AN EXPANSION OF THE LAGUERRE POLYNOMIALS, $L_n^{\alpha}(z)$

Max Wyman

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SECTION 1

Introduction

By a result due to Tricomi[1], it is known that the Laguerre polynomials have an expansion in terms of the Bessel functions, $J_{\nu}(z)$, of the form

$$(1.1)\quad L_n^{\alpha}(z) = \frac{e^{\frac{1}{2}x}\,\Gamma(\alpha+n+1)}{n!\,(Nz)^{\alpha/2}} \sum_{m=0}^{\infty} A_m (z/4N)^{\frac{1}{2}m} J_{\alpha+m}(2(Nz)^{1/2}) ,$$

where the coefficients A_m are determined by

$$(1.2)\quad (m+2)\,A_{m+2} = (m+\alpha+1)\,A_m - 2N\,A_{m-1}\,,\quad A_0 = 1,\ A_1 = 0,$$

$$A_2 = \frac{1}{2}(\alpha+1) ,$$

and

$$(1.3)\quad N = n + \frac{1}{2}(\alpha+1) .$$

When the complex variable z is bounded, the series converges uniformly. It is also known[2] that (1.1) behaves like an asymptotic expansion, when $N \to \infty$, provided that $z = O(N^{\lambda})$, $\lambda < 1/3$.

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1. Higher Transcendental Functions, Bateman Manuscript Project, McGraw Hill, 1953, Vol. 2, page 199.
2. Loc. Cit.

Recently[3], Erdélyi has obtained two asymptotic formulae which completely determine the asymptotic behavior of $L_n^{\alpha}(4Nz)$ when z is real. One of the interesting features of one of these formulae is that it retains its asymptotic validity as $z \to 0$ in an unrestricted manner.

If we replace z by $4Nz$ in the Tricomi result, the range of asymptotic validity reduces to $z = 0(N^{-\beta})$, $\beta > 2/3$. Thus the range of approach of $z \to 0$ becomes restricted. One of the reasons for the restriction is that the coefficients A_m are all polynomials in N whose degree increases after every third term.

The object of the present paper is to establish an expansion of Tricomi type that incorporates the interesting feature of the Erdélyi result and at the same time extends the result to hold for complex z confined to a neighborhood of $z = 0$.

SECTION 2

Derivation of the Expansion

We start with the well-known integral representation

$$(2.1)\quad L_n^{\alpha}(z) = \frac{\Gamma(n+\alpha+1)}{n!\; 2\pi i} \int_{-\infty}^{0+} t^{-(n+\alpha+1)} (t-z)^n e^t \, dt \; ,$$

where the path of integration is the usual loop starting at $-\infty$, enclosing the origin, and ending at $-\infty$.

If we replace z by $4Nz$ and t by $2N(t+z)$, we obtain

3. Asymptotic Solutions of Differential Equations with Transition Points or Singularities, Journal of Math. Physics, Vol. 1, No. 1 pp. 12-20, 1960.

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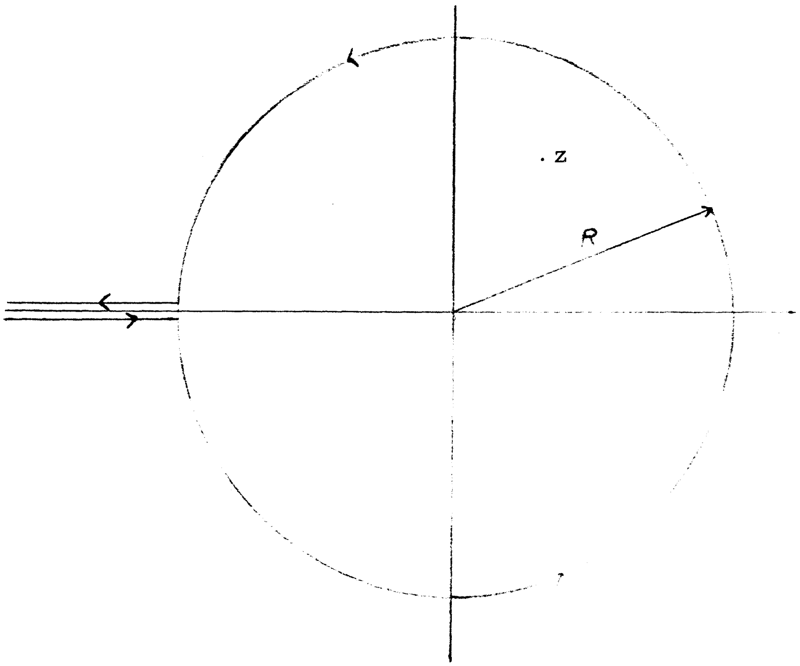
$$(2.2)\quad L_n^{\alpha}(4Nz) = \frac{\Gamma(n+\alpha+1)\, e^{2Nz}}{n!\,(2N)^{\alpha}\, 2\pi i} \int_C \frac{(t-z)^n \exp(2Nt)\, dt}{(t+z)^{n+\alpha+1}}$$

