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Sharp Inequalities for Differentially Subordinate Harmonic Functions and Martingales

Adam Osękowski

Abstract. We determine the best constants $C_{p,\infty}$ and $C_{1,p}$, $1 < p < \infty$, for which the following holds. If u, v are orthogonal harmonic functions on a Euclidean domain such that v is differentially subordinate to u, then

$$\|v\|_p \le C_{p,\infty}\|u\|_\infty, \quad \|v\|_1 \le C_{1,p}\|u\|_p.$$

In particular, the inequalities are still sharp for the conjugate harmonic functions on the unit disc of $\mathbb{R}^2$. Sharp probabilistic versions of these estimates are also studied. As an application, we establish a sharp version of the classical logarithmic inequality of Zygmund.

1 Introduction

The objective of this paper is to study some sharp inequalities for orthogonal harmonic functions. Let us introduce the necessary background. Suppose that N is a fixed positive integer, D is an open connected subset of $\mathbb{R}^N$ and let u and v be real-valued harmonic functions on D. Following Burkholder [5], we say that v is differentially subordinate to u if for all $x \in D$ we have

$$|\nabla v(x)| \le |\nabla u(x)|. \tag{1.1}$$

The functions u, v are said to be *orthogonal* if

$$\nabla u \cdot \nabla v = 0 \quad \text{on } D, \tag{1.2}$$

where the dot $\cdot$ stands for the standard scalar product in $\mathbb{R}^N$. As an example for which (1.1) and (1.2) are valid, take $N = 2$, D equal to the unit disc of $\mathbb{R}^2$ and u, v satisfying the Cauchy–Riemann equations.

Fix a point $\xi \in D$ and let D_0 be a bounded connected subdomain of D, satisfying $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$. We will consider those u, v for which

$$|v(\xi)| \le |u(\xi)|. \tag{1.3}$$

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The conditions (1.1), (1.2), and (1.3) imply many interesting estimates involving u and v. Denote by $\mu_{D_0}^{\xi}$ the harmonic measure on ∂D_0 with respect to ξ. If $1 \leq p < \infty$, define p-th norm and weak p-th norm of u by

$$\|u\|_p = \Big[\sup_{D_0} \int_{\partial D_0} |u(x)|^p \, d\mu_{D_0}^{\xi}(x)\Big]^{1/p},$$

$$\|u\|_{p,\infty} = \sup_{\lambda>0} \lambda \Big[\sup_{D_0} \mu_{D_0}^{\xi}(\{x \in \partial D_0 : |u(x)| \geq \lambda\})\Big]^{1/p},$$

where the supremum is taken over all D_0 as above. If D is the unit disc of $\mathbb{R}^2$, $\xi = (0,0)$ and v is assumed to be the harmonic conjugate of u with $v(\xi) = u(\xi)$, the problem of comparing the p-th norms of u and v goes back the work by M. Riesz [14], who showed that for some universal c_p, $1 < p < \infty$, we have

$$\|v\|_p \leq c_p \|u\|_p. \tag{1.4}$$

Then it was shown by Pichorides [12] and Cole (see Gamelin [10]) that the optimal constant c_p above is equal to $\cot(\pi/2p^*)$, where $p^* = \max\{p, p/(p-1)\}$. Finally, Bañuelos and Wang [1] proved the following.

Theorem 1.1 *Suppose that u, v satisfy* (1.1), (1.2), *and* (1.3). *Then for* $1 < p < \infty$ *the inequality* (1.4) *is valid, with c_p equal to the Pichorides–Cole constant.*

If one drops the orthogonality assumption, inequality (1.4) remains true, with some different constant c_p. Precisely, we have the following result of Burkholder [5].

Theorem 1.2 *Let u, v satisfy* (1.1) *and* (1.3). *Then for* $1 < p < \infty$,

$$\|v\|_p \leq (p^* - 1)\|u\|_p.$$

It is not known whether the constant $p^* - 1$ is the best possible (except for the case $p = 2$, when the inequality is sharp).

It is natural to question what happens in the case $p = 1$. Let us first consider the setting of conjugate harmonic functions on the unit disc. It turns out that the p-th norms of u and v are not comparable, but, as proved by Kolmogorov, the following weak-type estimate is valid: for some universal $c_{1,\infty} < \infty$,

$$\|v\|_{1,\infty} \leq c_{1,\infty} \|u\|_1. \tag{1.5}$$

Then it was shown by Davis [7], that the optimal $c_{1,\infty}$ above equals

$$\frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots}.$$

Finally, the paper [6] by Choi contains the proof of the following result for orthogonal harmonic functions.

Theorem 1.3 *If u, v satisfy* (1.1), (1.2), *and* (1.3), *then inequality* (1.5) *is valid with $c_{1,\infty}$ equal to the Davis constant.*

Without the orthogonality assumption, we have the following fact, proved by Burkholder [5].

Theorem 1.4 *Let u, v satisfy* (1.1) *and* (1.3). *Then $\|v\|_{1,\infty} \leq 2\|u\|_1$, and the constant 2 is the best possible.*

All the inequalities discussed above have their counterparts in martingale theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space filtered by a nondecreasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$. Assume, in addition, that $\mathcal{F}_0$ contains all the events of probability 0. Let $X = (X_t)$, $Y = (Y_t)$ be two real valued martingales adapted to $(\mathcal{F}_t)$. Let $[X, Y]$ denote the quadratic covariance process between X and Y (see [8]).

