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HALF-TURNS AND INFINITE CHAINS OF CLIFFORD CONFIGURATIONS

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1. Introduction. In a recent paper [**7**] Longuet-Higgins and Parry prove that, given a general Clifford configuration of degree 5 (abbreviated to CL5), $\mathbf{C}_0$ say, there exist points P and Q such that the inverses of P in the circles of $\mathbf{C}_0$ form the points of another CL5 $\mathbf{C}_1$, whilst the inverses of Q in the circles of $\mathbf{C}_1$ are the points of $\mathbf{C}_0$; also the inverses of Q in the circles of $\mathbf{C}_0$ form the points of a CL5 $\mathbf{C}_{-1}$, whilst the inverses of P in the circles of $\mathbf{C}_{-1}$ are the points of $\mathbf{C}_0$. This leads to an infinite chain $\ldots, \mathbf{C}_{-2}, \mathbf{C}_{-1}, \mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots$ of CL5s, each connected to the next by means of the same two points P and Q, called the poles of the chain.

Longuet-Higgins and Parry start with a CL5 $\mathbf{K}$ and construct (using theorems of de Longchamps, Steiner, Kantor and Bath) an infinite chain as just described in which the pole P is one of the points of $\mathbf{K}$. They show later that any CL5 is contained in some infinite chain of this type. In the present paper we show how to construct an infinite chain starting from $\mathbf{C}_0$, without reference to $\mathbf{K}$; moreover $\mathbf{C}_0$ can be a Clifford configuration of any degree n, subject to a certain restriction on the configuration when $n \geqq 6$. We also prove the existence of half-turns (Möbius involutions), with P and Q as mates, that map the various links of the chain to themselves or to other links. These half-turns provide alternative, and perhaps simpler, proofs of the existence of the chains and of their properties (some new properties, and some already proved in [**7**]). The basic tool in the construction is the notion of a pair of conjugate pentads [**7**] and the associated half-turn [**8**], described here in Section 3.

Theorems about opposite poles of a CLn are proved in Section 6, from which infinite chains and their properties are derived in Section 7. Numerical identities connecting three constants associated with a chain are derived in Section 8, and the converse of an earlier result about CL6s is proved in Section 9.

I am grateful to the referee for suggesting some extra notation, for providing an alternative proof of Theorem 7.3, and for asking questions that resulted in the writing of Sections 10 and 11.

2. Half-turns and involutions. Let a, b be orthogonal circles meeting at A, B. The product of the inversions in a and b is the *half-turn* (or Möbius

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involution) α with poles A and B. If we take B at infinity, we have the familiar half-turn of Euclidean geometry, the product of reflections in the perpendicular lines a and b; this shows that α is determined by A and B only, and is independent of the choice of the orthogonal circles a and b.

If $P\alpha = Q$, then P, Q are *mates* with respect to α. If $P \neq Q$, then P, Q are concylic with A, B, and P, Q harmonically separate A, B. We can take $a = ABPQ$; now take P to be at infinity, and we have Fig. 1: α is the product of the reflection in a line through Q and the inversion in a circle with centre Q.

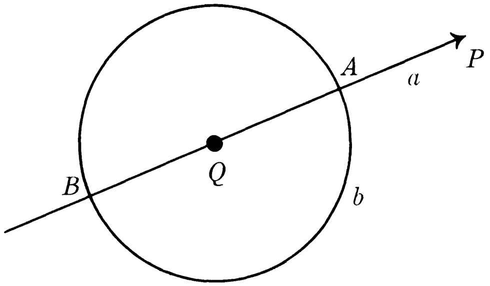

FIGURE 1

LEMMA 2.1 (a) *Let α, β, γ be half-turns with a common pair of distinct mates P, Q. Then $\alpha\beta\gamma$ is a half-turn with mates P, Q.*

(b) *Let α, β, γ be half-turns with a common pole A. Then $\alpha\beta\gamma$ is a half-turn with A as one pole.*

Proof. If we take P at infinity in (a), the result is obvious. In (b) take A at infinity; then $\alpha\beta$, the product of two Euclidean half-turns, is a translation, and $\alpha\beta\gamma$ is a Euclidean half-turn.

LEMMA 2.2. *If $P \neq R$, S and $Q \neq R$, S, there exists a unique half-turn with P, Q as mates and R, S as mates.*

Proofs of the case when P, Q, R, S are all distinct can be found in [**5**, p. 232] or [**8**, p. 522]; the other cases are easily dealt with.

Let A_0, A_1, A_2, A_3 be distinct points. Let X be a point not lying on the circumcircle of any three of the four points, and let the circles XA_0A_i, XA_jA_k meet again at X_i (where i, j, k is any permutation of $1, 2, 3$). Let τ_i denote the mapping defined by $X\tau_i = X_i$. It can be shown that τ_1, τ_2, τ_3 are mutually commutative involutory mappings, and that $\tau_j\tau_k$ is the half-turn with mates A_0, A_i and A_j, A_k. We call $\tau_1\tau_2\tau_3$ the *involution* $A_0A_1A_2A_3$; it depends only on the four points, and not on the particular order of the points. If A_0, A_1, A_2, A_3 are concyclic, this involution is just the inversion in $A_0A_1A_2A_3$. If they are not concyclic, take one of the points, A_0 say, at infinity; then the involution maps any point to its isogonal conjugate with respect to the triangle $A_1A_2A_3$ [**3**, p. 113]. Hence the image of X in the involution $A_0A_1A_2A_3$ is called the *generalized isogonal conjugate* of X with respect to A_0, A_1, A_2, A_3.

These results are not difficult to prove (the fact that $\tau_j\tau_k$ is a half-turn is an immediate consequence of Collings's theorem [**8**, Theorem 5]). The previous paragraph is based on [**1**, p. 57], where various references are given, but I have not found any single convenient reference giving all the proofs.