The contour C is shown below



t-plane

We consider the integral, I, defined by

$$(2.3)\quad I = (2\pi i)^{-1} \int_C \frac{(t-z)^n \exp(2Nt)\, dt}{(t+z)^{n+\alpha+1}}$$

$$= (2\pi i)^{-1} \int_C (t^2-z^2)^{-\frac{1}{2}(\alpha+1)} \exp[2N(t+\tfrac{1}{2}\log\{(t-z)/(t+z)\})]\, dt ,$$

where

$$(2.4)\quad N = n + \tfrac{1}{2}(\alpha+1) .$$

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For convenience we introduce G by

$$(2.5)\quad G = t + \frac{1}{2}\log\{(1-zt^{-1})/(1+zt^{-1})\} ,$$

and write

$$(2.6)\quad I = (2\pi i)^{-1} \int_C (t^2 - z^2)^{-\frac{1}{2}(\alpha+1)} \exp(2NG)\, dt .$$

The assumption $|z| < R$, $R > 0$ implies that G is a regular function of t and z for $|t| \geq R$.

We introduce a change of variable, $t = t(u)$, into our integral representation implicitly defined by

$$(2.7)\quad G = u + \tau^2 u^{-1} ,$$

where τ is, for the moment, an undetermined bounded complex parameter. Solving (2.7) for u, we obtain a solution

$$(2.8)\quad u = \frac{1}{2}(G + (G^2 - 4\tau^2)^{1/2}) ,$$

where the branch of $(G^2 - 4\tau^2)^{1/2}$ is taken to be the one that behaves like t when t is real, large, and positive. Under these conditions, u has an expansion of the form

$$(2.9)\quad u = t \sum_{k=0}^{\infty} a_k t^{-2k} , \quad a_0 = 1 ,$$

valid, uniformly in z, τ^2, and t, for $|t| \geq R$. This implies that a constant M exists such that

$$(2.10)\quad |a_k| \leq M R^{2k} .$$

If we let $t = \xi^{-1}$, and $u = \zeta^{-1}$, we can write (2.9) in the form

$$(2.11)\quad \xi = \zeta\,(1 + \sum_{k=1}^{\infty} a_k \xi^{2k}) .$$

By the Lagrange theorem for the reversion of a power series, there will exist a unique solution of (2.11) of the form

$$(2.12)\quad \zeta = \sum_{k=0}^{\infty} h_{2k+1}\, \xi^{2k+1} / (2k+1)! \ ,$$

where

$$(2.13)\quad h_{2k+1} = \frac{d^{2k}}{d\xi^{2k}} \left[1 + \sum_{k=1}^{\infty} a_k\, \xi^{2k} \right]^{2k+1} \Bigg|_{\xi = 0} .$$

The bound $a_k = 0(R^{2k})$ and the Cauchy inequalities imply the existence of a fixed $r > 0$ such that

$$(2.14)\quad h_{2k+1} / (2k+1)! = 0(1/r^{2k+1}), \quad \text{uniformly in the arguments.}$$

If we replace ξ by t^{-1} and ξ by u^{-1}, we are lead to an expansion of the form

$$(2.15)\quad t = u(1 + \sum_{k=1}^{\infty} c_k\, u^{-2k}) \ ,$$

which is uniform for some fixed $K > 0$ providing $|u| \geq K$. Although K must be fixed, its choice is arbitrary and can be taken as large as we like.

We can, by taking K sufficiently large, calculate $(t^2 - z^2)^{-\frac{1}{2}(\alpha+1)}$ dt from (2.15) and find

$$(2.16)\quad (t^2 - z^2)^{-\frac{1}{2}(\alpha+1)}\, dt = u^{-(\alpha+1)}\, du\, (1 + \sum_{k=1}^{\infty} g_k\, u^{-2k}) \ ,$$

where each g_k is a polynomial in z and τ^2.

