Three Problems on Exponential Bases

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Abstract. We consider three special and significant cases of the following problem. Let $D \subset \mathbb{R}^d$ be a (possibly unbounded) set of finite Lebesgue measure. Let $E(\mathbb{Z}^d) = \{e^{2\pi i x \cdot n}\}_{n \in \mathbb{Z}^d}$ be the standard exponential basis on the unit cube of $\mathbb{R}^d$. Find conditions on D for which $E(\mathbb{Z}^d)$ is a frame, a Riesz sequence, or a Riesz basis for $L^2(D)$.

1 Introduction

We are interested in the following problem. Let $D \subset \mathbb{R}^d$ be a set of Lebesgue measure $|D| < \infty$. Let $E(\mathbb{Z}^d) = \{e^{2\pi i n \cdot x}\}_{n \in \mathbb{Z}^d}$ be the standard exponential basis for the unit cube $Q_d = \left[-\frac{1}{2}, \frac{1}{2}\right]^d$. Can $E(\mathbb{Z}^d)$ be a frame, a Riesz sequence, or a Riesz basis for $L^2(D)$?

We have recalled definitions and general facts about frames, Riesz sequences and Riesz bases in Section 2.

Our investigation was motivated by the following problems,

Problem 1 (The broken interval)   Let $J = [0, \alpha) \cup [\alpha + r, L + r)$, with $0 < \alpha < L$ and $r > 0$. For which values of the parameters is the set $E(\mathbb{Z})$ a Riesz basis, a Riesz sequence, or a frame in $L^2(J)$?

It is easy to verify that $E(\mathbb{Z})$ is a frame on J when $L + r \leq 1$, and it is a Riesz sequence when either $\alpha \geq 1$ of $L - \alpha \geq 1$ (see also Lemma 2.2). It is proved in [24] that $E(\mathbb{Z})$ is an orthonormal basis for J if and only if the measure of J is $L = 1$ and the "gap" r is a non-negative integer.

Problem 2 (The rotated square)   Let $Q_h = \left[-\frac{h}{2}, \frac{h}{2}\right] \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ be a square with side $h > 0$. For $\theta \in [0, 2\pi)$, we let $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation

$$\rho_\theta(x, y) = (x \cos \theta - y \sin \theta, \ x \sin \theta + y \cos \theta).$$

For which values of θ is $E(\mathbb{Z}^2)$ a Riesz basis, a Riesz sequence, or a frame on $\rho_\theta(Q_h)$?

The solution to this problem is trivial only for certain values of the parameters (for example, when θ is an integer multiple of $\frac{\pi}{2}$).

The next problem was kindly suggested by Chun Kit Lai.

Problem 3 (The translated parallelepiped)   Let $P \subset \mathbb{R}^d$ be a parallelepiped with sides parallel to the vectors $v_1, \ldots, v_d \in \mathbb{R}^d$. Find conditions on these vectors for which the set $E(\mathbb{Z}^d)$ is a Riesz basis, a Riesz sequence, or a frame in $L^2(P)$.

We recall that a *lattice* is the image of $\mathbb{Z}^d$ by a linear invertible transformation $B \colon \mathbb{R}^d \to \mathbb{R}^d$ and we observe that Problem 3 is equivalent to the following: for which lattices $\Lambda = B\mathbb{Z}^d$ is the set $E(B\mathbb{Z}^d) = \{e^{2\pi i B n \cdot x}\}_{n \in \mathbb{Z}^d}$ a Riesz basis, or a Riesz sequence, or a frame in $L^2(\mathbb{Q}_d)$?

Problem 3 is related to certain optimization problems on lattices that have deep applications in computer sciences and in cryptography. See Section 7.1 for details and references.

We first prove necessary and sufficient conditions for which $E(\mathbb{Z}^d)$ is a Riesz sequence or a frame on a given domain $D \subset \mathbb{R}^d$, and then we completely solve Problems 1, 2, and 3.

We let

$$(1.1) \qquad \Phi(x) = \sum_{m \in \mathbb{Z}^d} \chi_D(x + m),$$

where χ_D denotes the characteristic function of D. Note that $\Phi(x)$ only takes non-negative integer values. Our first result is the following theorem.

Theorem 1.1   *$E(\mathbb{Z}^d)$ is a Riesz sequence in $L^2(D)$ if and only if there exist constants $0 < A \le B < \infty$ for which $A \le \Phi(x) \le B$ for a.e. $x \in \mathbb{Q}_d$.*

That is, we prove that $E(\mathbb{Z}^d)$ is a Riesz sequence in $L^2(D)$ if and only if the integer translates of D (*i.e.*, the sets $D + n = \{x + n, x \in D\}$, with $n \in \mathbb{Z}^d$) cover $\mathbb{R}^d$ with the possible exception of a set of measure zero.

It is interesting to compare Theorem 1.1 with results in [5,17,21]. In these papers the authors consider domains that *multi-tile* $\mathbb{R}^d$, *i.e.*, bounded measurable sets $S \subset \mathbb{R}^d$ for which there exist a set of translations Λ and an integer $h > 0$ such that $\sum_{\lambda \in \Lambda} \chi_{S+\lambda}(x) \equiv h$ a.e.; if $h = 1$, we say that S *tiles* $\mathbb{R}^d$. It is proved in [17, Theorem 1] and in [21, Theorem 1] that bounded domains that multi-tile $\mathbb{R}^d$ with a lattice of translation have an exponential basis; in the recent [5, Theorem 4.4], the converse of [21, Theorem 1] is proved.

If Φ is as in (1.1) and $\Phi(x) \equiv k$ a.e., then D multi-tiles $\mathbb{R}^d$ with lattice of translations $\mathbb{Z}^d$. By Theorem 1.1, $E(\mathbb{Z}^d)$ is a Riesz sequence on $L^2(D)$; when D is bounded, it is shown in [21, Theorem 1] that $E(\mathbb{Z}^d)$ can be completed to an exponential basis for $L^2(D)$, but when D is not bounded an example in [5] shows that that may not be possible.

Next, we investigate conditions for which $E(\mathbb{Z}^d)$ is a frame on D. The following result is proved in [16, Lemma 2.10]. See also the recent [7, Theorem 2].

Theorem 1.2   *$E(\mathbb{Z}^d)$ is a frame on $L^2(D)$ if and only if for every $m, s \in \mathbb{Z}^d$, with $m \neq s$, we have that*

$$(1.2) \qquad |(D + m) \cap (D + s)| = 0.$$

In other words, $E(\mathbb{Z}^d)$ is a frame in $L^2(D)$ if and only if the integer translates of D only overlap on sets of measure zero. Equivalently, $E(\mathbb{Z}^d)$ is a frame on $L^2(D)$ if and only if $\Phi(x) \leq 1$ for a.e. $x \in \mathbb{R}^d$.

We finally prove the following theorem.