Following [1, 16], we say that Y is *differentially subordinate* to X if the process $[X, X] - [Y, Y]$ is nondecreasing and nonnegative as a function of t. In particular, if this is the case, then we have $|Y_0| \leq |X_0|$ which can be obtained simply by comparing $[X, X]_0$ and $[Y, Y]_0$.

Here is the martingale version of Theorem 1.2 and Theorem 1.4 taken from [16] (see also [4]). We write $\|X\|_p = \sup_t \|X_t\|_p$ and $\|X\|_{1,\infty} = \sup_t \sup_\lambda \lambda\mathbb{P}(|X_t| \geq \lambda)$.

Theorem 1.5 *Let X and Y be two martingales such that Y is differentially subordinate to X. Then for $1 < p < \infty$, we have*

$$\|Y\|_p \leq (p^* - 1)\|X\|_p.$$

Furthermore, $\|Y\|_{1,\infty} \leq 2\|X\|_1$. Both inequalities are sharp.

We say that X and Y are orthogonal if the process $[X, Y]$ is constant. Under the assumption of differential subordination and orthogonality, Bañuelos and Wang [1–3] proved the following fact.

Theorem 1.6 *Let X and Y be two continuous-time orthogonal martingales such that Y is differentially subordinate to X. Then for $1 < p < \infty$,*

$$\|Y\|_p \leq \cot(\pi/2p^*)\|X\|_p.$$

Furthermore,

$$\|Y\|_{1,\infty} \leq \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots}\|X\|_1.$$

Both inequalities are sharp.

In the present paper we continue the research in this direction and find the optimal constants in related inequalities for orthogonal harmonic functions and martingales. Let

$$C_{p,\infty} = \begin{cases} 1 & \text{if } 1 < p \leq 2, \\ \left[\dfrac{2^{p+2}}{\pi^{p+1}}\Gamma(p+1)\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{p+1}}\right]^{1/p} & \text{if } p > 2, \end{cases}$$

and for $1 < p < \infty$, $C_{1,p} = C_{p/(p-1),\infty}$.

Theorem 1.7 *Let u, v satisfy* (1.1), (1.2), *and* (1.3). *Then for* $1 < p < \infty$,

$$\|v\|_1 \leq C_{1,p}\|u\|_p, \tag{1.6}$$

$$\|v\|_p \leq C_{p,\infty}\|u\|_\infty. \tag{1.7}$$

Both inequalities are sharp, even if D *is a unit disc in* $\mathbb{R}^2$, $\xi = (0,0)$, *and* u, v *are assumed to satisfy the Cauchy–Riemann equations.*

Theorem 1.8 *Let* X *and* Y *be two continuous-time orthogonal martingales such that* Y *is differentially subordinate to* X. *Then for* $1 < p < \infty$,

$$\|Y\|_1 \leq C_{1,p}\|X\|_p, \tag{1.8}$$

$$\|Y\|_p \leq C_{p,\infty}\|X\|_\infty. \tag{1.9}$$

Both inequalities are sharp.

As an application, we present sharp versions of some classical inequalities for conjugate harmonic functions on the unit disc which may seem more natural in our context. Let $\Phi, \Psi\colon [0,\infty) \to \mathbb{R}$ be the Young functions given by $\Phi(t) = e^t - t - 1$ and $\Psi(t) = (t+1)\log(t+1) - t$.

Theorem 1.9 *Let* u, v *be conjugate harmonic functions on the unit disc.*

(i) *If* $\|u\|_\infty \leq 1$, *then for* $\gamma < \pi/2$,

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \Phi\big(\gamma|v(re^{i\theta})|\big)\, d\theta \leq 8\int_1^\infty \frac{t^{2\gamma/\pi} - \frac{2\gamma}{\pi}\log t - 1}{t^2+1}\, dt. \tag{1.10}$$

(ii) *For* $K > 2/\pi$,

$$\sup_{0<r<1} \int_{-\pi}^{\pi} |u(re^{i\theta})|\, d\theta \leq \sup_{0<r<1} \int_{-\pi}^{\pi} \Psi\big(K|u(re^{i\theta})|\big)\, d\theta + 8\int_1^\infty \frac{t^{2/(K\pi)} - \frac{2\log t}{K\pi} - 1}{t^2+1}\, dt.$$

Both inequalities are sharp.