3. Conjugate pentads. Let A, B, C, D, E be distinct points, no four concyclic. Let A' be the mate of A in the involution $BCDE$, and define B', C', D', E' similarly. We then have two pentads of points. In [**6**, Theorem (U) p. 209, and p. 212] it is shown that A, A' are mates in the involution $B'C'D'E'$, etc., and that any two points of either pentad are inverse points in the circle through the opposite triad (e.g. A and B are inverse points in the circle $C'D'E'$). In [**8**, p. 527] I showed that there exists a half-turn mapping A, B, C, D, E to A', B', C', D', E'. Such pentads are called *conjugate* pentads; we denote them by the symbol

$$\begin{pmatrix} A & B & C & D & E \\ A' & B' & C' & D' & E' \end{pmatrix}.$$

4. Clifford configurations. A *Clifford configuration of degree n*, abbreviated to CLn, consists of 2^{n-1} points A, A_{ij}, A_{ijkl}, ... and 2^{n-1} circles c_i, c_{ijk}, ..., where the suffixes run through all (unordered) combinations of 1, 2, ..., n; a point lies on a circle if its suffix is obtained by either removing one symbol from, or adjoining one symbol to, the suffix of the circle (e.g. in a CL5 the circle c_{123} contains the points A_{12}, A_{13}, A_{23}, A_{1234}, A_{1235}); the suffix of A is the empty set, so A lies on $c_1, \ldots, c_n$. Any n circles in general position through a point A determine a unique CLn (see for instance [**4**, p. 90]).

The notation can also be reversed, so that the circles are labelled c, c_{ij}, c_{ijkl}, ... and the points A_i, A_{ijk}, ..., but c and the points $A_1, \ldots, A_n$ lying on it do not determine the configuration uniquely.

This notation obscures the fact that a CLn is transitive on points and lines; it is a self-dual $(2^{n-1}{}_n)$ configuration.

We obtain an alternative notation by writing, for instance, c_{11010} (with 1's in the 1st, 2nd and 4th places) for c_{124} in a CL5, and A_{01010} for A_{24} etc. Then any permutation of the suffix symbols, followed by the replacement of 0 by 1 and vice versa in one or more (or in none) of the suffix-places, together with the interchange of the letters A and c if necessary, gives an automorphism or duality of the CLn, and all $n!2^n$ automorphisms and dualities can be obtained in this way.

Both the above notations are sometimes cumbersome; we shall denote the points of a CLn by A, B, C, ... when this seems preferable.

Circles or points whose suffixes (in the first notation) are complemen-

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tary are *opposite* (e.g. in a CL5 A_{12} is opposite to c_{345}). Thus every point or circle has an opposite point or circle respectively when n is even, and every point has an opposite circle when n is odd.

5. Inverses in concurrent circles. Let $c_1, \ldots, c_n$ be circles all concurrent at a point A, no three meeting at any other point. Under what circumstances will there exist a point P, distinct from A, whose inverses in $c_1, \ldots, c_n$ are concyclic? When $n = 3$, every point $P \neq A$ trivially satisfies this condition. When $n > 3$ we invert A to infinity, so that $c_1, \ldots, c_n$ become lines, no three concurrent. As is pointed out in [**7**, p. 553], the reflections of P in $c_1, \ldots, c_n$ are concyclic if and only if the feet of the perpendiculars from P to $c_1, \ldots, c_n$ are concyclic; this occurs if and only if P is a focus of a conic touching $c_1, \ldots, c_n$, and the feet of the perpendiculars then lie on the auxiliary circle of the conic.

Thus when $n = 4$ there is a single infinity of positions for P. When $n = 5$ there are in general two positions for P: the two foci of the unique conic touching $c_1, \ldots, c_5$. When $n \geqq 6$ there will in general be no point P satisfying the condition, but if $c_1, \ldots, c_n$ touch a conic there will in general be two positions for P.

When $n \geqq 5$ there will be just one position for P if and only if the conic is a circle or a parabola, but the second possibility cannot occur if $c_1, \ldots, c_n$ are lines of a CLn. For if $c_1, \ldots, c_n$ touch a parabola with focus P, the circles $c_{123}, c_{124}, \ldots$ (being circumcircles of triangles touching the parabola) all pass through P; so $P = A_{1234} = A_{1235} = \ldots$ and the CLn is degenerate.

6. The poles of a CLn. Let $\mathbf{C}$ be a CLn, and P any point. The inverses of P in the circles of $\mathbf{C}$ will in general be 2^{n-1} distinct points. If the inverses of P in concurrent circles of $\mathbf{C}$ are always concyclic, then the 2^{n-1} inverses of P and the circles on which they lie form another CLn, $\mathbf{C}'$ say. We say then that P is a *pole* of $\mathbf{C}$, and that $\mathbf{C}'$ is the *transform of* $\mathbf{C}$ *by* P; we write $\mathbf{C}' = \mathbf{C}^P$.

We shall denote the inverse of P in c_λ by A_λ', where λ denotes a subset of $\{1, 2, \ldots, n\}$, and shall write $c_\lambda{}^P = A_\lambda'$. If the circles $c_\lambda, c_\mu, \ldots$ are concurrent in A_ν, where $\lambda, \mu, \ldots, \nu$ are subsets of $\{1, 2, \ldots, n\}$, we shall denote the circle containing $A_\lambda', A_\mu', \ldots$ by c_ν', and shall write $A_\nu{}^P = c_\nu'$.

At the risk of obscuring the distinction between circles and points, we can use the symbol (λ) to denote c_λ or A_λ (depending on the number of elements in λ), and $(\lambda)'$ to denote A_λ' or c_λ'. If $\bar{\lambda}$ denotes the complement of λ in the set $\{1, 2, \ldots, n\}$, then (λ) and $(\bar{\lambda})$ are opposite elements of $\mathbf{C}$. If $(\lambda) \in \mathbf{C}$ then $(\bar{\lambda})'$ is the *opposite* element of $\mathbf{C}'$. Note that when n is odd and (λ) is a point of $\mathbf{C}$, then $(\bar{\lambda})'$ is also a point, the *opposite point of* $\mathbf{C}'$.

THEOREM 6.1. *Let A be a point of a* CLn $\mathbf{C}$, *and $c_1, \ldots, c_n$ the circles of* $\mathbf{C}$ *through A; let P be a point whose inverses in $c_1, \ldots, c_n$ are concyclic. Then*

(i) *P is a pole of* $\mathbf{C}$;

(ii) *writing* $\mathbf{C}^P = \mathbf{C}'$ *and* $(\lambda)^P = (\lambda)'$, *there exists a point* Q *such that* $(\lambda)'^Q = (\lambda)$ *for all subsets* λ, *so that* $\mathbf{C}'^Q = \mathbf{C}$;

(iii) *if n is odd there exists a half-turn with mates P, Q mapping each point of* $\mathbf{C}$ *to the opposite point of* $\mathbf{C}'$, *and if n is even there exists a half-turn with mates P, Q mapping the points and circles of* $\mathbf{C}$ *to their opposite points and circles (in* $\mathbf{C}$*)*;

(iv) *Q is also a pole of* $\mathbf{C}$ *(as well as being a pole of* $\mathbf{C}'$*)*;

(v) $(\mathbf{C}^Q)^P = \mathbf{C}$.