Theorem 1.3   *Assume that $|D| = 1$. The following are equivalent in $L^2(D)$:*

(i) $E(\mathbb{Z}^d)$ *is a frame.*
(ii) $E(\mathbb{Z}^d)$ *is a complete.*
(iii)   *The integer translates of D tile $\mathbb{R}^d$.*
(iv) $E(\mathbb{Z}^d)$ *is an orthonormal Riesz basis.*
(v) $E(\mathbb{Z}^d)$ *is a Riesz sequence.*

We recall that a set $\{w_i\}_{i \in I}$ is *complete* in a Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ if and only if $\langle u, w_i \rangle_H = 0$ for every $i \in I$ implies $u = 0$. A frame is complete, but the converse is not necessarily true.

B. Fuglede proved in [15] that if $\Lambda = A\mathbb{Z}^d$ is a lattice in $\mathbb{R}^d$, the set $E(A\mathbb{Z}^d)$ is an orthogonal exponential basis in $L^2(D)$ if and only if $\{D + \mu\}_{\mu \in (A^t)^{-1}\mathbb{Z}^d}$ tiles $\mathbb{R}^d$. Here, $(A^t)^{-1}$ denotes the inverse of the transpose of A. Thus, the equivalence of (iii) and (iv) in Theorem 1.3 is a special case of Fuglede's theorem. The connections between tiling and exponential bases are deep and interesting and have been intensely investigated. We refer the reader to the introduction and to the references cited in [7]. See also [22].

This paper is organized as follows: in Section 2 we present preliminary definitions and known results. We prove Theorems 1.1, 1.2, and 1.3 in Sections 3 and 4. We solve Problems 1, 2, and 3 in Sections 5, 6, and 7.

2   Preliminaries and Notation

We denote by $x \cdot y = x_1 y_1 + \cdots + x_d y_d$ the inner product of $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$.

We let $\|f\|_2 = (\int_{\mathbb{R}^d} |f(x)|^2 dx)^{\frac{1}{2}}$ be the standard norm in $L^2(\mathbb{R}^d)$; we let $\mathbf{c} = \{c_j\}_{j \in \mathbb{Z}^d}$, and we denote by $\|\mathbf{c}\|_{\ell^2} = (\sum_{j \in \mathbb{Z}^d} |c_j|^2)^{\frac{1}{2}}$ the standard norm in $\ell^2(\mathbb{Z}^d)$. We denote by $\langle f, g \rangle_2 = \int_{\mathbb{R}^d} f(x) \bar{g}(x) dx$ the inner product in $L^2(\mathbb{R}^d)$. When there is no ambiguity, we will also use the same notation for the inner product in $\ell^2(\mathbb{Z}^d)$.

The Fourier transform of a function $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ is

$$\widehat{f}(x) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i x \cdot t} dt.$$

We will often say that a family of sets $\{D_\lambda\}_{\lambda \in \Lambda}$ covers $\mathbb{R}^d$ with the understanding that $\mathbb{R}^d - \bigcup_{\lambda \in \Lambda} D_\lambda$ may be a nonempty set of measure zero.

We use the notation τ_w to denote the translation operator $g \to g(\cdot + w)$.

2.1   Frames and Riesz Bases

We have used the excellent textbooks [11, 19] for most of the definitions and preliminary results presented in this section.

Let H be a separable Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \| = \sqrt{\langle \cdot , \cdot \rangle}$. A sequence of vectors $\mathcal{V} = \{v_j\}_{j \in \mathbb{Z}} \subset H$ is a *frame* if there exist constants $0 < A, \ B < \infty$ such that the following inequality holds for every $w \in H$:

$$(2.1) \qquad A\|w\|^2 \leq \sum_{j \in \mathbb{Z}} |\langle w, v_j \rangle|^2 \leq B\|w\|^2.$$

We say that $\mathcal{V}$ is a *tight frame* if $A = B$ and a *Parseval frame* if $A = B = 1$.

The left inequality in (2.1) implies that $\mathcal{V}$ is complete in H but it may not be linearly independent. A *Riesz basis* is a linearly independent frame.

An equivalent definition of Riesz basis is the following: the set $\mathcal{V}$ is a *Riesz sequence* if there exists constants $0 < A \leq B < \infty$ such that, for every finite set of coefficients $\{a_j\}_{j \in J} \subset \mathcal{C}$, we have that

$$(2.2) \qquad A \sum_{j \in J} |a_j|^2 \leq \Big\| \sum_{j \in J} a_j v_j \Big\|^2 \leq B \sum_{j \in J} |a_j|^2,$$

and it is *Riesz basis* if it also satisfies (2.1). If $\mathcal{V}$ is a Riesz basis, the constants A and B in (2.1) and (2.2) are the same (see [11, Proposition 3.5.5]).

An orthonormal basis is a Riesz basis; we can write $w = \sum_{j \in \mathbb{Z}} \langle v_j, w \rangle v_j$ for every $v \in H$, and this representation formula yields the following important identities. For every $w, \ z \in H$,

$$(2.3) \qquad \|w\|^2 = \sum_{n \in \mathbb{Z}} |\langle v_n, w \rangle|^2, \quad \langle w, z \rangle = \sum_{n \in \mathbb{Z}} \langle v_n, w \rangle \overline{\langle v_n, z \rangle}.$$

The following useful proposition can be found in [11, Prop. 3.2.8].

Proposition 2.1 *A sequence of unit vectors in H is a Parseval frame if and only if it is an orthonormal Riesz basis.*

Let $D \subset \mathbb{R}^d$ be a measurable set, with $|D| < \infty$. An *exponential basis* of $L^2(D)$ is a Riesz basis made of functions in the form of $e^{2\pi i x \cdot \lambda}$, where $\lambda \in \mathbb{R}^d$. Exponential bases are important in the applications, because they allow one to represent functions in $L^2(D)$ in a stable manner, with coefficients that are easy to calculate.

The following lemma is easy to prove (see *e.g.*, [14, Prop. 2.1]).

Lemma 2.2 *Let $D_1 \subset D \subset D_2$ be measurable sets of $\mathbb{R}^d$, with $|D_2| < \infty$. Let $\mathcal{V} = \{e^{2\pi i x \cdot \lambda_n}\}_{n \in \mathbb{Z}}$ be Riesz basis of $L^2(D)$ with frame constants $0 < A \leq B < \infty$; then $\mathcal{V}$ is a Riesz sequence on $L^2(D_2)$ and a frame on $L^2(D_1)$ with the same frame constants.*

2.2 The Beurling Density

In [8, 9] A. Beurling characterized sampling sets by means of their density.