The logarithmic estimate above is related to the classical inequality of Zygmund [17] ($\|v\|_1 \leq A\int_{-\pi}^{\pi} u\log^+ u + B$ for some $A, B > 0$). This should also be compared to the results of Pichorides [12] and Essen, Shea, and Stanton [9]. Pichorides showed that there is $L = L(K) < \infty$ such that

$$\|v\|_1 \leq K \sup_{0<r<1} \int_{-\pi}^{\pi} |u(re^{i\theta})| \log|u(re^{i\theta})| \frac{d\theta}{2\pi} + L(K)$$

if and only if $K > 2/\pi$. He also determined the sharp version of this estimate under an additional assumption that the function u is nonnegative. Essen, Shea, and Stanton studied the limit case $K = 2/\pi$, and showed that for some absolute constants C_1

and C_2,

$$\begin{aligned}\|v\|_1 \leq \frac{2}{\pi} \sup_{0<r<1} \int_{-\pi}^{\pi} |u(re^{i\theta})| \log(e+|u(re^{i\theta})|)\,\frac{d\theta}{2\pi} \\ + \frac{4}{\pi} \sup_{0<r<1} \int_{-\pi}^{\pi} |u(re^{i\theta})| \log\log(e+|u(re^{i\theta})|)\,\frac{d\theta}{2\pi} + C_1\|u\|_1 + C_2.\end{aligned}$$

In addition, the constant $2/\pi$ is the best, and $4/\pi$ cannot be replaced by a constant smaller than $2/\pi$. See [9] for details and for other related results.

The paper is organized as follows. The proofs of the announced estimates are based on the existence of certain special superharmonic functions. We study (1.7) and (1.9) in the next section, while (1.6) and (1.8) are established in Section 3. The final section is devoted to the proof of Theorem 1.9.

2 On Inequalities (1.7) and (1.9)

If $1 \leq p \leq 2$, the estimates (1.7) and (1.9) are straightforward. Indeed, we have

$$\|v\|_p \leq \|v\|_2 \leq \|u\|_2 \leq \|u\|_\infty,$$

and a similar chain of inequalities yields the martingale inequality. Obviously, the constant 1 is the best possible. Therefore, we may restrict ourselves to the case when p lies in the interval $(2,\infty)$.

Let $\mathcal{H} = \mathbb{R} \times (0,\infty)$ denote the upper half-plane and let $\mathcal{U} = \mathcal{U}_p \colon \mathcal{H} \to \mathbb{R}$ be given by the Poisson integral

$$\mathcal{U}(\alpha,\beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta \left| \frac{2}{\pi} \log|t| \right|^p}{(\alpha-t)^2+\beta^2}\,dt.$$

The function $\mathcal{U}$ is harmonic on $\mathcal{H}$ and satisfies

$$\lim_{(\alpha,\beta)\to(z,0)} \mathcal{U}(\alpha,\beta) = \left(\frac{2}{\pi}\right)^p \big|\log|z|\big|^p, \quad z \neq 0. \tag{2.1}$$

Let S denote the strip $(-1,1)\times\mathbb{R}$ and consider a conformal mapping $\varphi(z) = ie^{-i\pi z/2}$, or

$$\varphi(x,y) = \left(e^{\pi y/2}\sin\left(\frac{\pi}{2}x\right), e^{\pi y/2}\cos\left(\frac{\pi}{2}x\right)\right), \quad (x,y) \in \mathbb{R}^2.$$

One easily verifies that φ maps S onto $\mathcal{H}$. Define $U = U_p$ on S by

$$U(x,y) = \mathcal{U}(\varphi(x,y)).$$

The function U is harmonic on S and by (2.1) can be extended to the continuous function on the closure $\overline{S}$ of S by $U(\pm 1, y) = |y|^p$.

Further properties of U are investigated in the lemma below.

Lemma 2.1 (i) *The function U satisfies $U(x,y)=U(-x,y)$ on $\overline{S}$.*
(ii) *We have $U(x,y)\geq |y|^p$ for all $(x,y)\in\overline{S}$.*
(iii) *For any $(x,y)\in S$ we have $U_{xx}(x,y)\leq 0$ and $U_{yy}(x,y)\geq 0$.*
(iv) *If $(x,y)\in S$ and $y>0$, then $U_{yyy}(x,y)\geq 0$.*
(v) *For any $(x,y)\in\overline{S}$ such that $|y|\leq|x|$, we have $U(x,y)\leq C^p_{p,\infty}$.*
(vi) *For any $(x,y)\in S$ we have $U(x,y)\leq 2^{p-1}|y|^p+2^{p-1}C^p_{p,\infty}$.*

Proof (i) This is a consequence of the equality $\mathcal{U}(\alpha,\beta)=\mathcal{U}(-\alpha,\beta)$, $(\alpha,\beta)\in\mathcal{H}$: simply substitute $s=-t$ in the integral defining $\mathcal{U}$.

(ii) This follows from Jensen's inequality: we get, after a change of variables $t=s\exp(\pi y/2)$,

$$\begin{aligned}(2.2)\qquad U(x,y)&=\int_{-\infty}^{\infty}\Big|\frac{2}{\pi}\log|s|+y\Big|^p\cdot\frac{1}{\pi}\frac{\cos(\frac{\pi}{2}x)}{(s-\sin(\frac{\pi}{2}x))^2+\cos^2(\frac{\pi}{2}x)}\,ds\\ &\geq\Big|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\frac{\pi}{2}x)\big(\frac{2}{\pi}\log|s|+y\big)}{(s-\sin(\frac{\pi}{2}x))^2+\cos^2(\frac{\pi}{2}x)}\,ds\Big|^p=|y|^p.\end{aligned}$$

(iii) In view of the harmonicity of U, it suffices to deal with the second estimate. Using Fubini's theorem we verify that

$$U_{yy}(x,y)=\frac{p(p-1)}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\frac{\pi}{2}x)\,\big|\frac{2}{\pi}\log|s|+y\big|^{p-2}}{(s-\sin(\frac{\pi}{2}x))^2+\cos^2(\frac{\pi}{2}x)}\,ds,$$

and it is evident that the expression on the right is nonnegative.

