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INVERSE IMAGES OF SECTORS BY FUNCTIONS IN WEIGHTED BERGMAN–ORLICZ SPACES

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Abstract

For $\varepsilon > 0$, let $\Sigma_\varepsilon = \{z \in \mathbb{C} : |\arg z| < \varepsilon\}$. It has been proved (D. E. Marshall and W. Smith, *Rev. Mat. Iberoamericana* **15** (1999), 93–116) that $\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)|\, dA(z) \simeq \int_{\mathbb{D}} |f(z)|\, dA(z)$ for every $\varepsilon > 0$, uniformly for every univalent function f in the classical Bergman space A^1 that fixes the origin. In this paper, we extend this result to those conformal maps on $\mathbb{D}$ belonging to weighted Bergman–Orlicz classes such that $f(0) = |f'(0)| - 1 = 0$.

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1. Introduction and main results

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. For $p > 0$ and $\alpha > -1$, the weighted Bergman space A^p_α is defined as the class of all holomorphic functions f in $\mathbb{D}$ for which

$$\|f\|^p_{\alpha,p} = \int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) < \infty, \tag{1.1}$$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \, dA(z)$ and $dA(z) = (1/\pi)\, dx\, dy = (r/\pi)\, dr\, d\theta$ is the normalized two-dimensional Lebesgue measure, and $z = x + iy = re^{i\theta}$. Notice that dA_α is a probability measure. It is well known that, if $p \geq 1$, A^p_α is a Banach space with the norm defined in (1.1) while, for $p \in (0, 1)$, A^p_α becomes a quasi-Banach space. For the classical and modern theory of Bergman spaces, we refer to the monographs [2, 6, 18].

In [7], Marshall and Smith, trying to solve a conjecture by Ortel and Smith [9] concerning extremal dilations for quasiconformal mappings, proved that the inverse

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images of sectors $\Sigma_\varepsilon = \{w \in \mathbb{C} : |\arg w| < \varepsilon\}$ where $\varepsilon > 0$, by univalent functions $f \in A_0^1$ with $f(0) = 0$ have mass enough so that the norms $\|f\|_{1,0}$ can be reached essentially, in a uniform way, by integration over $f^{-1}(\Sigma_\varepsilon)$. More precisely we have the following result.

THEOREM A. *Given any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that*

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)|\, dA(z) > \delta \int_{\mathbb{D}} |f(z)|\, dA(z), \tag{1.2}$$

for any univalent function f in A_0^1 fixing the origin.

In [11], we proved that Theorem A does hold for weighted Bergman spaces A_α^p, whenever $p \geq 1$ and $\alpha > 2p - 1$, and an example was given there showing that estimate (1.2) is not true for any $\varepsilon > 0$ and $\alpha < 2p - 2$. We do not know what happens for values of α between $2p - 2$ and $2p - 1$. It is another open question to see whether or not the univalence hypothesis in Theorem A can be dropped.

Our aim in this paper is to extend Theorem A to weighted Bergman–Orlicz classes. Following the classical paper by Mazur and Orlicz [8], for $\varphi : [0, \infty) \to [0, \infty)$ a continuous increasing function such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$, we define the set $\mathcal{L}_\alpha^\varphi$ of all measurable functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} \varphi(\lambda |f(z)|)\, dA_\alpha(z) < \infty,$$

for some $\lambda > 0$ depending on f. It is easy to see that $\mathcal{L}_\alpha^\varphi$ is a linear space. A good reference for the theory of Orlicz spaces is the book by Rao and Ren [14]. Let $H(\mathbb{D})$ denote the space of all holomorphic functions in $\mathbb{D}$; we say that $A_\alpha^\varphi := \mathcal{L}_\alpha^\varphi \cap H(\mathbb{D})$ is the *Bergman–Orlicz class associated to φ*.