For $h > 0$ and $x \in \mathbb{R}^d$, we let $Q_h(x)$ denote the closed cube centered at x with side length h. Let $\Lambda = \{\lambda_j\}_{j \in \mathbb{Z}} \subset \mathbb{R}^d$ be *uniformly discrete*; *i.e.*, we assume that $|\lambda_j - \lambda_k| \geq \delta > 0$ whenever $\lambda_j \neq \lambda_k$. Following [12] we denote by

$$\mathcal{D}^+(\Lambda) = \limsup_{h \to \infty} \frac{\sup_{x \in \mathbb{R}^d} |\Lambda \cap Q_h(x)|}{h^d} \ \text{ and } \ \mathcal{D}^-(\Lambda) = \liminf_{h \to \infty} \frac{\inf_{x \in \mathbb{R}^d} |\Lambda \cap Q_h(x)|}{h^d}$$

the upper and lower density of Λ. If $\mathcal{D}^-(\Lambda) = \mathcal{D}^+(\Lambda)$ we say that Λ has *uniform Beurling density* $\mathcal{D}(\Lambda)$.

Theorem 2.3 is a generalization of theorems of Landau and Beurling [8, 25] in dimension $d \geq 1$. See also [26] and [27, Sect. 2].

Theorem 2.3   *If $E(\Lambda) = \{e^{2\pi i \lambda_j \cdot x}\}_{j \in \mathbb{Z}}$ is a frame in $L^2(D)$, then $\mathcal{D}^-(\Lambda) \geq |D|$. If $E(\Lambda)$ is a Riesz sequence in $L^2(D)$, then $\mathcal{D}^+(\Lambda) \leq |D|$.*

Thus, a necessary condition for $E(\Lambda)$ to be a Riesz basis in $L^2(D)$ is that $\mathcal{D}(\Lambda) = |D|$. In the special case where $\Lambda = \mathbb{Z}^d$, we have the following corollary.

Corollary 2.4   *If $E(\mathbb{Z}^d)$ is a frame in $L^2(D)$, then $|D| \leq 1$; if $E(\mathbb{Z}^d)$ is a Riesz sequence in $L^2(D)$, then $|D| \geq 1$.*

2.3 Shift Invariant Spaces

We let

$$V^2(\varphi) := \overline{\mathrm{span}\{\tau_k \varphi\}_{k \in \mathbb{Z}^d}},$$

where $\varphi \in L^2(\mathbb{R}^d)$ and "bar" denotes the closure in $L^2(\mathbb{R}^d)$. The space $V^2(\varphi)$ is *shift-invariant*, i.e., if $f \in V^2(\varphi)$, then also $\tau_m f \in V^2(\varphi)$ for every $m \in \mathbb{Z}^d$. Shift-invariant spaces of functions appear naturally in signal theory and in other branches of applied sciences. Following [4, 10, 13], we say that the translates $\{\tau_k \varphi\}_{k \in \mathbb{Z}^d}$ form a Riesz basis in $V^2(\varphi)$ if there exist constants $0 < A, B < \infty$ such that, for every finite set of coefficients $\mathbf{d} = \{d_j\} \subset \mathbb{C}$, we have that

$$(2.4) \qquad A\|\mathbf{d}\|_{\ell^2}^2 \leq \left\| \sum_j d_j \tau_j \varphi \right\|_2^2 \leq B\|\mathbf{d}\|_{\ell^2}^2.$$

If (2.4) holds, then $V^2(\varphi) = \{f = \sum_{k \in \mathbb{Z}^d} d_k \tau_k \varphi, \ \mathbf{d} \in \ell^2\}$, and the sequence $\{d_k\}_{k \in \mathbb{Z}^d}$ is uniquely determined by f.

The following theorem is well known; see *e.g.,* [23] or [3, Prop. 1.1].

Theorem 2.5   *The set $\{\tau_m \varphi\}_{m \in \mathbb{Z}^d}$ is a Riesz basis in $V^2(\varphi)$ with frame constants $0 < A, B < \infty$ if and only if*

$$A = \inf_{y \in \mathcal{Q}_d} \sum_{m \in \mathbb{Z}^d} |\widehat{\varphi}(y + m)|^2 \leq \sup_{y \in \mathcal{Q}_d} \sum_{m \in \mathbb{Z}^d} |\widehat{\varphi}(y + m)|^2 = B.$$

3   Proof of Theorem 1.1

Let $\ell_0^2(\mathbb{Z}^d) \subset \ell^2(\mathbb{Z}^d)$ be the set of sequences $\mathbf{a} = (a_n)_{n \in \mathbb{Z}^d}$ such that $a_n = 0$ whenever $|n| \geq N$, with $N = N(\mathbf{a}) \geq 0$. Let $S(\mathbf{a}) = \sum_{n \in \mathbb{Z}^d} a_n e^{2\pi i n \cdot x}$. Recall that $E(\mathbb{Z}^d)$ is a Riesz sequence in $L^2(D)$ if and only if there exists constants $0 < A, B < \infty$ such that

$$(3.1) \qquad A\|\mathbf{a}\|_2^2 \leq \|S(\mathbf{a})\|_{L^2(D)}^2 \leq B\|\mathbf{a}\|_2^2$$

for every $\mathbf{a} \in \ell_0^2(\mathbb{Z}^d)$. We conclude that

$$(3.2) \quad \|S(\mathbf{a})\|_{L^2(D)}^2 = \int_D \Big| \sum_{n \in \mathbb{Z}^d} a_n e^{2\pi i n \cdot x} \Big|^2 dx = \int_D \Big(\sum_{n,m \in \mathbb{Z}^d} a_n \overline{a_m} e^{2\pi i (n-m) \cdot x} \Big) dx$$

$$= \sum_{n,m \in \mathbb{Z}^d} a_n \overline{a_m} \int_D e^{2\pi i (n-m) \cdot x} dx = \sum_{n,m \in \mathbb{Z}^d} a_n \overline{a_m} \widehat{\chi_D}(n-m).$$

Let T_D be the operator, initially defined in $\ell_0^2(\mathbb{Z}^d)$, as

$$(3.3) \qquad\qquad T_D(\mathbf{a})_m = \sum_{n \in \mathbb{Z}^d} a_n \widehat{\chi_D}(n-m), \ m \in \mathbb{Z}^d.$$

The calculation above shows that $\|S(\mathbf{a})\|_{L^2(D)}^2 = \langle T_D(\mathbf{a}), \mathbf{a} \rangle_2$, where $\langle \cdot, \cdot \rangle_2$ denotes the inner product in $\ell^2(\mathbb{Z}^d)$. We can easily verify that $T_D(\mathbf{a})$ is self-adjoint and, in view of (3.2), that $\langle T_D(\mathbf{a}), \mathbf{a} \rangle_2 \geq 0$ for every $\mathbf{a} \in \ell_0^2(\mathbb{Z}^d)$; thus, (3.1) holds if and only if

$$(3.4) \qquad\qquad A\|\mathbf{a}\|_2^2 \leq \langle T_D(\mathbf{a}), \mathbf{a} \rangle_2 \leq B\|\mathbf{a}\|_2^2, \quad \mathbf{a} \in \ell_0^2(\mathbb{Z}^d).$$

To prove (3.4) we need the following lemma.

