \epsilon N} \{x(N', \nu)\}) = r(A) = |N| .$$

This implies (8) and (9).

4. The rank of sets of any cardinal. We use the same notation as in the preceding section. Let $r(A)$ be a rank function defined in M. Let L be a subset of M. A subset L^* of L is called a base of L, if L^* is a maximal independent subset of L, i.e. if

$$f(L^*) = 1 \ ; \ f(L^* + \{x\}) = 0. \qquad (x \epsilon L - L^*).$$

I shall now prove that the concepts of independence and base relating to sets of any arbitrary cardinal have some of the usual properties well known in the case of finite sets, in particular that (i) given two independent sets, the second of higher cardinal than the first, it is possible to enlarge the first by the addition of one more element of the second set in such a way that independence is not lost; (ii) every independent subset of a set is contained in some base of this set; (iii) all bases of a set are of the same cardinal.

THEOREM. (i) *If* $|L| < |L'|$; $f(L) = f(L') = 1$, *then there exists* $x' \epsilon L' - LL'$ *satisfying* $f(L + \{x'\}) = 1$.

(ii) *If* $L_1 \subset L; f(L_1) = 1$, *then there exists a base of* L *which contains* L_1. *In particular* (L_1 *the empty set*) *every set* L *possesses at least one base.*

(iii) *If* L' *and* L'' *are bases of* L, *then* $|L'| = |L''|$.

In view of (ii) and (iii), one can define the *rank cardinal* $r(L)$ as the largest cardinal of independent subsets of L, or, which is equivalent, as the common cardinal of all bases of L. If L is finite, this definition is consistent with the given definition of r.

[8] [1], Theorem 3.

[9] [3], (3.3) and Lemma 1.

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Proof. For every integer $n \geqslant 0$ and every ordered system $x_1, x_2, \ldots, x_n$ of n elements of M, define a number $I(x_1, \ldots, x_n)$ as follows:

$$I(x_1, \ldots, x_n) = 1, \text{ if } r(\{x_1, \ldots, x_n\}) = n,$$
$$I(x_1, \ldots, x_n) = 0 \text{ otherwise.}$$

According to Whitney,[10] the function I satisfies conditions (i)-(iv) of [1], in particular

$$(11) \qquad I(y_1, \ldots, y_m)\, I(x'_1, \ldots, x'_{m+1}) \leq \sum_{\mu=1}^{m+1} I(y_1, \ldots, y_m, x'_\mu).$$

In order to prove (i) of the Theorem, assume that

$$(12) \qquad f(L + \{x\}) = 0, \quad (x \in L' - LL').$$

Let $x' \in L'$. Then there exists a set $\{x_1, \ldots, x_n\} \underset{\neq}{\subset} L$ satisfying

$$(13) \qquad I(x_1, \ldots, x_n, x') = 0.$$

For if $x' \in L$, this follows from (12). If $x' \in L$, (13) holds for $n = 1$, $x_1 = x'$. Put, for some fixed choice of n and $x_1, \ldots, x_n$, $\{x_1, \ldots, x_n\} = A(x')$. I want to apply Lemma 2 to the system of sets $A(x')$ $(x' \in L')$, using the cardinal number as the rank function occurring in Lemma 2. In order to verify (7) assume that

$$\{x'_1, \ldots, x'_k\} \underset{\neq}{\subset} L',$$
$$(14) \qquad A(x'_1) + \ldots + A(x'_k) = \{y_1, \ldots, y_m\} \underset{\neq}{\subset} L,$$
$$m < k.$$

This should lead to a contradiction. By (13),

$$I(y_{m_1}, \ldots, y_{m_l}, x'_1) = 0$$

for suitable m_λ; $1 \leq m_1 < \ldots < m_l \leq m$. Then $I(y_1, \ldots, y_m, x'_1) = 0$ and, by symmetry,

$$(15) \qquad I(y_1, \ldots, y_m, x'_\kappa) = 0 \qquad (1 \leqslant \kappa \leqslant k).$$

Since $f(L) = 1$,

$$(16) \qquad I(y_1, \ldots, y_m) = 1.$$

One deduces from (14), (16), (15), (11) that $I(x_1', \ldots, x'_k) = 0$, which contradicts $f(L') = 1$. This proves that

$$|A(x'_1) + \ldots + A(x'_k)| \geqslant k$$

whenever $\{x'_1, \ldots, x'_k\} \underset{\neq}{\subset} L'$. Hence, by Lemma 2, one can find elements $\phi(x')$ satisfying

$$\phi(x') \in A(x') \subset L \qquad (x' \in L'),$$
$$\phi(x') \neq \phi(x'') \text{ for } x' \neq x''.$$

This contradicts the fact that $|L'| > |L|$. Hence (i) holds.

In order to prove (ii), suppose that $L_1 \subset L$; $f(L_1) = 1$. Let Λ be the aggregate whose elements are all sets L' satisfying $L_1 \subset L' \subset L$; $f(L') = 1$. Thus $L_1 \in \Lambda$. Then Zorn's Lemma [4] applies to Λ. For let Λ' be any subaggregate of Λ which has the property that, whenever $L' \in \Lambda'$, $L'' \in \Lambda'$, at least one of the

[10] [3], 6.

relations $L' \subset L''$, $L'' \subset L'$ holds; denote by L''' the union of all sets $L' \epsilon \Lambda'$. Then, clearly, any finite subset A of L''' is subset of some element of Λ', and hence satisfies $f(A) = 1$. Therefore $f(L''') = 1$, $L''' \epsilon \Lambda'$. By Zorn's Lemma, Λ possesses a maximal element L^*. This means that

$$L_1 \subset L^* \subset L;\ f(L^*) = 1,$$
$$f(L^* + \{x\}) = 0 \qquad (x \epsilon L - L^*),$$

i.e. that L^* is a base of L containing L_1. This proves (ii).

Finally, suppose that L' and L'' are bases of L, $|L'| < |L''|$. Then $f(L') = f(L'') = 1$. Hence, by (i), there exists $x' \epsilon L'' - L'L''$ such that $f(L' + \{x'\}) = 1$. This contradicts the fact that L' is a base of L and hence, by definition, not a proper subset of an independent subset of L. This completes the proof of the theorem.

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