For u large, (2.15) gives $t \sim u$. Thus the path of integration in the u-plane has the same form as the one in the

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t-plane. When $|u| \geq K$, K fixed and sufficiently large, the validity of term by term integration is easily obtained and we find

$$(2.17) \quad I = \tau^{-\alpha} [I_\alpha (4N\tau) + \sum_{k=1}^{\infty} g_k \tau^{-2k} I_{\alpha+2k} (4N\tau)] ,$$

where $I_\nu(z)$ is the modified Bessel function of order ν.
We therefore obtain

$$(2.18) \quad L_n^\alpha(4Nz) = \frac{\Gamma(n+\alpha+1) e^{2Nz}}{n! (2N\tau)^\alpha} [I_\alpha(4N\tau) + \sum_{k=1}^{\infty} g_k \tau^{-2k} I_{\alpha+2k}(4N\tau$$

This is an expansion that is similar to the one obtained by Tricomi except that it contains an arbitrary bounded parameter τ and the coefficients $g_k \tau^{-2k}$ are independent of N. The range of validity of (2.18) requires z to be bounded.

SECTION 3

Behavior as an Asymptotic Expansion

It has already been noted that the choice of the parameter τ is arbitrary. For example the choice $\tau = 0$ gives

$$(3.1) \quad L_n^\alpha(4Nz) = \frac{\Gamma(n+\alpha+1) e^{2Nz}}{n!} \left[\frac{1}{\Gamma(\alpha+1)} + \sum_{k=1}^{\infty} g_k \frac{(2N)^{2k}}{\Gamma(\alpha+2k+1)} \right] .$$

Since $g_1 = 0(z)$, (3.1) will not act as an asymptotic expansion unless $z = 0(N^{-\gamma})$; $\gamma > 2$. Thus the choice $\tau = 0$, would mean that asymptotic validity would require a restriction on the approach of $z \to 0$ that is more stringent than the one required by the Tricomi expansion. The object of our paper is to show that a choice of τ exists which will reverse this conclusion. We attempt to accomplish this objective by making the domains of validity of the various series involved in the derivation as large as possible.

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The function

$$(3.2)\quad G = t + \frac{1}{2}\log\{(1-zt^{-1})/(1+zt^{-1})\},$$

has a domain of regularity $|t| > |z|$. However

$$(3.3)\quad \frac{dG}{dt} = (t^2-z^2+z)/(t^2-z^2)$$

and $\frac{dG}{dt} = 0$ when $t = \pm t_0$, $t_0 = (z^2-z)^{1/2}$. It is therefore possible that a branch point may result in the solution of (2.1) for t as a function of u. Similarly from (2.8) branch points may occur at solutions of $G^2 = 4\tau^2$. We coalesce these points by choosing

$$(3.4)\quad 2\tau = G(t_0, z).$$

Since $|t_0| > |z|$ when $\mathrm{Re}\ z < 1/2$, we can write

$$(3.5)\quad \tau = t_0 + \frac{1}{2}\log\{(1-z\,t_0^{-1})/(1+z\,t_0^{-1})\}$$

$$= t_0\,(1 - \sum_{k=1}^{\infty} z^{2k-1}/(2k-1)\,t_0^{2k}), \quad \mathrm{Re}\ z < 1/2$$

$$= t_0\,(1 - \sum_{k=1}^{\infty} z^{k-1}/(2k-1)(z-1)^k).$$

Further the expansion of G about $t = t_0$ yields

$$(3.6)\quad G = 2\tau + \frac{1}{2}G''(t_0, z)\,(t-t_0)^2 + \ldots$$

$$= 2\tau - \frac{t_0}{z}(t-t_0)^2 + \ldots$$

and

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$$(3.7)\quad G^2 = 4\tau^2 - \frac{4\tau t_0}{z}(t-t_0)^2 + \ldots .$$

From

$$(3.8)\quad \frac{\tau t_0}{z} = (z-1)\,(1 - \sum_{k=1}^{\infty} z^{k-1} / (2k-1)(z-1)^k) ,$$

$\tau t_0 / z \to -1$ as $z \to 0$.