Lemma 3.1   *Assume that $\|T_D\|_{\ell^2 \to \ell^2} = \sup_{\|\mathbf{a}\|_2 = 1} \|T_D(\mathbf{a})\|_2 < \infty$. The inequality below holds for every $\mathbf{a} \in \ell_0^2(\mathbb{Z}^d)$ such that $\|\mathbf{a}\|_2 = 1$:*

$$(3.5) \qquad\qquad \frac{\|T_D(\mathbf{a})\|_2^2}{\|T_D\|_{\ell^2 \to \ell^2}} \leq \langle T_D(\mathbf{a}), \mathbf{a} \rangle_2 \leq \|T_D\|_{\ell^2 \to \ell^2}.$$

Proof of Theorem 1.1   Let $\Phi(x)$ be as in (1.1). We show that if there exist constants $0 < A' \leq B' < \infty$ such that $A' \leq \Phi(x) \leq B'$ a.e. in $\mathcal{Q}_d$, then $E(\mathbb{Z}^d)$ is a Riesz sequence in $L^2(D)$. Since $\Phi(x) = \sum_{m \in \mathbb{Z}^d} \chi_D(x+m) = \sum_{m \in \mathbb{Z}^d} |\chi_D(x+m)|^2$, by Theorem 2.5 the set $\{\tau_m \widehat{\chi_D}\}_{m \in \mathbb{Z}^d}$ is a Riesz basis of $V^2(\widehat{\chi_D})$ with frame constants $0 < A', B' < \infty$. In view of (2.4) and (3.3), the inequality

$$A'\|\mathbf{a}\|_2^2 \leq \|T_D \mathbf{a}\|_2^2 \leq B'\|\mathbf{a}\|_2^2$$

holds for every $\mathbf{a} \in \ell_0^2(\mathbb{Z}^d)$. By Lemma 3.1, we have (3.4), as required.

   If $E(Z^d)$ is a Riesz sequence on D, we argue as in the proof of [6, Theorem 3.1]. Using Plancherel's identity and the Poisson summation formula, from (3.2) we obtain

$$(3.6) \quad \|S(\mathbf{a})\|_{L^2(D)}^2 = \sum_m \Big| \sum_{n \in \mathbb{Z}^d} a_n \widehat{\chi_D}(n-m) \Big|^2$$

$$= \int_{\mathcal{Q}_d} \Big| \sum_{m \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} a_n \widehat{\chi_D}(n-m) e^{2\pi i x \cdot m} \Big|^2 dx$$

$$= \int_{\mathcal{Q}_d} \Big| \Big(\sum_{n \in \mathbb{Z}^d} a_n e^{2\pi i x \cdot n} \Big) \sum_{m \in \mathbb{Z}^d} \widehat{\chi_D}(n-m) e^{2\pi i x \cdot (m-n)} \Big|^2 dx$$

$$= \int_{\mathcal{Q}_d} \Big| \sum_{n \in \mathbb{Z}^d} a_n e^{2\pi i n \cdot x} \Big|^2 |\Phi(x)|^2 dx.$$

By assumption, the integral in (3.6) is finite and so $\Phi(x) < \infty$ a.e.. To show that $\Phi(x) > 0$ a.e., we argue by contradiction. Suppose that there exists $\Omega \subset D$, with $|\Omega| > 0$, where $\Phi(x) \equiv 0$. We can assume that $\Omega \subset \mathcal{Q}_d$. Since $E(\mathbb{Z}^d)$ is a Riesz

basis in $L^2(\mathcal{Q}_d)$, we can write $\chi_{\mathcal{Q}_d - \Omega}(x) = \sum_{n \in \mathbb{Z}^d} b_n e^{2\pi i n \cdot x}$, with $\vec{b} \in \ell^2(\mathbb{Z}^d)$. Thus, $\int_{\mathcal{Q}_d} |\Phi(x)|^2 |\sum_{n \in \mathbb{Z}^d} b_n e^{2\pi i n \cdot x}|^2 dx = 0$, which, together with (3.6), contradicts (3.1). ∎

Proof of Lemma 3.1 The right-hand inequality in (3.5) is [18, Theorem 13.8], so we only need to prove the left-hand inequality. Let $\alpha = \sup_{\|\mathbf{a}\|_2 = 1} |\langle T_D(\mathbf{a}), \mathbf{a} \rangle|$ and $U = \alpha I - T_D$, where I is the identity operator in $\ell^2(\mathbb{Z}^d)$. It is easy to verify that U is positive and that

$$(3.7) \qquad\qquad T_D U T_D + U T_D U = \alpha^2 T_D - \alpha T_D^2.$$

The operators $T_D U T_D$ and $U T_D U$ are positive too; indeed, for every $\mathbf{a} \in \ell^2$, we have that $\langle T_D U T_D \mathbf{a}, \mathbf{a} \rangle_2 = \langle U(T_D \mathbf{a}), T_D \mathbf{a} \rangle_2 \geq 0$ and $\langle U T_D U \mathbf{a}, \mathbf{a} \rangle_2 = \langle T_D(U \mathbf{a}), U \mathbf{a} \rangle_2 \geq 0$, because T_D and U are both positive. By (3.7), the operator $\alpha T_D - T_D^2$ is also positive. For every $\mathbf{a} \in \ell^2$ with $\|\mathbf{a}\|_2 = 1$, we have that

$$\left\langle (\alpha T_D - T_D^2) \mathbf{a}, \mathbf{a} \right\rangle_2 = \alpha \langle T_D \mathbf{a}, \mathbf{a} \rangle_2 - \langle T_D^2 \mathbf{a}, \mathbf{a} \rangle_2 = \alpha \langle T_D \mathbf{a}, \mathbf{a} \rangle_2 - \| T_D \mathbf{a} \|_2^2 \geq 0,$$

and the left inequality in (3.5) is proved. ∎

Remark From the identity (3.6), it follows that the constants A and B in (3.1) are the minimum and maximum of $\Phi(x)$ on the unit square $\mathcal{Q}_d$. Thus, A and B are integers.

When $|D| = 1$, Theorem 1.3 shows that $E(\mathbb{Z}^d)$ is a Riesz sequence if and only the integer translates of D tile $\mathbb{R}^d$, and so $A = B = 1$. In general, if $k \leq |D| < k + 1$ for some positive integer k, we can easily verify that the integer translates of D cover $\mathbb{R}^d$ k times but not $k + 1$ times. Thus, $A \leq |D|$ and $B \geq |D|$.