(iv) We have

$$U_y(x,y)=\frac{p}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\frac{\pi}{2}x)\,\big|\frac{2}{\pi}\log|s|+y\big|^{p-2}\big(\frac{2}{\pi}\log|s|+y\big)}{(s-\sin(\frac{\pi}{2}x))^2+\cos^2(\frac{\pi}{2}x)}\,ds.$$

Therefore, for $\varepsilon\in(0,y)$ we have

$$2U_y(x,y)-U_y(x,y-\varepsilon)-U_y(x,y+\varepsilon)=\frac{p}{\pi}\int_{-\infty}^{\infty}\frac{f_{y,\varepsilon}\big(\frac{2}{\pi}\log|s|\big)\cos(\frac{\pi}{2}x)}{(s-\sin(\frac{\pi}{2}x))^2+\cos^2(\frac{\pi}{2}x)}\,ds=I,$$

where

$$f_{y,\varepsilon}(h)=2|y+h|^{p-2}(y+h)-|y-\varepsilon+h|^{p-2}(y-\varepsilon+h)-|y+\varepsilon+h|^{p-2}(y+\varepsilon+h).$$

The expression I, after being split into integrals over the nonpositive and nonnegative half-line and substituting $s=\pm e^r$, can be written in the form

$$I=\frac{p}{\pi}\int_{-\infty}^{\infty}f_{y,\varepsilon}\Big(\frac{2}{\pi}r\Big)g^x(r)\,dr,$$

where

$$g^x(r)=\frac{\cos(\frac{\pi}{2}x)e^r}{(e^r-\sin(\frac{\pi}{2}x))^2+\cos^2(\frac{\pi}{2}x)}+\frac{\cos(\frac{\pi}{2}x)e^r}{(e^r+\sin(\frac{\pi}{2}x))^2+\cos^2(\frac{\pi}{2}x)}.$$

Observe that $f_{y,\varepsilon}(h) \leq 0$ for $h \geq -y$ and that we have $f_{y,\varepsilon}(-y+h) = -f_{y,\varepsilon}(-y-h)$ for all h. Furthermore, g^x is even and for $r > 0$,

$$(g^x)'(r) = \frac{\cos(\frac{\pi}{2}x)e^r(1-e^r)}{[(e^r - \sin(\frac{\pi}{2}x))^2 + \cos^2(\frac{\pi}{2}x)]^2} + \frac{\cos(\frac{\pi}{2}x)e^r(1-e^r)}{[(e^r + \sin(\frac{\pi}{2}x))^2 + \cos^2(\frac{\pi}{2}x)]^2} \leq 0.$$

This implies $I \leq 0$ and, since $\varepsilon \in (0,x)$ was arbitrary, the function $U(x,\cdot)\colon y \mapsto U_y(x,y)$ is convex on $(0,\infty)$.

(v) First we show that

$$U_{xy}(x,y) \leq 0 \quad \text{for } x \in (0,1),\ y > 0. \tag{2.3}$$

Since U is harmonic on S, so is U_y and hence we have $U_{xxy}(x,y) = -U_{yyy}(x,y) \leq 0$ for $x \in (0,1)$ and $y > 0$. Since $U_x(0,y) = 0$, which is a consequence of (i), we see that $U_{xy}(0,y) = 0$ and therefore (2.3) follows.

Let $0 \leq y \leq x \leq 1$ and consider a function $\Phi(t) = U(tx,ty)$, $t \in [-1,1]$. Then Φ is even and by (iii) and (2.3),

$$\begin{aligned}\Phi''(t) &= x^2U_{xx}(tx,ty) + 2xyU_{xy}(tx,ty) + y^2U_{yy}(tx,ty)\\ &\leq x^2\Delta U(tx,ty) + 2xyU_{xy}(tx,ty) \leq 0\end{aligned}$$

for $t \in (-1,1)$. This implies

$$\begin{aligned}U(x,y) = \Phi(1) \leq \Phi(0) = U(0,0) = \mathfrak{U}(0,1) &= \frac{2^{p+1}}{\pi^{p+1}}\int_0^\infty \frac{|\log t|^p}{t^2+1}\,dt\\ &= \frac{2^{p+1}}{\pi^{p+1}}\int_{-\infty}^\infty \frac{|s|^p e^s}{e^{2s}+1}\,ds = \frac{2^{p+2}}{\pi^{p+1}}\int_0^\infty s^pe^{-s}\sum_{k=0}^\infty(-e^{-2s})^k\,ds\\ &= \frac{2^{p+2}}{\pi^{p+1}}\Gamma(p+1)\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{p+1}} = C_{p,\infty}^p.\end{aligned}$$