It is natural to assume that the symbol φ enjoys some reasonable properties in such a way that the classes A_α^φ contain classical weighted Bergman spaces A_α^p, $p > 0$, as particular cases. This leads us to consider two types of symbols, corresponding to the settings $1 \leq p < \infty$ and $0 < p < 1$.

Hence, for $\psi : [0, \infty) \to [0, \infty)$ a strictly increasing convex function satisfying $\psi(0) = 0$, we will say that ψ satisfies the Δ_2 *condition near infinity* if there exist constants $t_0 \geq 0$ and $K_\Delta > 1$ such that

$$\psi(2t) \leq K_\Delta \psi(t),$$

for all $t \geq t_0$. In the case $t_0 = 0$ we will say that ψ satisfies the *global* Δ_2 *condition*. The Δ_2 condition plays an important role in the theory of A_α^ψ spaces. Furthermore, if we define

$$M_{\alpha,\psi}(f) := \int_{\mathbb{D}} \psi(|f(z)|)\, dA_\alpha(z),$$

then A_α^ψ is a Banach space equipped with the Luxemburg norm

$$\|f\|_{\alpha,\psi} = \inf\{k > 0 : M_{\alpha,\psi}(f/k) \leq 1\},$$

and A_α^ψ coincides with the class of all holomorphic functions f in $\mathbb{D}$ for which $M_{\alpha,\psi}(f) < \infty$, if and only if ψ satisfies the Δ_2 condition near infinity.

Bergman–Orlicz spaces with symbol satisfying these properties have been considered by He [5] and by Wang and Xu [17] to study problems of different nature. Examples of such symbol functions ψ are: $\psi_1(t) = t^p$, $p \geq 1$; $\psi_2(t) = \exp[\log(t+1)^{3/2}] - 1$; and $\psi_3(t) = \exp[\log(t+2)\log\log(t+2)] - 2^{\log\log 2}$.

Our first result extends Theorem A to this type of weighted Bergman–Orlicz space.

Theorem 1.1. *Let $\psi : [0,\infty) \to [0,\infty)$ be a strictly increasing convex function satisfying $\psi(0) = 0$ and the global Δ_2 condition, and suppose that $\alpha > 2K_\Delta - 1$. Then, given $\varepsilon > 0$, there exists a constant $\delta > 0$, depending on ψ and α, such that*

$$\int_{f^{-1}(\Sigma_\varepsilon)} \psi(|f(z)|)\, dA_\alpha(z) \geq \delta\varepsilon \|f\|_{\alpha,\psi},$$

for any univalent function $f \in A_\alpha^\psi$ satisfying $f(0) = |f'(0)| - 1 = 0$.

If we ask for an *appropriate* symbol φ so that the corresponding A_α^φ space behaves as the classical Bergman space A_α^p, $0 < p < 1$, one may consider φ to be a *modulus function*, that is a nonnegative function defined on $[0,\infty)$ that is a strictly increasing, subadditive, right-continuous at $t = 0$, and such that $\varphi(t) = 0$ if and only if $t = 0$. Given a modulus function φ and $\alpha > -1$, we define the *weighted Bergman–Orlicz space A_α^φ associated to φ* to be the class of all holomorphic functions f on the unit disk $\mathbb{D}$ such that

$$\|f\|_{\alpha,\varphi} := \int_{\mathbb{D}} \varphi(|f(z)|)\, dA_\alpha(z) < \infty.$$

Then A_α^φ is a metric topological space with metric given by $d(f,g) = \|f - g\|_{\alpha,\varphi}$ (see [15, 16]). Furthermore, if $\varphi(|f|)$ is subharmonic in $\mathbb{D}$ for every $f \in A_\alpha^\varphi$, then A_α^φ is a complete metric space.

For this last type of symbol function, we prove the following result.