4 Proof of Theorem 1.2

Let $D \subset \mathbb{R}^d$ be measurable, with $|D| \leq 1$. By Lemma 2.2, the theorem is trivial when $D \subset \mathcal{Q}_d$, so we assume that $D - \mathcal{Q}_d$ has positive measure. Let $D_1, \ldots, D_N, \ldots$ be a (possibly infinite) family of disjoint sets of positive measure such that $D - \mathcal{Q}_d = \bigcup_j D_j$. We can choose the D_j in such way that, for certain vectors $v_1, \ldots, v_N, \cdots \in \mathbb{Z}^d$, we have that $D_j + v_j \subset \mathcal{Q}_d$. Let $D_0 = D \cap \mathcal{Q}_d$ and $v_0 = 0$. We prove the following lemma.

Lemma 4.1 $E(\mathbb{Z}^d)$ *is frame for $L^2(D)$ if and only, for every $v \in \mathbb{Z}^d$ and every $k \neq j$,*

$$(4.1) \qquad\qquad |(v + D_j) \cap D_k| = 0.$$

It is easy to verify that (4.1) is equivalent to (1.2), and so Theorem 1.2 is equivalent to Lemma 4.1.

Proof Assume that $|(D_1 + v) \cap D_0| > 0$ for some $v \in \mathbb{Z}^d$ (the proof is similar in the other cases). We can assume without loss of generality that $D_1 + v \subset \mathcal{Q}_d$ (see Figure 1); otherwise, we let $D_1 = D_1' \cup D_1''$, with $D_1' + v \subset \mathcal{Q}_d$, and we replace D_1 with D_1'. We show that $E(\mathbb{Z}^d)$ is not a frame on $L^2(D)$.
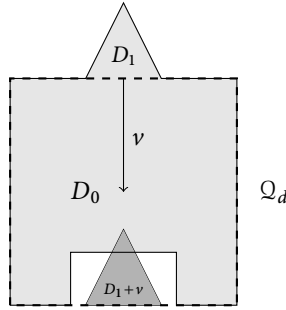
Figure 1:

Every $f \in L^2(D)$ can be written as $f = f_0 + f_1$, where $f_0 = f\chi_{D-D_1}$ and $f_1 = f\chi_{D_1}$. Recall that $\tau_w g(x) = g(x+w)$. It follows that

$$
(4.2) \quad |\langle e^{2\pi i n \cdot x}, f\rangle_{L^2(D)}|^2 = |\langle e^{2\pi i n \cdot x}, f_0\rangle_{L^2(D-D_1)} + \langle e^{2\pi i n \cdot x}, f_1\rangle_{L^2(D_1)}|^2
$$
$$
= |\langle e^{2\pi i n \cdot x}, f_0\rangle_{L^2(D-D_1)} + \langle e^{2\pi i n \cdot (x-v)}, \tau_{-v}f_1\rangle_{L^2(D_1+v))}|^2
$$
$$
= |\langle e^{2\pi i n \cdot x}, f_0\rangle_{L^2(\mathcal{Q}_d)} + \langle e^{2\pi i n \cdot x}, \tau_{-v}f_1\rangle_{L^2(\mathcal{Q}_d)}|^2
$$
$$
= |\langle e^{2\pi i n \cdot x}, f_0\rangle_{L^2(\mathcal{Q}_d)}|^2 + |\langle e^{2\pi i n \cdot x}, \tau_{-v}f_1\rangle_{(\mathcal{Q}_d)}|^2
$$
$$
+ 2\operatorname{Re}\left(\langle e^{2\pi i n \cdot x}, f_0\rangle_{L^2(\mathcal{Q}_d)}\overline{\langle e^{2\pi i n \cdot x}, \tau_{-v}f_1\rangle_{L^2(\mathcal{Q}_d)}}\right).
$$

We have used the change of variables $x \to x - v$ in the second inner product in (4.2) and the fact that $e^{2\pi i n \cdot v} = 1$. Since $E(\mathbb{Z}^d)$ is an orthonormal basis in $\mathcal{Q}_d$, the identities (2.3) and the calculation above yield

$$
(4.3) \quad \sum_{n \in \mathbb{Z}^d} |\langle e^{2\pi i n \cdot x}, f\rangle_{L^2(D)}|^2 = \|f_0\|^2_{L^2(\mathcal{Q}_d)} + \|\tau_{-v}f_1\|^2_{L^2(\mathcal{Q}_d)} + 2\operatorname{Re}\langle f_0, \tau_{-v}f_1\rangle_{L^2(\mathcal{Q}_d)}.
$$

If we let $B = (D - D_1) \cap (D_1 + v)$, we can choose $f = f_1 + f_0$, with $f_1(x) = \chi_B(x+v)$ and $f_0(x) = -\chi_B(x)$; from (4.3) it readily follows that $\sum_{n \in \mathbb{Z}^d} |\langle e^{2\pi i n \cdot x}, f\rangle_{L^2(D)}|^2 = 0$, which contradicts (2.1).

We now assume that $|(w + D_j) \cap D_k| = 0$ for every $k \neq j$ and every $w \in \mathbb{Z}^d$; we prove that $E(\mathbb{Z}^d)$ is a tight frame in $L^2(D)$. We assume for simplicity that $D_1 + v \subset \mathcal{Q}_d$ for some $v \in \mathbb{Z}^d$. Let $f = f_0 + f_1$ be as in the first part of the proof. By assumption, $|(D_1 + v) \cap (D - D_1)| = 0$, and so (4.3) yields

$$
\sum_{n \in \mathbb{Z}^d} |\langle e^{2\pi i n \cdot x}, f\rangle_{L^2(D)}|^2 = \|f_0\|^2_{L^2(\mathcal{Q}_d)} + \|\tau_{-v}f_1\|^2_{L^2(\mathcal{Q}_d)}
$$
$$
= \|f\chi_{D-D_1}\|^2_{L^2(\mathcal{Q}_d)} + \|f\chi_{D_1+v}\|^2_{L^2(\mathcal{Q}_d)} = \|f\|_{L^2(D)}.
$$

Thus, $E(\mathbb{Z}^d)$ is a tight frame in $L^2(D)$, as required.   ■

The proof of Theorem 1.2 shows that if (4.1) is not satisfied, we can produce a function $f \in L^2(D)$ for which $\langle f, e^{2\pi i x \cdot n}\rangle_{L^2(D)} = 0$ for every $n \in \mathbb{Z}^d$, and so $E(\mathbb{Z}^d)$ is not complete. This observation proves the following corollary.

Corollary 4.2 $E(\mathbb{Z}^d)$ *is complete in* $L^2(D)$ *if and only if the integer translates of* D *intersect on sets of measure* 0.

Proof of Theorem 1.3   We have proved that (i) $\Leftrightarrow$ (ii); we show that (ii) $\Leftrightarrow$ (iii). By Corollary 4.2, $E(\mathbb{Z}^d)$ is complete in $L^2(D)$ if and only if the integer translates of D overlap only on sets of measure zero. Thus, (iii) $\Rightarrow$ (ii). Let us prove that (ii) $\Rightarrow$ (iii); let $D_0, D_1, \ldots, D_N, \ldots$ and $v_0, v_1, \ldots, v_N, \ldots$ be as in the proof of Lemma 4.1. Since $|(D_j + v_j) \cap (D_k + v_k)| = 0$ when $k \neq j$, and

$$(4.4) \qquad 1 = |\mathcal{Q}_d| = |D| = |D_0| + \sum_j |D_j + v_j|$$

necessarily $\bigcup_j (D_j + v_j) = \mathcal{Q}_d$ and the integer translates of D tile $\mathbb{R}^d$.

