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THE PHRAGMÉN-LINDELÖF PRINCIPLE

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The theorem below is one version of the Phragmén-Lindelöf principle [4], which extends the maximum modulus theorem. The theorem has many applications, including the proof of a better-known but less general result [3], which is sometimes attributed to Phragmén and Lindelöf.

The usual proofs of the principle have a function $\omega(z)$ regular in D and use analytic continuation [2, p. 394], or a branch of log $\omega(z)$ [1, p. 138], in order to define $[\omega(z)]^\eta$. So D is required to be simply connected and $\omega(z)$ is required to be non-vanishing. This note removes these restrictions and gives a more elementary proof.

THEOREM. *Let $f(z)$ be regular in the bounded domain D and let M be a constant. Suppose there is a regular function $\omega(z)\neq 0$ in D with $|\omega(z)|\leq 1$ such that the boundary of D is the union of two sets A and B, and*
(a) *whenever z is sufficiently close to A, $|f(z)|\leq M$, and*
(b) *for every $\eta>0$, whenever z is sufficiently close to B, $|f(z)|\,|\omega(z)|^\eta\leq M$.*
Then, for all z in D, $|f(z)|\leq M$.

Proof. For each positive integer n, $F_n(z)=[f(z)]^n\omega(z)$ is regular in D. If z is sufficiently close to A, $|F_n(z)|\leq M^n$, by (a). If z is sufficiently close to B, $|F_n(z)|\leq M^n$, by (b), with $\eta=1/n$. The boundary of D is $A\cup B$, so by the maximum modulus theorem, $|F_n(z)|\leq M^n$, for all z in D. That is, $|f(z)|\,|\omega(z)|^{1/n}\leq M$, for all n. Hence, $|f(z)|\leq M$, for all z such that $\omega(z)\neq 0$. Since $|f(z)|$ is continuous and the zeros of $\omega(z)$ are isolated, $|f(z)|\leq M$, for all z in D.

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