We shall call Q the *opposite pole of P with respect to* $\mathbf{C}$; (v) states that P is the opposite pole of Q with respect to $\mathbf{C}$.

COROLLARY (of Theorem 6.1 and Section 5). *A* CL4 *has a single infinity of poles; a* CL5 *has in general two poles, but in special cases only one. There exist* CLn*'s with poles for* $n \geqq 6$; *such* CLn*'s have in general two poles, but in special cases only one.*

Proof of Theorem 6.1. (a) $n = 3$. A CL3 is a trivial configuration, consisting of four non-concyclic points A, B, C, D and the four circles $a = BCD$, $b = CDA$, $c = DAB$, $d = ABC$; parts (i) and (iv) are trivial.

Let A', B', C', D' be the inverses of P in a, b, c, d, and let Q^* be the mate of P in the involution $ABCD$; we then have the conjugate pentads

$$\begin{pmatrix} A & B & C & D & Q^* \\ A' & B' & C' & D' & P \end{pmatrix},$$

so, as we remarked in Section 3, A, B, C, D are the inverses of Q^* in $B'C'D'$, . . . , and there exists a half-turn mapping A to A', . . . , and Q^* to P. Thus Q, the opposite pole of P, is Q^*, the mate of P in the involution $ABCD$; hence the opposite pole of Q is its mate, namely P. Thus the case $n = 3$ is simply a restatement of results in Section 3.

(b) $n = 4$. Label the points and circles of a CL4 as in Fig. 2, denote the inverses of P in a, b, . . . by A', B', . . . , and assume that D', C', B', H' are concyclic (in the figure P is at infinity). Consider the CL3 $(ABCD)$ and its transform $(ABCD)^P = (A'B'C'D')$. We have seen that there exists Q whose inverses in $B'C'D'$, $A'C'D'$, . . . are A, B, ... Consider similarly the CL3 $(ABEF)$ and its transform $(ABEF)^P = (G'H'C'D')$; there exists a point R whose inverses in $H'C'D'$, $G'C'D'$, . . . are A, B, ... Now the circles $B'C'D'$, $H'C'D'$ coincide by assumption. Hence $R = Q$. Hence B and Q are inverses in the circles $A'C'D'$ and $G'C'D'$; these last two circles must therefore coincide, i.e., A', C', D', G' are concyclic. Proceeding step by step in this way we obtain all the sets of concyclic points needed for the existence of $\mathbf{C}'$ and we have thus proved (i) and (ii).

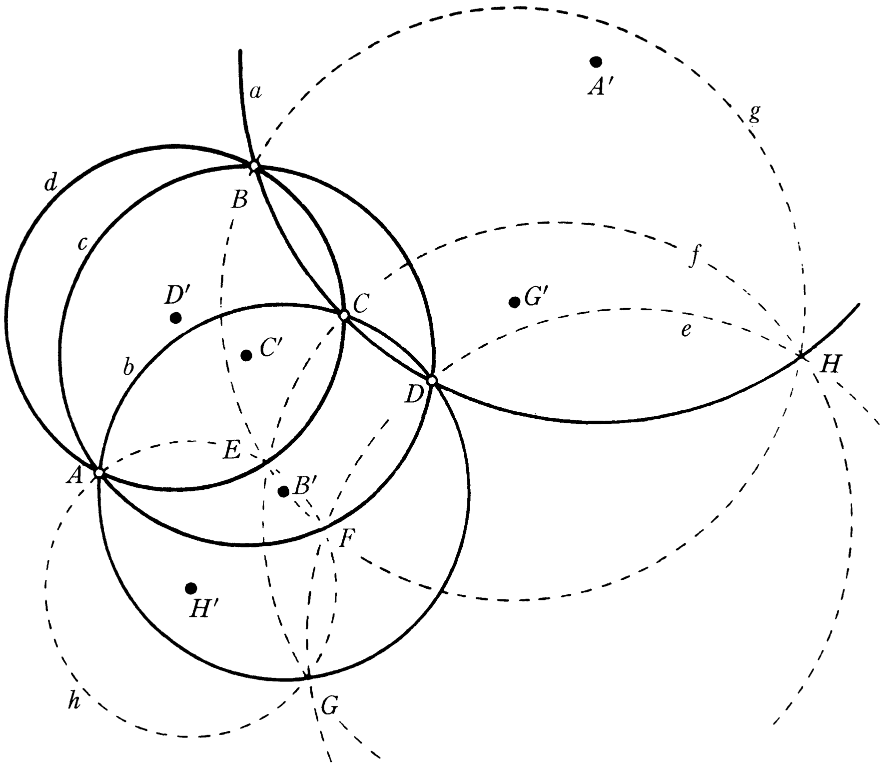

FIGURE 2

From (a) there exist a half-turn α mapping A, B, C, D, P to A', B', C', D', Q and a half-turn β mapping A, B, E, F, P to G', H', C', D', Q. Let γ be the half-turn mapping E, P to F, Q. Then $\theta = \alpha\beta\gamma$ is a half-turn (by Lemma 2.1) mapping C, D, P to F, E, Q; note that F, E are opposite to C, D respectively in the CL4.

Similarly there exist a half-turn ϕ mapping C, B, P to F, G, Q and a half-turn ψ mapping C, A, P to F, H, Q. By Lemma 2.2 $\theta = \phi = \psi$, so θ maps A, B, C, D, P to H, G, F, E, Q. Thus we have proved (iii). (Note. This part of the proof will be generalized later for $n = 6$ etc., but a different proof can be given when $n = 4$ only: I proved in [**8**, Theorem 3] that there exists a half-turn θ' mapping A, B, C, D to H, G, F, E, and we have $\theta' = \theta$ by Lemma 2.2.)