(vi) It is clear from the formula for U appearing in (2.2) that

$$\begin{aligned}U(x,y) &\leq 2^{p-1}|y|^p + 2^{p-1}\int_{-\infty}^\infty \left|\frac{2}{\pi}\log|s|\right|^p \cdot \frac{1}{\pi}\frac{\cos(\frac{\pi}{2}x)}{(s-\sin(\frac{\pi}{2}x))^2+\cos^2(\frac{\pi}{2}x)}\,ds\\ &= 2^{p-1}|y|^p + 2^{p-1}U(x,0) \leq 2^{p-1}|y|^p + 2^{p-1}U(0,0).\end{aligned}$$

Here in the last passage we have used (i) and (iii). Now use part (v) to complete the proof. ■

To establish the martingale inequalities (1.7) and (1.9), we will need the following auxiliary facts. Recall that for any semi-martingale X there exists a unique continuous local martingale part X^c of X satisfying

$$[X,X]_t = |X_0|^2 + [X^c,X^c]_t + \sum_{0<s\leq t}|\triangle X_s|^2$$

for all $t \geq 0$. Here $\triangle X_s = X_s - X_{s-}$ denotes the jump of X at time s. Furthermore, we have that $[X^c, X^c] = [X, X]^c$, the pathwise continuous part of $[X, X]$. Here is [3, Lemma 2.1].

Lemma 2.2 *If X and Y are semi-martingales, then Y is differentially subordinate and orthogonal to X if and only if Y^c is differentially subordinate and orthogonal to X^c, $|Y_0| \leq |X_0|$, and Y has continuous paths.*

Now we are ready to prove the martingale inequality.

Proof of (1.9) With no loss of generality, we may assume that $\|X\|_\infty = 1$. Let $t \in (0, \infty)$. Since U is of class C^∞ on S, we may apply Itô's formula to obtain

$$U(X_t, Y_t) = U(X_0, Y_0) + I_1 + \frac{1}{2} I_2 + \frac{1}{2} I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \int_{0+}^t U_x(X_{s-}, Y_s)\, dX_s + \int_{0+}^t U_y(X_{s-}, Y_s)\, dY_s, \\ I_2 &= 2\int_{0+}^t U_{xy}(X_{s-}, Y_s)\, d[X^c, Y]_s, \\ I_3 &= \int_{0+}^t U_{xx}(X_{s-}, Y_s)\, d[X^c, X^c]_s + \int_{0+}^t U_{yy}(X_{s-}, Y_s)\, d[Y, Y]_s, \\ I_4 &= \sum_{0<s\leq t} \left[U(X_s, Y_s) - U(X_{s-}, Y_s) - U_x(X_{s-}, Y_s)\Delta X_s \right]. \end{aligned} \tag{2.4}$$

Note that we have used above the equalities $Y_{s-} = Y_s$ and $Y = Y^c$ which are due to the continuity of paths of Y. By Lemma 2.1(v) and Lemma 2.2 we have $U(X_0, Y_0) \leq C_{p,\infty}^p$. The term I_1 has zero expectation, as the stochastic integrals are martingales. We have $I_2 = 0$ in view of the orthogonality of X and Y. The differential subordination together with Lemma 2.1(iii) give

$$I_3 \leq \int_0^t U_{xx}(X_s, Y_s)\, d[X^c, X^c]_s + \int_0^t U_{yy}(X_s, Y_s)\, d[X^c, X^c]_s = 0.$$

Finally, we have that $I_4 \leq 0$ by the concavity of $U(\,\cdot\,, y)$ for any fixed $y \in \mathbb{R}$ (see Lemma 2.1 (iii)). Therefore, by Lemma 2.1(ii), $\mathbb{E}|Y_t|^p \leq \mathbb{E}U(X_t, Y_t) \leq C_{p,\infty}^p$, and it suffices to take the supremum over t to obtain (1.8). ■

Proof of the inequality (1.7) It suffices to show that for any bounded subdomain D_0 of D satisfying $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$ we have

$$\int_{\partial D_0} |v(x)|^p \, d\mu_{D_0}^\xi(x) \leq C_{p,\infty}^p \|u\|_\infty^p.$$

Let $B = (B_t)_{t\geq 0}$ be an N-dimensional Brownian motion starting from ξ and let τ denote the first moment B hits the boundary of D_0. Consider martingales X, Y given by $X_t = u(B_{\tau\wedge t})$ and $Y_t = v(B_{\tau\wedge t})$, $t \geq 0$. We have

$$[X,X]_t = u^2(\xi) + \int_0^{\tau\wedge t} |\nabla u(B_s)|^2\, ds,$$

$$[Y,Y]_t = v^2(\xi) + \int_0^{\tau\wedge t} |\nabla v(B_s)|^2\, ds,$$

$$[X,Y]_t = u(\xi)v(\xi) + \int_0^{\tau\wedge t} \nabla u(B_s)\cdot\nabla v(B_s)\, ds,$$

and we see that the assumptions on u and v imply that Y is differentially subordinate to X and that X, Y are orthogonal. Therefore, by (1.9),

$$\int_{\partial D_0} |v(x)|^p\, d\mu^{\xi}_{D_0}(x) = \|Y\|_p^p \leq C^p_{p,\infty}\|X\|^p_\infty \leq C^p_{p,\infty}\|u\|^p_\infty. \qquad \blacksquare$$