In a neighborhood of $t = t_0$, both branches of $(G^2-4\tau^2)^{1/2}$ are regular functions of t. Similar analysis yields the same result at $t = -t_0$. As $z \to 0$, it is easily established that the only solutions of $G(t, z) = 2\tau$ are $t = \pm t_0$. For this reason, the function u defined by (2.8) is a regular function of t as long as $|t| > |z|$. On the circle $|t| = K|z|$, $K > 1$, $|u| = 0(|z|)$. Hence $|u| / |t| = 0(1)$ uniformly in z. This is sufficient to prove that the a_k of (2.9) satisfy

$$(3.9)\quad a_k = 0((Kz)^{2k}) ,\quad \text{uniformly in } k ,\ \text{as } z \to 0 .$$

By tracing through the implication of (3.9) for the various series involved in the derivation of (2.18), we find

$$(3.10)\quad g_k = 0((Kz)^{2k}) ,\quad \text{uniformly in } k ,\ \text{as } z \to 0 .$$

Since $\tau = 0(z^{1/2})$,

$$(3.11)\quad g_k \tau^{-2k} = 0((K^2 z)^k) ,\quad \text{uniformly in } k ,\ \text{as } z \to 0 .$$

In the expansion

$$(3.12)\quad L_n^{\alpha}(4Nz) = \frac{\Gamma(n+\alpha+1)e^{2Nz}}{n!\ (2N\tau)^{\alpha}}\,[I_{\alpha}(4N\tau) + \sum_{k=1}^{\infty} g_k \tau^{-2k} I_{\alpha+2k}(4N\tau)$$

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only non-negative integral powers of τ^2 occur. Since τ^2 is a regular function of z, as $z \to 0$, we have no concern over the fact that τ is a multivalued function of z. For definiteness we shall require $|\arg \tau| \leq \pi/2$.

If we allow $N \to \infty$ and $z \to 0$ in such a way that $N\tau$ remains bounded, $(z = 0(1/N^2))$, we obviously have

$$(3.13)\quad L_n^{\alpha}(4Nz) = \frac{\Gamma(n+\alpha+1)e^{2Nz}}{n!\,(2N\tau)^{\alpha}}[I_{\alpha}(4N\tau) + 0(z)], \quad \text{as } z \to 0$$

If in addition, we require that $4N\tau$ be bounded away from the zero's of $I_{\alpha}(z)$, then we may write

$$(3.14)\quad L_n^{\alpha}(4Nz) = \frac{\Gamma(n+\alpha+1)e^{2Nz}}{n!\,(2N\tau)^{\alpha}}\, I_{\alpha}(4N\tau)\,[1 + 0(z)], \quad \text{as } z \to 0,$$

uniformly in N.

It is interesting to point out that we need not restrict $4N\tau$ to be bounded away from the zero's of I_{α}. For example, if $I_{\alpha}(a) = 0$, where "a" is a fixed zero. Then $4N\tau = a$, implies $\tau = a/4N \to 0$ as $N \to \infty$ and $z \to 0$. We would then obtain

$$(3.15)\quad L_n^{\alpha}(4Nz) = \frac{\Gamma(n+\alpha+1)e^{2Nz}\, g_1\, I_{\alpha+2}(a)}{n!\,(\frac{1}{2}a)^{\alpha}\,\tau^2}\,[1 + 0(z)],$$

uniformly in N and the asymptotic character of the series is retained.

When we allow $N \to \infty$ and $z \to 0$ in such a way that $4N\tau$ becomes unbounded, the proof that (3.12) still retains its asymptotic character is essentially the same as the proof required to show that the Tricomi result, (1.1), retains its asymptotic character even though $Nz \to \infty$. For this reason we shall not repeat the proof. It is possible to establish that (3.12) retains the character of a complete asymptotic expansion, under the single restriction $z \to 0$. We therefore provide extensions of both the results of Tricomi and Erdélyi in a neighborhood of $z = 0$.

SECTION 4

The Erdélyi Result

When z is real, we find from

$$(4.1) \quad \tau = \frac{1}{2}(t_0 + \frac{1}{2}\log\{(t_0 - z)/(t_0 + z)\}) ,$$

that

$$(4.2) \quad \tau = \frac{1}{2}[\sqrt{x^2 - x} + \sinh^{-1}(-x)^{1/2}] , \quad x \leq 0$$

$$= \frac{i}{2}[\sqrt{x - x^2} + \sin^{-1}(x^{1/2})] , \quad x \geq 0 .$$

Even though the argument of the modified Bessel function used in (2.18) agrees with the corresponding argument used by Erdélyi, the expansion (2.18) is not the same as the Erdélyi expansion. This latter expansion can be obtained from (2.18) in the following way.