Since θ maps **C** to itself, it clearly maps a pole of **C** to a pole of **C**. Hence $Q = P\theta$ is a pole of **C**, which proves (iv). Also the opposite pole of $P\theta$ is $Q\theta$, which proves (v).

(c) $n = 5$. In (b) we combined two CL3s to prove (i) and (ii) when $n = 4$. In a similar way we now combine two CL4s to prove (i) and (ii) when $n = 5$. In fact part of the proof can be made slightly easier when $n = 5$ (and when $n > 5$).

We observe that if Q is the opposite pole of P with respect to $\mathbf{C}$ then it is also the opposite pole of P with respect to every Clifford configuration of smaller degree contained in $\mathbf{C}$. From (iii) applied to the CL4 determined by c_1, c_2, c_3, c_4, there exists a half-turn α with mates P, Q mapping A, A_{12} to A_{1234}, A_{34}. From (iii) applied to the CL3 (A_{1234}, A_{34}, A_{1345}, A_{2345}), there exists a half-turn β with mates P, Q mapping A_{1234}, A_{34} to A_{345}', A_{12345}'. Let γ be the unique half-turn with mates P, Q interchanging A_{345}', A_{12345}'. Then $\alpha\beta\gamma$ is a half-turn with mates P, Q mapping A, A_{12} to their opposite points A_{12345}', A_{345}' in $\mathbf{C}'$. By applying this method to various pairs of points of $\mathbf{C}$ and using the uniqueness of a half-turn with two given pairs of mates, we prove (iii).

From (iv) applied to the CL4s determined by c_1, c_2, c_3, c_4 and c_1, c_2, c_3, c_5, we see that the inverses of Q in c_1, c_2, c_3, c_4, c_5 are concyclic, so Q is a pole of $\mathbf{C}$ by (i).

The opposite pole of Q with respect to $\mathbf{C}$ is the same as its opposite pole with respect to any CL4 contained in $\mathbf{C}$, namely P.

(d) We build up the proof for larger values of n by induction.

We have seen as a corollary of Theorem 6.1 that a general CL5, and certain CLn's when $n > 5$, have two poles, P and P' say. From Theorem 6.1 (iv), the opposite pole of P is either P itself or P'. We now show that the first possibility cannot occur when P and P' are distinct.

THEOREM 6.2. *Suppose that the* CLn $\mathbf{C}$ ($n \geqq 5$) *has two distinct poles* P *and* P'. *Then* P *and* P' *are opposite poles.*

Proof. Let $c_1, \ldots, c_n$ be the circles through the point A of $\mathbf{C}$. Take P at infinity; then $A_1', \ldots, A_n'$ are the centres of $c_1, \ldots, c_n$, and they lie on the circle $c' : A^P = c'$. Suppose the opposite pole of P is P itself. Then $c'^P = A$, i.e., A is the centre of c'. Thus $AA_1' = \ldots = AA_n'$ and the circles $c_1, \ldots, c_n$ have equal radii. Invert with respect to A : $c_1, \ldots, c_n$ become lines $c_1^*, \ldots, c_n^*$ equidistant from A, so the unique conic touching $c_1^*, \ldots, c_n^*$ is a circle. Thus from Section 5 we see that $\mathbf{C}$ has only one pole, a contradiction. Hence the opposite pole of P is P'.

If we construct $\mathbf{C}$ by starting with circles $c_1, \ldots, c_n$ of equal radii passing through A, then we can take P at infinity. The same method of proof shows that P is its own pole and that all the circles of $\mathbf{C}$ and $\mathbf{C}'$ have equal radii [**2**; see also **7**, p. 556].

THEOREM 6.3 *Let* $c_1, \ldots, c_n$ *be the circles through a point* A *of a* CLn $\mathbf{C}$ *with opposite poles* P *and* Q. *Then*

(i) $\mathbf{C}$ *can be embedded in a* CLm *with opposite poles* P *and* Q, *for any* $m > n$, *and*

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(ii) *if we invert A to infinity,* $c_1 \ldots, c_n$ *become lines touching a conic with foci P and Q.*

Proof. (i) The inverses of P in $c_1, \ldots, c_n$ are concyclic. Draw further circles $c_{n+1}, \ldots, c_m$ through A such that the inverses of P in $c_1, \ldots, c_m$ are concyclic. Then $c_1, \ldots, c_m$ determine a unique CLm with pole P. Let Q^* be the opposite pole of the CLm. Then P, Q^* are opposite poles of $\mathbf{C}$ also, so that $Q^* = Q$.

(ii) When $n \geqq 5$ this follows immediately from the discussion in Section 5 and Theorem 6.2. When $n = 3$ or 4, we know that P is the focus of a conic touching $c_1, \ldots, c_n$; to show that Q is the other focus of the same conic, embed the CLn in a CL5.

7. Infinite chains of CLn**'s.** We now modify the notation of Theorem 6.1 and write $\mathbf{C} = \mathbf{C}_0$, $\mathbf{C}' = \mathbf{C}^P = \mathbf{C}_1$, $\mathbf{C}^Q = \mathbf{C}_{-1}$. Since P and Q are opposite poles of $\mathbf{C}_1$ and of $\mathbf{C}_{-1}$, let us write $\mathbf{C}_1{}^P = \mathbf{C}_2$, $\mathbf{C}_{-1}{}^Q = \mathbf{C}_{-2}$ and define inductively $\mathbf{C}_{r+1} = \mathbf{C}_r{}^P$, $\mathbf{C}_{-(r+1)} = \mathbf{C}_{-r}{}^Q$; then $\mathbf{C}_r$ and $\mathbf{C}_{-r}$ have P, Q as opposite poles, as is easily proved by induction, so $\mathbf{C}_r{}^P = \mathbf{C}_{r+1}$ and $\mathbf{C}_r{}^Q = \mathbf{C}_{r-1}$ for both positive and negative values of r. This is the infinite chain of CLn's derived in [**7**] for $n = 5$. If $P = Q$ there are only two distinct links in the chain, and we have the situation described in the paragraph before Theorem 6.3.

We may extend the present notation and denote the typical element of $\mathbf{C}_r$ by $(\lambda)_r$, in such a way that $(\lambda)_r{}^P = (\lambda)_{r+1}$ and $(\lambda)_r{}^Q = (\lambda)_{r-1}$. We shall say that $(\lambda)_r$ and $(\lambda)_s$ are *corresponding* elements of $\mathbf{C}_r$ and $\mathbf{C}_s$, whilst $(\lambda)_r$ and $(\bar{\lambda})_s$ are *opposite* elements, thus extending our previous use of the word "opposite".