Sharpness It suffices to prove the optimality of $C_{p,\infty}$ in (1.7). First we provide an example for D equal to the strip S and $\xi = (0,0)$; to treat the case when D is the unit disc of $\mathbb{R}^2$, we will use a conformal mapping from the disc to S (see below). Let $u(x,y) = x$ and $v(x,y) = y$ for $(x,y) \in S$. We have that $\|u\|_\infty \leq 1$ and u, v satisfy the Cauchy–Riemann equations. Let $B = (B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion starting from $(0,0)$. For $n \geq 2$, let $D_n = (-1+1/n, 1-1/n)\times(-n,n)$ and $\tau_n = \inf\{t : B_t \notin D_n\}$, $\tau = \inf\{t : B_t \notin D\}$. We will show that

$$\|v\|_p \geq \|B^{(2)}_\tau\|_p = C^p_{p,\infty}.$$

The inequality above is a consequence of

$$\|v\|_p^p \geq \int_{\partial D_n} |v(x,y)|^p\, d\mu^{\xi}_{D_n}(x,y) = \mathbb{E}|B^{(2)}_{\tau_n}|^p,$$

the almost sure convergence $B_{\tau_n} \to B_\tau$, and Fatou's lemma. To prove $\|B^{(2)}_\tau\|_p = C^p_{p,\infty}$, note that by the harmonicity of U, Itô's formula yields

$$C^p_{p,\infty} = U(0,0) = \mathbb{E}U(B_{\tau\wedge t}), \qquad t \geq 0.$$

By Burkholder–Davis–Gundy inequalities, we have, for some universal c_p and c'_p,

$$\sup_t \|B^{(2)}_{\tau\wedge t}\|_p \leq c_p\|\tau^{1/2}\|_p \leq c'_p \sup_t \|B^{(1)}_{\tau\wedge t}\|_p = c'_p.$$

Therefore the martingale $(B^{(2)}_{\tau\wedge t})_{t\geq 0}$ converges in L^p and hence, by Lemma 2.1(vi) and Lebesgue's dominated convergence theorem,

$$C^p_{p,\infty} = \lim_{t\to\infty} \mathbb{E}U(B_{\tau\wedge t}) = \mathbb{E}U(B_\tau) = \|B^{(2)}_\tau\|^p_p.$$

This proves the optimality of (1.7) for $D = S$. If D is the unit disc of $\mathbb{R}^2$, let $F = F_1 + iF_2$, $F(0) = 0$, be a conformal mapping from D onto S and let $\overline{u} = u\circ F = F_1$, $\overline{v} = v\circ F = F_2$. Then $\overline{u}$, $\overline{v}$ satisfy the Cauchy–Riemann equations, $\|\overline{u}\|_\infty \leq 1$, and $\|\overline{v}\|^p_p = \|v\|^p_p \geq C^p_{p,\infty}$. $\blacksquare$

3 On Inequalities (1.6) and (1.8)

We start with the observation that for $p \geq 2$ the inequalities are trivial. For example, (1.6) follows from

$$\|v\|_1 \leq \|v\|_2 \leq \|u\|_2 \leq \|u\|_p,$$

and, clearly, the inequality is sharp. Therefore, we assume that $1 < p < 2$ throughout this section.

As we have seen, the crucial role in the proof of (1.7) and (1.9) was played by the special function U. Here we will also need such an object. However, things are more complicated. First, we will not work with (1.6) and (1.8) directly, but rather with the following modifications of these estimates:

$$\int_{\partial D_0} |v(x)|\, d\mu^{\xi}_{D_0}(x) \leq \int_{\partial D_0} |u(x)|^p\, d\mu^{\xi}_{D_0}(x) + L,$$

where D_0 is as before, and $\|Y\|_1 \leq \|X\|_p^p + L$. Here L is a fixed positive number. In order to establish these inequalities, we will use the value function of the following optimal stopping problem. Let $B = (B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion starting from $(0, 0)$ and introduce $V\colon \mathbb{R}^2 \to (-\infty, \infty]$ by

$$V(x, y) = \sup \mathbb{E}G(x + B^{(1)}_\tau, y + B^{(2)}_\tau), \tag{3.1}$$

where $G(x, y) = |y| - |x|^p$ and the supremum is taken over all stopping times of B satisfying $\mathbb{E}\tau^{p/2} < \infty$.

The key properties of V are listed in the lemma below.