By Fuglede's theorem, (iii) $\Leftrightarrow$ (iv). Clearly (iv) $\Rightarrow$ (v); to finish the proof of the theorem we show that (v) $\Rightarrow$ (iii). By Theorem 1.1, the integer translates of D cover $\mathbb{R}^d$; thus, $\bigcup_j (D_j + v_j) = \mathcal{Q}_d$ and from (4.4) it follows that the $D_j + v_j$'s can only intersect on sets of measure zero. Thus, the integer translates of D can only intersect on sets of measure zero, and (iii) is proved.   ∎

5   The Broken Interval

In this section we solve Problem 1. We let $J = [0, \alpha) \cup [\alpha + r, L + r) \subset \mathbb{R}$, with $0 < \alpha < L$ and $r > 0$.

By Lemma 4.1 and Theorem 1.1, $E(\mathbb{Z})$ is a frame on $L^2(J)$ if and only if the integer translates of J do not overlap in $[0, 1]$, and it is a Riesz sequence if and only if the integer translates of J cover $\mathbb{R}$.

Let $[r]$ be the integer part of r, *i.e.*, the largest integer $n \leq r$; let $\{r\} = r - [r]$ be the fractional part of r.

Theorem 5.1   (i) $E(\mathbb{Z})$ *is a frame on* J *if and only if* $L + \{r\} \leq 1$.

(ii) $E(\mathbb{Z})$ *is a Riesz sequence on* J *if and only if one of the following is true:*

  (a) $\alpha \geq 1$ *or* $L - \alpha \geq 1i$;
  (b) $\{r\} = 0$ *and* $L \geq 1$;
  (c) $1 \leq L < 2$ *and* $L + \{r\} \geq 2$.

To prove (ii) we will need the following lemma.

Lemma 5.2   *The integer translates of* J *cover* $\mathbb{R}$ *if and only if the integer translates of* $J' = [0, \alpha) \cup [\alpha + \{r\}, L + \{r\})$ *cover* $\mathbb{R}$.

Proof   If the integer translates of J cover $\mathbb{R}$, then for $x \in \mathbb{R}$, there is an integer m such that either $x \in (m, \alpha + m)$ or $x \in (\alpha + r + m, L + r + m)$. If $x \in (m, \alpha + m)$, then clearly $x \in J' + m$ as well. If $x \in (\alpha + r + m, L + r + m)$, then $x \in (\alpha + \{r\} + [r] + m, L + \{r\} + [r] + m)$, *i.e.*, x is in the translation of J' by $[r] + m$. The converse is similar.   ∎

Proof of Theorem 5.1   By Theorem 1.2 and Lemma 5.2, $E(\mathbb{Z})$ is a frame on J if and only if the integer translates of $[0, \alpha) \cup [\alpha + \{r\}, L + \{r\})$ do not intersect in $[0, 1]$,

that is, if and only if $(0, \alpha) \cap (\alpha + \{r\} - 1, L + \{r\} - 1) = \varnothing$. This is equivalent to having either $\alpha \le \alpha + \{r\} - 1$, which is impossible, or $L + \{r\} \le 1$. That proves part (i).

Let us prove part (ii). By Lemma 5.2 we can assume that $r = \{r\}$, *i.e.,* that $0 \le r < 1$.

By Theorem 1.1, $E(\mathbb{Z})$ is a Riesz sequence on J if and only if the integer translates of J cover $\mathbb{R}$. If one of the connected components $[0, \alpha)$ or $[\alpha + r, L + r)$ covers $\mathbb{R}$ by integer translations, we have that either $\alpha \ge 1$ or $L - \alpha \ge 1$, and (a) is proved.

If neither component covers $\mathbb{R}$ by integer translations, *i.e.,* if $\alpha < 1$ and $L - \alpha < 1$ both hold, we can consider 2 sub-cases:

- If $r = 0$, we have that $J = [0, L)$, and the integer translates of J cover $\mathbb{R}$ if and only if $L \ge 1$; this proves (b).
- Suppose next that $r > 0$. The integer translates of J cover $\mathbb{R}$ if and only if the "gap" $(\alpha, \alpha + r)$ is covered by integer translates of J. This is possible if and only if $(1, \alpha + 1) \cup (\alpha + r - 1, L + r - 1) \supset (\alpha, \alpha + r)$ (see Figure 2).
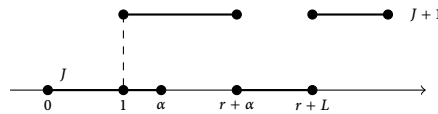

Figure 2:

We have $(1, \alpha + 1) \cap (\alpha, \alpha + r) = (1, \alpha + r)$, because $\alpha, r < 1$. Thus, J covers $\mathbb{R}$ if and only if $(\alpha + r - 1, L + r - 1) \cap (\alpha, \alpha + r) \supset (\alpha, 1)$. This is equivalent to the conditions

$$\begin{cases} r - 1 \le 0 \\ L + r - 1 \ge 1 \\ \alpha + r \ge 1 \end{cases} \iff \begin{cases} r \le 1 \\ L + r \ge 2 \\ \alpha + r \ge 1 \end{cases} .$$

Since $\alpha < 1$ and $L - \alpha < 1$ by assumption, and recalling that $r < 1$, we can see at once that the condition $L + r \ge 2$ implies $\alpha + r \ge 1$. Indeed, if $\alpha + r < 1$, then

$$L + r = L - \alpha + \alpha + r < L - \alpha + 1 < 2.$$

Thus, the integer translates of J covers $\mathbb{R}$ if and only if $L + r \ge 2$, and we have (c). The theorem is proved. ∎

6 The Rotated Square

Let $Q_h = Q_h(0) = \left[-\frac{h}{2}, \frac{h}{2}\right] \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ be the square in $\mathbb{R}^2$ centered at the origin with sides of length h. Let $A_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$ be the matrix of a rotation by an angle θ, and let $Q_{h,\theta} = A_\theta Q_h(0)$ be the square obtained from the rotation of $Q_h(0)$. The following theorem offers a complete solution to Problem 2.

Theorem 6.1

(i) $E(\mathbb{Z}^2)$ *is a Riesz sequence on* $L^2(Q_{h,\theta})$ *if and only if* $h \ge 1 - \sin(2\theta)$.

(ii) $E(\mathbb{Z}^2)$ *is a frame on* $L^2(Q_{h,\theta})$ *if and only if* $h \le \frac{1}{\sin\theta+\cos\theta}$.