Next we investigate how the various links in a chain are connected by half-turns. A transformation that maps elements of $\mathbf{C}_r$ to the corresponding elements of $\mathbf{C}_s$ maps $\mathbf{C}_r$ to $\mathbf{C}_s$ *directly*; a transformation that maps elements of $\mathbf{C}_r$ to the opposite elements of $\mathbf{C}_s$ maps $\mathbf{C}_r$ to $\mathbf{C}_s$ *in reverse*, or *reverses* $\mathbf{C}_r$ if $r = s$.

THEOREM 7.1. *Suppose n is odd. Let* α *and* β *denote the half-turns with mates P, Q mapping* $\mathbf{C}_0$ *to* $\mathbf{C}_1$ *and* $\mathbf{C}_0$ *to* $\mathbf{C}_{-1}$ *respectively, both in reverse. Then for* $r = \ldots, -1, 0, 1, \ldots$

(a) $\mathbf{C}_r\beta\alpha = \mathbf{C}_{r+2}$ *directly,*

(b) $\mathbf{C}_r(\alpha\beta)^r\alpha = \mathbf{C}_{r+1}$ *in reverse.*

Proof. (a) $\mathbf{C}_{-1}\beta\alpha = \mathbf{C}_1$ directly, so the result is true for $r = -1$. Suppose it is true for $r = k$; then $\beta\alpha$ maps $\mathbf{C}_k$ to $\mathbf{C}_{k+2}$ directly, and maps P to itself; hence $\beta\alpha$ maps $\mathbf{C}_k{}^P$ to $\mathbf{C}_{k+2}{}^P$ directly, i.e., $\mathbf{C}_{k+1}\,\beta\alpha = \mathbf{C}_{k+3}$ directly; hence the result is true for $r = k + 1$. The result now follows by induction for all $r \geqq -1$.

Similarly, using Q instead of P, we prove by induction that $\mathbf{C}_{-s}\alpha\beta =$

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$\mathbf{C}_{-(s+2)}$ for $s = -1, 0, 1, 2, \ldots$, which is equivalent to $\mathbf{C}_r\beta\alpha = \mathbf{C}_{r+2}$ for negative values of r.

(b) The result is true for $r = 0$. Suppose it is true for $r = k$; then $\mathbf{C}_k(\alpha\beta)^k\alpha = \mathbf{C}_{k+1}$ in reverse; hence

$$\mathbf{C}_{k+1}(\alpha\beta)^{k+1}\alpha = \mathbf{C}_k(\alpha\beta)^k\alpha(\alpha\beta)^{k+1}\alpha = \mathbf{C}_k\beta\alpha = \mathbf{C}_{k+2}$$

in reverse; hence the result is true for $r = k + 1$. The result now follows by induction for positive values of r. When $r = -s$ (s positive) we have $(\alpha\beta)^r\alpha = (\beta\alpha)^{s-1}\beta$, and we use a similar inductive proof on s.

THEOREM 7.2. *Suppose n is even. Let α and β denote the half-turns with mates P, Q reversing C_0 and C_{-1} respectively. Then for $r = \ldots, -1, 0, 1, \ldots$*

(a) $\mathbf{C}_r\beta\alpha = \mathbf{C}_{r+2}$ *directly;*

(b) $(\alpha\beta)^r\alpha$ *reverses* $\mathbf{C}_r$.

Proof. (a) Since α reverses $\mathbf{C}_0$ and maps Q to P, it follows that α maps $\mathbf{C}_0{}^Q$ to $\mathbf{C}_0{}^P$ in reverse, i.e., α maps $\mathbf{C}_{-1}$ to $\mathbf{C}_1$ in reverse. Hence $\mathbf{C}_{-1}\beta\alpha = \mathbf{C}_1$ directly. Since $\beta\alpha$ maps P and Q to themselves, we now proceed by induction as in the proof of Theorem 7.1(a).

(b) The result is true for $r = 0$. Suppose it is true for $r = k$; then $(\alpha\beta)^k\alpha$ reverses $\mathbf{C}_k$ and maps P to Q; hence it maps $\mathbf{C}_k{}^P$ to $\mathbf{C}_k{}^Q$ in reverse, i.e., $C_{k+1}(\alpha\beta)^k\alpha = \mathbf{C}_{k-1}$; hence

$$\mathbf{C}_{k+1}(\alpha\beta)^{k+1}\alpha = \mathbf{C}_{k-1}\beta\alpha = \mathbf{C}_{k+1};$$

hence the result is true for $r = k + 1$. The result now follows by induction for positive values of r. When $r = -s$ (s positive) we have

$$(\alpha\beta)^r\alpha = (\beta\alpha)^{s-1}\beta,$$

and we use a similar inductive proof on s.

It is proved in [**7**, Section 8] that, when P is at infinity, there is a dilation (or homothety) with centre Q mapping $\mathbf{C}_r$ to $\mathbf{C}_{r+2}$ directly; the proof is valid for all $n \geqq 3$. We have now shown that this dilation is independent of r and is equal to $\beta\alpha$. Here is an alternative proof of the existence of the dilation.

THEOREM 7.3. *When P is at infinity, $\beta\alpha$ is a dilation with centre Q.*

Proof. When P is at infinity, $\beta\alpha$ is a dilative rotation (a dilation with centre Q followed by a rotation about Q); this follows from the remarks preceding Lemma 2.1. Let $(\lambda)_0$ be any circle of $\mathbf{C}_0$. Then the points $(\lambda)_0{}^P = (\lambda)_1$ and $(\lambda)_0{}^Q = (\lambda)_{-1}$ are corresponding points of $\mathbf{C}_1$ and $\mathbf{C}_{-1}$; P and $(\lambda)_1$ are inverse with respect to $(\lambda)_0$, and so are Q and $(\lambda)_{-1}$. Hence P, $(\lambda)_1$, Q, $(\lambda)_{-1}$ lie on a circle orthogonal to $(\lambda)_0$. Hence Q, $(\lambda)_1$, $(\lambda)_{-1}$ are collinear when P is at infinity. Now $\beta\alpha$ maps $(\lambda)_{-1}$ to $(\lambda)_1$, and $(\lambda)_{-1}$ is any point of $\mathbf{C}_{-1}$. Hence the angle of rotation in $\beta\alpha$ is either 0^0 or 180^0; hence $\beta\alpha$ is a dilation with positive or negative constant.