Lemma 3.1 (i) *The function V is finite on $\mathbb{R}^2$.*
(ii) *The function V is a superharmonic majorant of G.*
(iii) *For any fixed $x \in \mathbb{R}$, the function $V(x, \cdot) : y \mapsto V(x, y)$ is convex.*
(iv) *If $|y| \leq |x|$, we have*

$$V(x, y) \leq \Big(\frac{C_{p/(p-1),\infty}}{p}\Big)^{p/(p-1)} \cdot (p - 1). \tag{3.2}$$

Proof (i) Take a stopping time $\tau \in L^{p/2}$ and note that the process $(B^{(2)}_{\tau\wedge t})$ is differentially subordinate and orthogonal to $(x + B^{(1)}_{\tau\wedge t})$. Therefore, by a theorem of Bañuelos and Wang, for any t,

$$\begin{aligned}\mathbb{E}|y + B^{(2)}_{\tau\wedge t}| &\leq |y| + \mathbb{E}|B^{(2)}_{\tau\wedge t}| \leq |y| + c + [\cot(\pi/2p^*)]^{-p}\|B^{(2)}_{\tau\wedge t}\|_p^p\\ &\leq |y| + c + \|x + B^{(1)}_{\tau\wedge t}\|_p^p,\end{aligned}$$

where $c = [\cot(\pi/2p^*)/p]^{p/(p-1)} \cdot (p - 1)$. Since $\tau \in L^{p/2}$, the Burkholder–Davis–Gundy inequality implies that the martingales $(B^{(1)}_{\tau\wedge t})$, $(B^{(2)}_{\tau\wedge t})$ converge in L^p to $B^{(1)}_\tau$ and $B^{(2)}_\tau$, respectively. Thus, letting $t \to \infty$ yields $V(x, y) \leq |y| + c$.

(ii) The inequality $V \geq G$ follows immediately by considering in (3.1) the stopping time $\tau \equiv 0$. The superharmonicity can be established using standard Markovian arguments (see [13, Chapter I]).

(iii) Fix $x, y_1, y_2 \in \mathbb{R}$ and $\lambda \in (0,1)$. For any $\tau \in L^{p/2}$, by the triangle inequality,

$$\begin{aligned}\mathbb{E}G\big(x + B^{(1)}_\tau, \lambda y_1 + (1-\lambda)y_2 + B^{(2)}_\tau\big) &\leq \lambda \mathbb{E}G(x + B^{(1)}_\tau, y_1 + B^{(2)}_\tau) \\ &\qquad + (1-\lambda)\mathbb{E}G(x + B^{(1)}_\tau, y_2 + B^{(2)}_\tau) \\ &\leq \lambda V(x, y_1) + (1-\lambda)V(x, y_2).\end{aligned}$$

It remains to take the supremum over τ to get the claim.

(iv) Fix a stopping time $\tau \in L^{p/2}$ and $t > 0$. We have

$$\mathbb{E}|y + B^{(2)}_{\tau\wedge t}| = \mathbb{E}(y + B^{(2)}_{\tau\wedge t})\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t}).$$

Consider a martingale $\zeta^t = (\zeta^t_r)_{r\geq 0}$ given by $\zeta^t_r = \mathbb{E}[\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t})|\mathcal{F}_{\tau\wedge r}]$. There exists an $\mathbb{R}^2$-valued predictable process $A = (A^{(1)}_r, A^{(2)}_r)_r$ such that for all r,

$$\zeta^t_r = \mathbb{E}\zeta^t_t + \int_0^{\tau\wedge r} A_s\, dB_s = \mathbb{E}\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t}) + \int_0^{\tau\wedge r} A_s\, dB_s$$

(see [15, Chapter V]). Therefore, using the properties of stochastic integrals, we may write

$$\begin{aligned}\mathbb{E}|y + B^{(2)}_{\tau\wedge t}| &= y\mathbb{E}\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t}) + \mathbb{E}B^{(2)}_{\tau\wedge t}\int_0^{\tau\wedge t} A_s\, dB_s \\ &= y\mathbb{E}\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t}) + \mathbb{E}\int_0^{\tau\wedge t}(0,1)\, dB_s \int_0^{\tau\wedge t} A_s\, dB_s \\ &= y\mathbb{E}\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t}) + \mathbb{E}\int_0^{\tau\wedge t} A^{(2)}_s\, ds \\ &= y\mathbb{E}\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t}) + \mathbb{E}\int_0^{\tau\wedge t}(1,0)\, dB_s \int_0^{\tau\wedge t}(A^{(2)}_s, -A^{(1)}_s)\, dB_s \\ &\leq |x||\mathbb{E}\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t})| + \mathbb{E}B^{(1)}_{\tau\wedge t}\int_0^{\tau\wedge t}(A^{(2)}_s, -A^{(1)}_s)\, dB_s \\ &= \mathbb{E}(x + B^{(1)}_{\tau\wedge t})\Big[\operatorname{sgn} x|\mathbb{E}\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t})| + \int_0^{\tau\wedge t}(A^{(2)}_s, -A^{(1)}_s)\, dB_s\Big] \\ &\leq \|x + B^{(1)}_{\tau\wedge t}\|_p \Big\|\operatorname{sgn} x|\mathbb{E}\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t})| + \int_0^{\tau\wedge t}(A^{(2)}_s, -A^{(1)}_s)\, dB_s\Big\|_{\frac{p}{p-1}}.\end{aligned}$$