Proof   We first prove part (i). Let $P_1 = (\frac{1}{2}, \frac{1}{2})$, $P_2 = (-\frac{1}{2}, \frac{1}{2})$, $P_3 = (-\frac{1}{2}, -\frac{1}{2})$, $P_4 = (\frac{1}{2}, -\frac{1}{2})$ be the vertices of $\mathcal{Q}_2$. We first find conditions on h and θ for which the points $P_1, \ldots, P_4$ lie on the sides of $Q_{h,\theta}$.
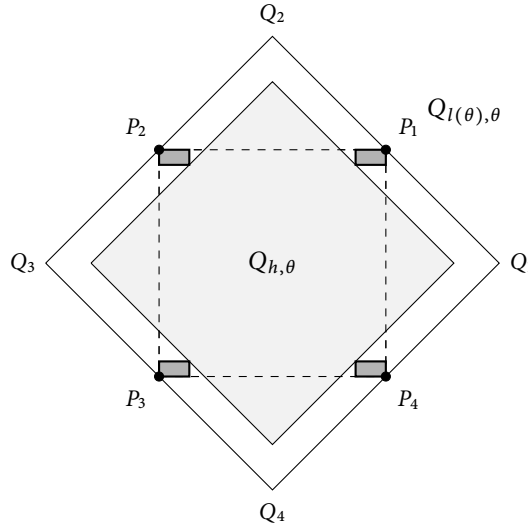

Figure 3:

Let $\ell_1 : y - \frac{1}{2} = \tan(\theta)(x - \frac{1}{2})$, $\ell_2 : y - \frac{1}{2} = -\frac{1}{\tan(\theta)}(x + \frac{1}{2})$, and $\ell_3 : y + \frac{1}{2} = \tan(\theta)(x + \frac{1}{2})$ be the equations of the sides of $Q_{h,\theta}$ that contain the points P_1, P_2 and P_3, respectively. It is easy to verify that ℓ_2 intersects ℓ_1 and ℓ_3 at the points

$$Q_2 = \left(-\frac{1}{2}\cos(2\theta), \frac{1}{2}\big(1-\sin(2\theta)\big)\right) \quad \text{and} \quad Q_3 = \left(-\frac{1}{2}\big(1-\sin(2\theta)\big), -\frac{1}{2}\cos(2\theta)\right),$$

and that the length of the segment that join Q_2 and Q_3 equals to $l(\theta) = 1 - \sin(2\theta)$. Thus, when $h \ge l(\theta)$, the set $E(\mathbb{Z}^2)$ is a Riesz sequence on $L^2(Q_{h,\theta})$.

We show that when $h < l(\theta)$, the integer translates of $Q_{h,\theta}$ do not cover the plane anymore. Indeed, if $h < l(\theta)$, the four vertices of $\mathcal{Q}_2$ are outside the square $Q_{l(\theta),\theta}$ and have positive distance from the boundary of $Q_{h,\theta}$. We can find a small rectangle R with sides parallel to the sides of $\mathcal{Q}_2$ for which $R + P_j \subset \mathcal{Q}_2 - Q_{h,\theta}$ for every j (see Figure 3). The integer translates of $Q_{h,\theta}$ cannot cover the rectangles $R + P_j$, and so the condition of Theorem 1.1 is not verified.

Let us prove part (ii). The vertices $Q_1, \ldots, Q_4$ of $Q_{h,\theta}$ lie on the sides of $\mathcal{Q}_2$ if and only if there exists $0 < t \le 1$ for which $Q_1 = (t - \frac{1}{2}, \frac{1}{2})$, $Q_2 = (-\frac{1}{2}, t - \frac{1}{2})$, $Q_3 = (\frac{1}{2} - t, -\frac{1}{2})$, and $Q_4 = (\frac{1}{2}, \frac{1}{2} - t)$. If we let $\tan(\theta)$ be the slope of the line that joins Q_3
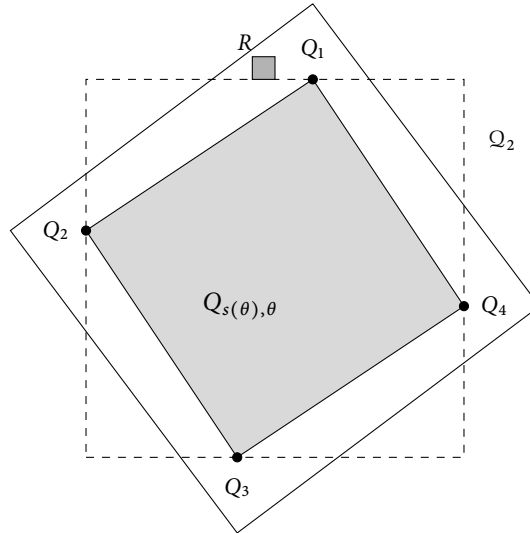
Figure 4:

and Q_4, we can see at once that $\tan(\theta) = \frac{1-t}{t}$; thus, $t = \frac{1}{1+\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)+\cos(\theta)}$. The length of the segment $[Q_3 Q_4]$ is then:

$$s(\theta) = \sqrt{t^2 + (1-t)^2} = \frac{1}{\sin(\theta) + \cos(\theta)}$$

and $E(\mathbb{Z}^2)$ is a frame on $Q_{h,\theta}$ whenever $h \leq s(\theta)$.

Let us show that $E(\mathbb{Z}^2)$ is not a frame on $Q_{h,\theta}$ whenever $h > s(\theta)$. Indeed, if $h > s(\theta)$, the set $Q_{h,\theta} - \mathcal{Q}_2$ has positive measure; we can find a small rectangle R with sides parallel to the sides of $\mathcal{Q}_2$ and vectors $n_1, \ldots, n_4 \in \mathbb{Z}^2$ such that $R + n_j \subset Q_{h,\theta} - \mathcal{Q}_2$ (see Figure 3). Thus, the integers translates of $Q_{h,\theta}$ overlap on the rectangles $R + n_j$ and by Theorem 1.2, $E(\mathbb{Z}^2)$ is not a frame on $L^2(Q_{h,\theta})$. ∎

7 The Translated Parallelepiped

In this section we solve Problem 3.

Let $P \subset \mathbb{R}^d$ be a parallelepiped with sides parallel to vectors $v_1, \ldots, v_d$. We let $A = \{a_{i,j}\}_{1 \leq i, j \leq d}$ be the matrix whose columns are $v_1, \ldots, v_d$; we let $A^{-1} = \{b_{i,j}\}_{1 \leq i, j \leq d}$. We prove the following theorem.