It is also proved in [**7**, Section 8] that, when P is at infinity, all the circles in all the links of a chain subtend the same angle (real or imaginary) at Q. Here is another proof of this result interpreted as a result in inversive (rather than Euclidean) geometry. Suppose the circle c subtends an angle 2λ at Q, and write $\mu = \frac{1}{2}\pi - \lambda$. Let the circle through P and Q orthogonal to c (i.e., the line joining Q to the centre of c) meet c at F and G. Then the cross-ratio (PQ, FG) has the value $\tan^2 \frac{1}{2}\mu$ or $\cot^2 \frac{1}{2}\mu$. We call this the *cross-ratio* (PQ, c) or (QP, c); it is an inversive invariant, and has two values which are inverses of each other. Thus the result at the beginning of this paragraph is equivalent to:

THEOREM 7.4. *The cross-ratio (PQ, c) is the same for all circles c in the links of an infinite chain.*

Proof. Given an infinite chain of CL3s or CL4s with opposite poles P and Q, we can embed this in a CL5 with opposite poles P and Q (Theorem 6.3); an infinite chain of CLn's ($n > 5$) contains "overlapping" infinite chains of CL 5s. Hence we need only prove the result for $n = 5$. (Note. It may be thought preferable to prove the result first in the more basic situation when $n = 3$, as in [**7**, Section 8]; then other infinite chains contain "overlapping" CL3s. However, the present proof comes naturally at this stage in the development. If P, Q are opposite poles of the CL3 $(ABCD)$, with P at infinity, then Q is the *isotopic* point of $ABCD$ [**9**]).

Consider any CL4 $\mathbf{D}$ contained in the CL5 $\mathbf{C}_0$, and let c, d be opposite circles in $\mathbf{D}$. There exists a half-turn γ with mates P, Q interchanging c and d. Since half-turns preserve cross-ratios, we have $(PQ, c) = (QP, d)$. Now $\mathbf{C}_0$ contains ten CL4s, and any two circles e, f in $\mathbf{C}_0$ are either opposite in some CL4 so that $(PQ, e) = (QP, f)$, or there exists a circle g such that e, g are opposite in a CL4 and g, f are opposite in a CL4 so that $(PQ, e) = (QP, g) = (PQ, f)$. Hence (PQ, c) is the same for all circles in $\mathbf{C}_0$.

Now $\mathbf{C}_0\alpha = \mathbf{C}_1$, and P, Q are mates in α. Hence $(PQ, c) = (QP, c\alpha)$ for all circles c in $\mathbf{C}_0$, so that (PQ, c) is the same for all circles in $\mathbf{C}_0$ and $\mathbf{C}_1$. The same is true for any two successive links in the chain, so the result is proved.

8. Numerical identities. Let $\mathbf{C}_0$ be a link in a chain with opposite poles P and Q; let A be any point of $\mathbf{C}_0$, and $c_1, \ldots, c_n$ the circles of $\mathbf{C}_0$ through A. If we invert A to infinity, $c_1, \ldots, c_n$ become tangents to a conic with foci P, Q (Theorem 6.3). Let e denote the eccentricity of this conic. Let d denote the dilation factor of the dilation $\beta\alpha$ (when P is at infinity), and let k denote the common cross-ratio (PQ, c) where c is any circle of $\mathbf{C}_0$.

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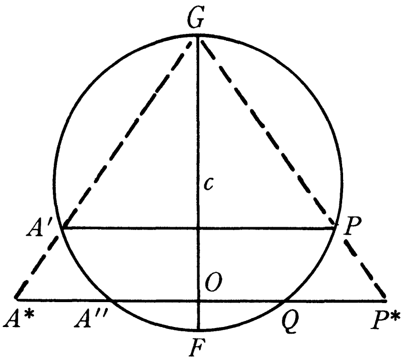

FIGURE 3

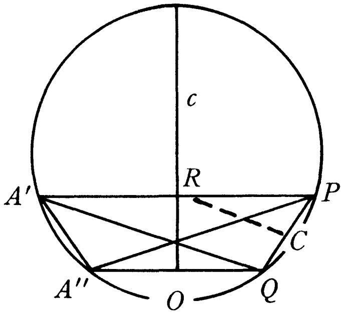

FIGURE 4

THEOREM 8.1.

$$d = \frac{(1+k)^2}{4k}; \quad e = \left| \frac{1+k}{1-k} \right| .$$

Proof. Let c be one of the circles through A; invert A to infinity, so that c is a line, and let the reflections of P, Q in c be A', A'' (Fig. 3). Then the circle $PQA''A'$ is orthogonal to c and meets c at F, G, say. Also A', A'' are the points of $\mathbf{C}_1$, $\mathbf{C}_{-1}$ corresponding to c in $\mathbf{C}_0$, so that $A''\beta\alpha = A'$. Hence $k = (PQ, FG) = (P^*Q, O\infty)$, and $d = (A'A'', QP)$ as we see by inverting P to infinity; hence $d = (A^*A'', QP^*)$. Write $OP^* = -OA^* = p$, $OQ = -OA'' = q$. Then

$$k = p/q \quad \text{and} \quad d = (p+q)^2/4pq = (1+k)^2/4k.$$

The centre of the conic with foci P, Q touching c is C (Fig. 4) and O, R lie on the auxiliary circle. Hence $e = CP/CR = QP/QA'$. Also

$$d = (A'A'', QP) = \frac{A'Q \cdot PA''}{QA'' \cdot A'P}.$$

By Ptolemy's theorem

$$QP \cdot A'A'' = PA'' \cdot QA' - A'P \cdot A''Q = (1 - d^{-1})PA'' \cdot QA',$$

i.e., $QP^2 = (1 - d^{-1})QA'^2$. Hence

$$e^2 = \frac{QP^2}{QA'^2} = (1 - d^{-1}) = \left(\frac{1-k}{1+k}\right)^2,$$

so

$$e = \left| \frac{1-k}{1+k} \right| .$$

Figures 3 and 4 show the case $e < 1$, with P and Q on the same side of c. When $e > 1$, P and Q are on opposite sides of c, and k and d are both negative; certain signs have to be changed in the above proof, but the results are the same.