Observe that the martingale

$$(\eta^t_r)_{r\geq 0} = \Big(\operatorname{sgn} x|\mathbb{E}\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t})| + \int_0^{\tau\wedge r}(A^{(2)}_s, -A^{(1)}_s)\, dB_s\Big)_{r\geq 0}$$

is differentially subordinate and orthogonal to ζ^t. Furthermore, we have $\|\zeta^t\|_\infty = \|\operatorname{sgn}(y + B^{(2)}_{\tau\wedge t})\|_\infty = 1$, so by (1.9) we see that $\|\eta^t\|_{p/(p-1)} \leq C_{p/(p-1),\infty}$. In consequence,

$$\mathbb{E}|y+B^{(2)}_{\tau\wedge t}| \leq C_{p/(p-1),\infty}\|x+B^{(1)}_{\tau\wedge t}\|_p \leq \mathbb{E}|x+B^{(1)}_{\tau\wedge t}|^p + \Big(\frac{C_{p/(p-1),\infty}}{p}\Big)^{p/(p-1)} \cdot (p-1),$$

and it suffices to let $t \to \infty$ to obtain (3.2), using the argument with the Burkholder–Davis–Gundy inequality . ■

Proof of (1.8) Fix $\delta > 0$, $\varepsilon > \delta\sqrt{2}$, and convolve G and V with a nonnegative C^∞ function g^δ, supported on the ball with center $(0,0)$ and radius δ, satisfying $\|g^\delta\|_1 = 1$. As the result, we obtain C^∞ functions G^δ and V^δ, such that $G^\delta \leq V^\delta$ and V^δ is superharmonic. Furthermore, by Lemma 3.1(iii), we have $V^\delta_{yy} \geq 0$ and, by superharmonicity, $V^\delta_{xx} \leq 0$. Let $\varepsilon > 0$, $t \geq 0$ and apply Itô's formula to obtain

$$V^\delta(\varepsilon + X_t, Y_t) = V^\delta(\varepsilon + X_0, Y_0) + I_1 + \frac{1}{2}I_2 + \frac{1}{2}I_3 + I_4,$$

where I_1, I_2, I_3 and I_4 are as in (2.4) (just replace U by V^δ and X by $\varepsilon + X$ there). Now we may repeat the arguments from the proof of (1.9) and thus obtain that $\mathbb{E}I_1 = 0$ and I_2, I_3, I_4 are nonpositive. Furthermore, since $\varepsilon > \delta/2$, the assumption on the support of g^δ, together with $|Y_0| \leq |X_0|$ and (3.2), imply

$$V^\delta(\varepsilon + X_0, Y_0) \leq \Big(\frac{C_{p/(p-1),\infty}}{p}\Big)^{p/(p-1)} \cdot (p-1).$$

Therefore we have proved that

$$\mathbb{E}G^\delta(\varepsilon + X_{\tau\wedge t}, Y_{\tau\wedge t}) \leq \Big(\frac{C_{p/(p-1),\infty}}{p}\Big)^{p/(p-1)} \cdot (p-1).$$

Obviously, we have $|G^\delta(x,y)| \leq |x| + \| y| + \delta|^p \leq 2^{p-1}(|y|^p + \delta^p)$. Hence, by Lebesgue's dominated convergence theorem, if we let $\varepsilon \to 0$ and $\delta \to 0$, we get

$$\mathbb{E}|Y_{\tau\wedge t}| \leq \mathbb{E}|X_{\tau\wedge t}|^p + \Big(\frac{C_{p/(p-1),\infty}}{p}\Big)^{p/(p-1)} \cdot (p-1).$$

By Burkholder-Davis-Gundy inequalities, we may replace $\tau \wedge t$ by τ in the above estimate. Applying it to the pair $(X', Y') = (X/\lambda, Y/\lambda)$ with

$$\lambda = \frac{\|X\|_p p^{1/(p-1)}}{C^{1/(p-1)}_{p/(p-1),\infty}}$$

(clearly, the differential subordination and orthogonality remain valid) yields (1.8). ■

Sharpness We may restrict ourselves to the unit disc of $\mathbb{R}^2$ and u, v satisfying the Cauchy–Riemann equations. Then the claim follows immediately by duality. ■

4 Proof of Theorem 1.9

Proof (i) This is straightforward. For any $k = 2, 3, \dots$ we have by (1.7),

$$\|v\|_k^k \le C_{k,\infty}^k = \frac{2^{k+1}}{\pi^{k+1}} \int_0^\infty \frac{|\log|t||^k}{t^2+1}\,dt = \frac{4}{\pi}\int_1^\infty \frac{\left(\frac{2}{\pi}\log t\right)^k}{t^2+1}\,dt, \tag{4.1}$$

so for $\gamma < \pi/2$,

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \Phi(\gamma|v(re^{i\theta})|)\frac{d\theta}{2\pi} = \sum_{k=2}^{\infty} \frac{\gamma^k \|v\|_k^k}{k!} \le \frac{4}{\pi}\int_1^\infty \frac{t^{2\gamma/\pi} - \frac{2\gamma}{\pi}\log t - 1}{t^2+1}\,dt$$

as desired. To see that the bound on the right is the best possible, consider the pair (u, v) studied at the end of Section 2. Then we have equality in (4.1) for all $k \ge 2$ and hence also (1.10) is sharp.

(ii) This follows from (i) by standard duality arguments, since the functions Φ and Ψ are conjugate to each other (in the sense that Φ' is the inverse to Ψ' on $(0, \infty)$). We omit the details. ■

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Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland
e-mail: ados@mimuw.edu.pl