Theorem 7.1   (i)   *The set $E(\mathbb{Z}^d)$ is a frame on $L^2(P)$ if and only if*

$$\det(A) \leq 1 \quad and \quad \max_{\substack{1 \leq i,k,j \leq d \\ j \neq k}} |a_{i,j}| + |a_{i,k}| \leq 1.$$

(ii) *The set $E(\mathbb{Z}^d)$ is a Riesz sequence on $L^2(P)$ if and only if*

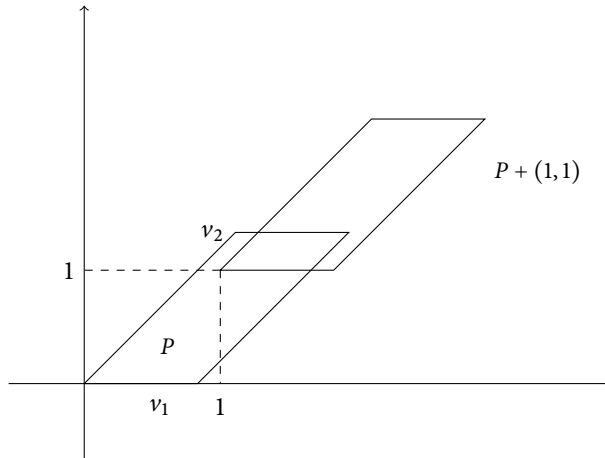$$\det(A) \geq 1 \quad and \quad \max_{1 \leq i,j \leq d} |b_{i,j}| \leq 1.$$

Figure 5:

Proof Observe that if $|P| = \det(A) > 1$, the set $E(\mathbb{Z}^d)$ cannot be a frame on $L^2(P)$, so in part (i) we assume $\det(A) \leq 1$. Similarly, for part (ii) we assume that $\det(A) \geq 1$.

We prove part (i) by induction on the dimension d.

By Theorem 1.2, the set $E(\mathbb{Z}^d)$ is a frame on $L^2(P)$ if and only if the integer translates of P overlap only on sets of zero measure. In dimension $d = 2$, we let $v_1 = (a_{1,1}, a_{2,1})$ and $v_2 = (a_{1,2}, a_{2,2})$ be the vectors that are parallel to the sides of P. When the components of v_1 and v_2 are non-negative, we can easily verify that P overlaps with $P + (1,1)$ if and only if the sum of the projections of v_1 and v_2 on the x_1 and x_2 axes has measure ≥ 1 (see Figure 4). Thus, P overlaps with $P + (1,1)$ if and only if $a_{1,1} + a_{1,2} \geq 1$ and $a_{2,1} + a_{2,2} \geq 1$. These conditions imply that no pair of integer translates of P intersect. For general v_1 and v_2 we can similarly verify that the integer translates of P do not intersect if and only if $|a_{1,1}| + |a_{1,2}| \geq 1$ and $|a_{2,1}| + |a_{2,2}| \geq 1$.

We now assume that part (i) of the theorem is valid in dimension $d \geq 2$. We prove that is is valid also in dimension $d + 1$.

Let P be a parallelepiped in $\mathbb{R}^{d+1}$. The integer translates of P overlap on sets of positive measure in $\mathbb{R}^{d+1}$ if and only if the integer translates of the faces of P overlap on sets of positive measure in $\mathbb{R}^d$. Let P_h be the face of P spanned by the vectors $v_1, \ldots, v_{h-1}, v_{h+1}, \ldots, v_{d+1}$.

Let $e_1 = (1, 0, \ldots, 0), \ldots, e_{d+1} = (0, \ldots, 0, 1)$ be the standard orthonormal basis in $\mathbb{R}^{d+1}$ and let H_j be the orthogonal complement of e_j. Clearly, the integer translates of P_h overlap if and only if the integer translates of the orthogonal projections of P_h on the H_j's overlap.

The projection of P_h on H_k is a parallelepiped in $\mathbb{R}^d$ spanned by the vectors $w_1, \ldots, w_{h-1}, w_{h+1}, \ldots, w_{d+1}$, where w_j is the projection of v_j on H_k, *i.e.*, it is the vector v_j with the k-th component removed. By assumptions,

$$\max_{\substack{1 \le i, k, j \le d+1 \\ i \ne k \\ k \ne j \ne h}} \{|a_{i,j}| + |a_{i,k}|\} \le 1.$$

This inequality is valid for every face of P and for every projection, and so we have that

$$\max_{\substack{1 \le i, j, k \le d+1 \\ k \ne j}} \{|a_{i,k}| + |a_{i,j}|\} \le 1,$$

as required.

We now prove part (ii). By Theorem 1.1, the integer translates of P must cover $\mathbb{R}^d$. Since P is the image of the unit cube $[0,1]^d$ via the linear transformation $A(x) = Ax$, we can write $P = A([0,1]^d)$. Thus, $E(\mathbb{Z}^d)$ is a Riesz sequence in $L^2(P)$ if and only if $\bigcup_{n \in \mathbb{Z}^d} (A([0,1]^d) + n) = \mathbb{R}^d$, or

$$A^{-1}\Big(\bigcup_{n \in \mathbb{Z}^d} (A([0,1]^d) + n)\Big) = \bigcup_{n \in \mathbb{Z}^d} ([0,1]^d + A^{-1}n) = \mathbb{R}^d.$$

The translates of the unit cube $[0,1]^d$ cover $\mathbb{R}^d$ if and only if the components of the vectors $A^{-1}e_k$ are all ≤ 1, *i.e.*, if and only if $\max_{1 \le i, j \le d} |b_{i,j}| \le 1$. ∎

7.1 The Shortest Vector Problem

Let $A: \mathbb{R}^d \to \mathbb{R}^d$ be linear and invertible; consider the parallelepiped $P = A(Q)$, where $Q = [0,1]^d$. The sides of P are parallel to the columns of the matrix that represents A.

By Corollary 4.2, the set $E(\mathbb{Z}^d)$ is complete in $L^2(A(Q))$ if and only if the integer translates of $A(Q)$ do not intersect. The integer translates of $A(Q)$ intersect if and only if there are $x, y \in Q$ such that $Ax = Ay + n$, for some nonzero $n \in \mathbb{Z}^d$. We can also say that the translates of $A(Q_d)$ intersect if and only if there exist $x, y \in Q$ and $n \in \mathbb{Z}^d$ such that $A^{-1}n = x - y$, *i.e.*, if and only if there exists $n \in \mathbb{Z}^d$ such that $A^{-1}n \in D = \{w \mid \|w\|_\infty < 1\}$.

These considerations show that Problem 3 is related to the so-called *shortest vector problems* (SVP). Given a lattice $\mathcal{L}$ and a norm $\|\cdot\|$ on $\mathbb{R}^d$, find the minimum length $\lambda = \min_{0 \ne v \in \mathcal{L}} \|v\|$ of a nonzero lattice point. The SVP is known to be NP-hard (see [1]).

The conjectured intractability of the SVP and of other optimization problems on lattices is central in the construction of secure lattice-based cryptosystems. For more information on this problem, see, *e.g.*, [2,20] and the references cited in those papers.

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