We saw in Section 5 that the case $e = 1$ cannot occur.

COROLLARY 1. *The eccentricity of the conic is independent of the point A used to define it.*

COROLLARY 2. *When $k = -3 \pm 2\sqrt{2}$, we have $e = \sqrt{2}$ and $d = -1$. Thus the conic is a rectangular hyperbola, and $\beta\alpha$ is a half-turn. Hence the associated chain has only four distinct links.*

9. A converse result about CL6s. One of the results contained in Theorem 6.1 is: if a CL6 has poles, then there exists a half-turn reversing the CL6. We now prove the converse.

THEOREM 9.1. *If there exists a half-turn reversing a* CL6, *then the* CL6 *has poles.*

Proof. We use the standard notation described in Section 4. The CL6 contains a CL4

$$\begin{pmatrix} A_{15} & A_{25} & A_{35} & A_{45} \\ A_{2345} & A_{1345} & A_{1245} & A_{1235} \end{pmatrix},$$

where four points in the same row, or any two points in the top row and the points not beneath them in the bottom row, are concyclic. By Theorem 6.1 (iii) there exists a half-turn (which preserves cross-ratios) mapping each point of this CL4 to the opposite point; hence

$$(A_{15}A_{25}, A_{35}A_{45}) = (A_{2345}A_{1345}, A_{1245}A_{1235}).$$

By hypothesis there exists a half-turn mapping each point of the CL6 to to the opposite point, so

$$(A_{2345}A_{1345}, A_{1245}A_{1235}) = (A_{16}A_{26}, A_{36}A_{46}).$$

Hence

$$(1) \qquad (A_{15}A_{25}, A_{35}A_{45}) = (A_{16}A_{26}, A_{36}A_{46}).$$

Now take A to be at infinity, so that the circles $c_1, \ldots, c_6$ become lines. Then (1) states that the cross-ratios of the points in which c_1, c_2, c_3, c_4 cut c_5 and c_6 are equal. Hence $c_1, \ldots, c_6$ touch a conic. Hence by Section 5 and Theorem 6.1 (i) the CL6 has poles.

I do not know whether there is a corresponding result for CL$2m$'s when $m > 3$.

10. Half-turns associated with sub-configurations. Suppose n is even. Let $\mathbf{C}(n)$ be a CLn with opposite poles P and Q. Then there exists a half-turn $\alpha(n)$ that reverses $\mathbf{C}(n)$. Now $\mathbf{C}(n)$ contains sub-configurations $\mathbf{C}(m)$, each with opposite poles P and Q, and each with an associated half-turn $\alpha(m)$ when m is even. (Different $\mathbf{C}(m)$'s for the same value of m have different associated half-turns.) We shall show that there are many ways in which the half-turns of type $\alpha(m)$ can be multiplied together to

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give $\alpha(n)$. Such products must contain an odd number of half-turns, since each half-turn interchanges P and Q. The situation when one or both of m and n is odd will also be discussed briefly.

We first consider a method of writing down the points of any CLm contained in a CLn, then we consider the effect of the various half-turns on connected pairs of points, before investigating products of half-turns. It will frequently be much clearer if we illustrate a general situation by means of an example.

Any CLm contained in a CLn ($m < n$; here m and n can be odd or even) is completely determined by m of the circles through a point of the CLn. The CLn contains 2^{n-1} points, and the CLm contains 2^{m-1} points, so the number of CLm's contained in a CLn is

$$\binom{n}{m} 2^{n-1}/2^{m-1} = \binom{n}{m} 2^{n-m}.$$

Take $n = 10$, $m = 4$. Using the notation of Section 6, the points of a CL10 can be denoted by even subsets (i.e., subsets with an even number of elements) of the set of symbols $\{1, 2, \ldots, 8, 9, X\}$. Choose any four of these symbols, say 1, 2, 3, 4, and any fixed subset of the remaining symbols, say $579X$ .Then the subsets

$$\begin{matrix} (579X) & (12579X) & (13579X) & (14579X) \\ (1234579X) & (34579X) & (24579X) & (23579X) \end{matrix}$$

are the points of a CL4: we have adjoined to $579X$ all the even subsets of 1234. We denote this CL4 by $5\bar{6}7\bar{8}9X$, since 6 and 8 are excluded from the fixed subset. If the fixed subset is odd, say $78X$, then the points of the corresponding CL4, denoted by $\bar{5}\bar{6}78\bar{9}X$, are

$$\begin{matrix} (178X) & (278X) & (378X) & (478X) \\ (23478X) & (13478X) & (12478X) & (12378X). \end{matrix}$$

In the general case, we can choose m symbols out of n in $\binom{n}{m}$ ways, and the fixed subset of the remaining symbols in 2^{n-m} ways, giving $\binom{n}{m} 2^{n-m}$ CLm's. Thus all CLm's contained in a CLn are obtained in this way.

A *connected pair* (of points) in a CLn is a pair of points of the CLn lying on two circles of the CL n (no pair of points lies on just one circle, or on more than two circles); a connected pair and the two circles form a CL2, so connected pairs are typified by the two examples

$$(579X), (12579X) \quad \text{and} \quad (178X), (278X).$$

For convenience we shall deal only with connected pairs of the second type, in which the two subsets differ in one symbol only.

Within our CL10, the connected pair $(178X)$, $(278X)$, which is de-

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noted by $\bar{3}\bar{4}\bar{5}\bar{6}78\bar{9}X$, is contained in various CL6s, for instance in

(2) $\bar{3}4\bar{5}\bar{6}$, $\bar{3}\bar{5}\bar{6}7$, $\bar{4}78\bar{9}$ and $\bar{5}78X$.

If the CL10 has opposite poles P and Q, all the CL6s have the same opposite poles, and the half-turn associated with a CL6 maps the connected pair to the opposite connected pair in the CL6; in the four examples in (2) above, the pairs opposite to $(178X)$, $(278X)$ are

(29), (19); (2479), (1479); (235678), (135678);
$(2346789X)$, $(1346789X)$.

The effect of the half-turn in each case is to interchange 1 and 2 and to "change" four of the remaining symbols (i.e., to remove p symbols and add q new symbols, where $p + q = 4$); starting with this particular connected pair we cannot remove four symbols, but with other pairs we can clearly do this. It is easily seen that every $\alpha(6)$ has this type of effect on a connected pair.