Theorem 1.2. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a concave modulus function and $\alpha \geq 0$. Then, given $\varepsilon > 0$, there exists a constant $\delta > 0$, depending on α, φ and ε, such that*

$$\int_{f^{-1}(\Sigma_\varepsilon)} \varphi(|f(z)|)\, dA_\alpha(z) \geq \delta \|f\|_{\alpha,\varphi},$$

for any conformal function $f \in A_\alpha^\varphi$ satisfying $f(0) = |f'(0)| - 1 = 0$.

Simple examples of functions φ satisfying the statements in Theorem 1.2 are $\varphi_1(t) = t^p$, $0 < p \leq 1$, and $\varphi_2(t) = \log(1+t)$. Moreover, if φ_1 and φ_2 are modulus functions, it is easy to see that $\varphi_1 \circ \varphi_2$ is also a modulus function, whence new examples of such functions can be explicitly found.

As a consequence of Theorem 1.2 we can write the following result.

COROLLARY 1.3. *Let $p \in (0, 1]$. Given $\varepsilon > 0$, there exists a constant $\delta > 0$ such that*

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)|^p \, dA(z) > \delta \int_{\mathbb{D}} |f(z)|^p \, dA(z),$$

for any univalent function in A_0^p satisfying $f(0) = 0$.

Notice that Corollary 1.3 gives a genuine extension of Theorem A for any p, $0 < p \leq 1$, a setting that has not been considered so far. Observe also that, in particular, for $\varphi(t) = t$, $t \geq 0$ and $\alpha = 0$, Theorem A follows since, in this case, the restriction $|f'(0)| = 1$ can be omitted.

The paper is organized as follows. In Section 2, we explain a construction due to Marshall and Smith [7], which provides a covering of the domain image of a conformal map f from the unit disk $\mathbb{D}$ onto $\Omega = f(\mathbb{D})$, such that $f(0) = 0$ and $|f'(0)| = 1$. Such a covering and other auxiliary results in this section are needed for the proofs of Theorems 1.1 and 1.2 that occupy Sections 3 and 4, respectively.

Finally, let us point out that, throughout this work, C and K will denote constants whose value can change in each appearance. Many times we will write $C(\varepsilon, \alpha, \ldots)$ and $K_i(\varepsilon, \alpha, \varphi, \ldots)$ to highlight that these constants depend on the parameters within parentheses.

2. A covering for the domain image

For the proofs of our principal results, we need certain very precise estimates involving the hyperbolic distance and the harmonic measure. Recall that the *hyperbolic metric* on $\mathbb{D}$ is defined as

$$\beta(z_1, z_2) = \inf \left\{ \int_\gamma \frac{2\,|dz|}{1 - |z|^2} : \gamma \text{ is an arc in } \mathbb{D} \text{ from } z_1 \text{ to } z_2 \right\} \quad \forall z_1, z_2 \in \mathbb{D}.$$

Since the shortest distance from 0 to any point $z \in \mathbb{D}$ is the radius $[0, z]$, in particular we have

$$\beta(0, z) = \log\left(\frac{1 + |z|}{1 - |z|}\right). \tag{2.1}$$

The hyperbolic metric is invariant under conformal self-maps of $\mathbb{D}$ and we can also define it on any simple connected proper domain Ω of $\mathbb{C}$ as $\beta_\Omega(w_1, w_2) := \beta(z_1, z_2)$, where $w_i = h(z_i)$, $i = 1, 2$, and $h : \mathbb{D} \to \Omega$ is any conformal map. An exhaustive study of the hyperbolic metric on $\mathbb{D}$ can be found, for example, in [12]. Analogously, the *quasihyperbolic* metric on Ω is defined as

$$k_\Omega(w_1, w_2) = \inf \left\{ \int_\gamma \frac{|dw|}{\delta_\Omega(w)} : \gamma \text{ is an arc in } \Omega \text{ from } w_1 \text{ to } w_2 \right\},$$

where $\delta_\Omega(w)$ denotes the Euclidean distance from w to the boundary $\partial\Omega$ of Ω. As shown in [4], the hyperbolic and quasihyperbolic metrics are comparable. In fact,

$$\tfrac{1}{2}\beta_\Omega(w_1, w_2) \leq k_\Omega(w_1, w_2) \leq 2\beta_\Omega(w_1, w_2), \tag{2.2}$$

for all $w_1, w_2 \in \Omega$.