The point $t = t_0$ corresponds to $u = \tau$. If we differentiate

$$(4.3) \quad t + \frac{1}{2}\log\{(t-z)/(t+z)\} = (u + \tau^2 u^{-1})$$

twice with respect to u and place $u = \tau$, we obtain

$$(4.4) \quad (\frac{dt}{du})_{u=\tau} = \sqrt{(-z)/t_0 \tau} .$$

The branch of the function involved in (4.4) is the one that is real and positive at $z = 0$. From (3.8), $(\tau t_0/z)_{z=0} = -1$.

With the choice of τ that was made in this section, the result contained in (2.16) will hold for $|u| \geq K|z|$ for some fixed $K > 0$. In particular it will hold for $u = \tau$ when $z \to 0$. Combining (2.16) and (4.4) we obtain

$$(4.5)\quad \sqrt{(-z/t_0\tau)}\ \tau^{\alpha+1} / (-z)^{(\alpha+1)/2} = 1 + \sum_{k=1}^{\infty} g_k\, \tau^{-2k} .$$

When we multiply (4.5) by $I_\alpha(4N\tau)$ we are able to write

$$(4.6)\quad I_\alpha = I_\alpha \sqrt{(-z/t_0\tau)}\ \tau^{\alpha+1} / (-z)^{(\alpha+1)/2} - I_\alpha \sum_{k=1}^{\infty} g_k\, \tau^{-2k} .$$

Substituting for I_α in (2.18), we obtain

$$(4.7)\quad L_n^\alpha(4Nz) = \frac{\Gamma(n+\alpha+1)e^{2Nz}}{n!\ (2N\tau)^\alpha}\left[\frac{I_\alpha\sqrt{(-z/t_0\tau)}\,\tau^{\alpha+1}}{(-z)^{(\alpha+1)/2}} + \sum_{k=1}^{\infty} g_k\, \tau^{-2k}(I_{\alpha+2k}-I_\alpha)\right] .$$

Further

$$(4.8)\quad I_{\alpha+2k}-I_\alpha = \sum_{r=1}^{k} (I_{\alpha+2r}-I_{\alpha+2r-2}) = -\frac{1}{2N\tau}\sum_{r=1}^{k} (\alpha+2r-1)\, I_{\alpha+2r-1} .$$

If we use this result in (4.7), we can, by isolating the zero's of I_α in the usual way obtain

$$(4.9)\quad L_n^\alpha(4Nz) = \frac{\Gamma(n+\alpha+1)e^{2Nz}\sqrt{(-\tau z/t_0)}\ I_\alpha}{n!\ (2N)^\alpha\ (-z)^{(\alpha+1)/2}}\ (1 + 0(\frac{\sqrt{z}}{N})) .$$

This is precisely the Erdélyi result. It should be pointed out that Erdélyi proved that (4.9) holds for z real and $-\infty < z < b < 1$. When z is bounded away from zero, the order term is replaced by $0(1/N)$. For this reason, our result is not a generalization of the Erdélyi result. It is a somewhat slight extension in that we have shown that the same result also holds for complex values of z providing $z \to 0$.

SECTION 5

Conclusion

Although the result, (3.14), that we have obtained is somewhat simpler than the Erdélyi result, the range of validity, $z \to 0$, is much more restricted than the range of validity that can be obtained for (4.9). On the other hand our range of validity is more extensive than the range of validity of the Tricomi result.

One of the interesting, but not surprising, results is the lack of agreement in the estimates of error involved in the various expansions in the region of common validity. The only time one must have such agreement would be a situation in which we demand that the expansion take a specified form, for example the Poincaré form of asymptotic expansion. The lack of agreement in the estimates of error is a reason for discarding the Poincaré definition of an asymptotic expansion and replacing it by a general definition which would include, as special cases, all of the expansions contained in this paper. Such a definition has been given by Erdélyi[4].

The extension of the range of validity to include any manner of approach of $z \to 0$ in the complex plane might at first seem trivial. However when this extension is used on the usual form, $L_n^{\alpha}(z)$, for the Laguerre polynomials the region of validity becomes $z = o(N)$.

4. This definition has not as yet been published.