Similarly the effect of a half-turn of type $\alpha(m)$ (m even) on the pair $(1 \ldots)$, $(2 \ldots)$ is to interchange 1 and 2 and "change" $m - 2$ of the remaining symbols. Conversely, if two connected pairs (of the second type) are obtained from each other in this way, then there is a half-turn of type $\alpha(m)$ mapping them to each other.

THEOREM 10.1. *Let* $\mathbf{C}(n)$ *be a* CLn *with opposite poles* P *and* Q. *If* $n = 4k$, *then* $\alpha(n)$ *can be expressed as the product of* $2k - 1$ *half-turns of type* $\alpha(4)$; *if* $n = 4k + 2$, *then* $\alpha(n)$ *can be expressed as the product of* $2k + 1$ *half-turns of type* $\alpha(4)$.

Proof. Here again, examples will illustrate the general situation.

$n = 8$. The connected pair (18), (28) is mapped successively by three $\alpha(4)$'s, α_1, α_2, α_3 say, to (23), (13); (1345), (2345); (234567), (134567). All the half-turns interchange P and Q, so $\alpha_1\alpha_2\alpha_3$ is a half-turn (Lemma 2.1) mapping (18), (28) to (234567), (134567). Now $\alpha(8)$ has the same effect on (18), (28). Hence by Lemma 2.2 $\alpha_1\alpha_2\alpha_3 = \alpha(8)$.

$n = 10$. The connected pair $(1X)$, $(2X)$ is mapped successively by five $\alpha(4)$'s, $\alpha_1, \ldots, \alpha_5$ say, to (24), (14); (13), (23); (2345), (1345); (134567), (234567); (23456789), (13456789). Again $\alpha_1\alpha_2 \ldots \alpha_5$ is a half-turn (by a corollary of Lemma 2.1) mapping $(1X)$, $(2X)$ to (23456789), (13456789), and $\alpha(10)$ has the same effect on $(1X)$, $(2X)$. Hence by Lemma 2.2

$$\alpha_1\alpha_2 \ldots \alpha_5 = \alpha(10).$$

THEOREM 10.2. *Let* $C(n)$ *be a* CLn *with opposite poles* P *and* Q, n *even. Then* $\alpha(n)$ *can be expressed as the product of three* $\alpha(n - 2)$*'s.*

Proof. Take $n = 10$. Using three $\alpha(8)$'s, we can successively map $(1X)$, $(2X)$ to $(2456789X)$, $(1456789X)$; (19), (29); (23456789), (13456789). Now proceed as in the proof of Theorem 10.1.

Associated with $\mathbf{C}(n)$ and its sub-configurations $\mathbf{C}(m)$ are the configuration $\mathbf{C}(n)^P = \mathbf{C}(n)'$ and its sub-configurations $\mathbf{C}(m)^P = \mathbf{C}(m)'$. When m or n is odd, the associated half-turn $\alpha(m)$ or $\alpha(n)$ maps $\mathbf{C}(m)$ or $\mathbf{C}(n)$ to $\mathbf{C}(m)'$ or $\mathbf{C}(n)'$ in reverse. The previous analysis of the effects of half-turns is still valid, with suitable modifications. For instance, when $n = 10$ there exist two $\alpha(5)$'s, $\alpha_1(5)$ and $\alpha_2(5)$ say, and an $\alpha(4)$, mapping $(1X)$, $(2X)$ successively to $(2345X)'$, $(1345X)'$; $(1345678X)$, $(2345678X)$; (23456789), (13456789), so that

$$\alpha_1(5)\alpha_2(5)\alpha(4) = \alpha(10).$$

This discussion should be sufficient to indicate that many identities exist between products of $\alpha(m)$'s.

11. Successive links when n is even. When n is odd, successive links of an infinite chain are inversively isomorphic, since there is a half-turn mapping them to each other (Theorem 7.1). When n is even, alternate links are inversively isomorphic, since $\mathbf{C}_r\beta\alpha = \mathbf{C}_{r+2}$ (Theorem 7.2), but the example in figure 5 shows that successive links need not be isomorphic when n is even. The CL4 $\mathbf{C}$ has four pairs of opposite points A, A'; B, B'; C, C'; D, D'. The pole P is at infinity, and the points of $\mathbf{C}^P$ (the centres of the circles of $\mathbf{C}$) are W, W'; X, X'; Y, Y'; Z, Z'. Now A, A', B, B' are concyclic; so are C, C', D, D', and the two circles are orthogonal. The only concyclic pairs of opposite points of $\mathbf{C}^P$ are W, W', Z, Z' and X, X', Y, Y'; these two circles are not orthogonal, so C and C^P are not inversively isomorphic.

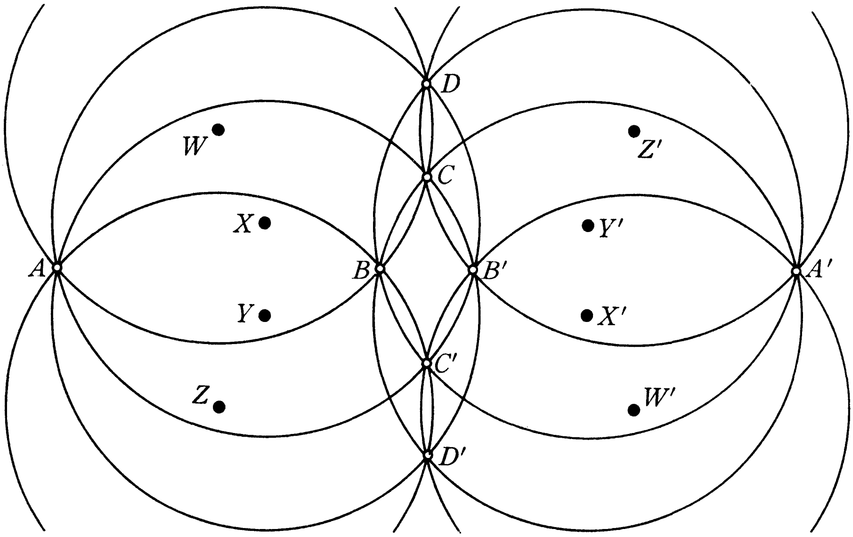

FIGURE 5

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