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Also, we will use some well-known results on distortion of conformal maps, which we collect in Theorem B below and which can be seen in [12].

THEOREM B. *Let f be a conformal map from $\mathbb{D}$ onto $\Omega = f(\mathbb{D}) \subset \mathbb{C}$. Then for any $z \in \mathbb{D}$ the following estimates hold:*

(i) $|f'(0)|\dfrac{|z|}{(1+|z|)^2} \le |f(z) - f(0)| \le |f'(0)|\dfrac{|z|}{(1-|z|)^2}$;

(ii) $|f'(0)|\dfrac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le |f'(0)|\dfrac{1+|z|}{(1-|z|)^3}$;

(iii) $\frac{1}{4}(1-|z|^2)|f'(z)| \le \delta_\Omega(f(z)) \le (1-|z|^2)|f'(z)|$;

(iv) $|f(z) - f(z_0)| \le 4\delta_\Omega(f(z_0))e^{2\beta(z,z_0)}$, $z_0 \in \mathbb{D}$;

(v) $\delta_\Omega(f(0))e^{-\beta(0,z)} \le (1+|z|)^2|f'(z)| \le 4\delta_\Omega(f(0))e^{3\beta(0,z)}$.

Let E be a closed subset of $\mathbb{C}$ with $\Omega \cap E \neq \emptyset$ and let Γ be the component of $\partial(\Omega \setminus E)$ contained in E; suppose that Γ is a finite union of Jordan arcs. We will denote by $\omega_\Omega(z, E)$ the harmonic measure of E evaluated at z. It is, by definition, the unique bounded harmonic function in $\Omega \setminus E$ that is identically 1 on E and vanishes on $\partial(\Omega \setminus E) \setminus E$. An excellent reference to study the harmonic measure is the recent monograph by Garnett and Marshall [3].

Next, we summarize a very clever analytic-geometric construction due to Marshall and Smith (see [7]), which provides a covering for the domain image $\Omega = f(\mathbb{D})$, where f is a conformal map on $\mathbb{D}$ with $f(0) = 0$. Notice that part (iii) of Theorem B implies that $|f'(0)|$ is comparable with $\delta_\Omega(0)$. Therefore, from now on, we will suppose that $\delta_\Omega(0) = 1$. Without loss generality, we will also assume that $\varepsilon < 1/10$.

We set $A_0 = \mathbb{D}$ and for $n \ge 1$ we put

$$A_n = \{w \in \mathbb{C} : (1+\varepsilon)^{n-1} < |w| < (1+\varepsilon)^n\}. \tag{2.3}$$

For each $n \in \mathbb{N} \cup \{0\}$, we choose, if possible, a Euclidean square $Q_n \subset A_n \cap \Sigma_\varepsilon \cap \Omega$ satisfying

$$\begin{cases} \text{(i)} \quad \operatorname{diam}(Q_n) \ge \dfrac{\varepsilon}{4}(1+\varepsilon)^{n-1}, \\ \text{(ii)} \quad \dfrac{1}{2} \le \dfrac{\operatorname{dist}(Q_n, \partial(A_n \cap \Sigma_\varepsilon \cap \Omega))}{\operatorname{diam}(Q_n)} \le 2. \end{cases} \tag{2.4}$$

Notice that the condition $\delta_\Omega(0) = 1$ implies that there is a square $Q_0 \subset \mathbb{D} \cap \Sigma_\varepsilon$ satisfying (i) and (ii) in (2.4). We will write $w_n = f(z_n)$ to denote the center of Q_n. For those $n \in \mathbb{N}$ for which we can construct a square Q_n satisfying properties in (2.4), we define the set

$$N(Q_n) = \{w \in \mathbb{C} : \beta_\Omega(w, w_n) < 100/\varepsilon\}.$$

Here $N(Q_n)$ is a hyperbolic neighborhood of Q_n and note that $0 \in N(Q_0)$. For any $w \in \Omega$ we denote by γ_w the hyperbolic geodesic from w to 0; then we will say that w

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is an element of Ω_n if $N(Q_n)$ is the first hyperbolic neighborhood that γ_w finds when crossing from w to 0, that is $w \in \Omega_n$ if and only if:

(a) $\beta_\Omega(\gamma_w, w_n) < 100/\varepsilon$; and
(b) if γ_w^n is the component of $\gamma_w \setminus N(Q_n)$ containing w, then either $\gamma_w^n = \emptyset$ or else $\beta_\Omega(\gamma_w, w_m) \geq 100/\varepsilon$ for all $n \neq m$.

If there is no Q_n in A_n satisfying the properties in (2.4), then we set $\Omega_n = \emptyset$. It is clear that $N(Q_n) \subset \Omega_n$, and since $0 \in N(Q_0)$, then the family $\{\Omega_n\}$ is a covering of Ω.

The sets Q_n and Ω_n enjoy some special properties that appear in [7, pp. 104–110]. For later reference, we include two results.

PROPOSITION C. *Let f be a conformal map from $\mathbb{D}$ onto the domain Ω such that $f(0) = 0$. Let the A_j, Q_n and Ω_n be as constructed above. Then, there exist positive constants C and $C(\varepsilon)$ such that*

$$e^{-\beta_\Omega(w_n, \Omega_n \cap A_j)} \omega_\Omega(w_n, \Omega_n \cap A_j) \leq C(\varepsilon)(1+\varepsilon)^{-|j-n|(1+C\varepsilon)}, \tag{2.5}$$

for all $j \in \mathbb{N} \cup \{0\}$ and for any $n \in \mathbb{N}$. When $n = 0$, we replace w_n by 0.

In the following proposition $C(\varepsilon)$ will denote a positive constant depending only on ε whose value can change from one line to another. The proof of Proposition D below appears in [7] (see also [10]).

PROPOSITION D. *Let f be a conformal map from $\mathbb{D}$ onto the domain Ω such that $f(0) = 0$. Let the A_j, Q_n and Ω_n be as constructed above. Then, there exists a constant $C(\varepsilon) > 0$ such that for every $j \in \mathbb{N} \cup \{0\}$ and each $n \in \mathbb{N}$ we have the following:*

(i) $\mathrm{Area}(f^{-1}(Q_n)) \leq C(\varepsilon) e^{-\beta_\Omega(0, Q_n)} \omega_\Omega(0, Q_n)$;

(ii) $\mathrm{Area}(f^{-1}(\Omega_n \cap A_j)) \leq C(\varepsilon) e^{-\beta_\Omega(0, \Omega_n \cap A_j)} \omega_\Omega(0, \Omega_n \cap A_j)$;

(iii) $e^{-\beta_\Omega(0, \Omega_n \cap A_j)} \leq C(\varepsilon) e^{-\beta_\Omega(0, Q_n)} e^{-\beta_\Omega(w_n, \Omega_n \cap A_j)}$; *and*

(iv) $\omega_\Omega(0, \Omega_n \cap A_j) \leq C(\varepsilon) \omega_\Omega(0, Q_n) \omega_\Omega(w_n, \Omega_n \cap A_j)$.

LEMMA 2.1. *Let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing function, $\alpha \geq 0$ and $\varepsilon > 0$. Then there exists a constant $C > 0$, depending only on φ, α and ε, such that*

$$\int_{f^{-1}(Q_0)} \varphi(|f|) \, dA_\alpha \geq C,$$

for any conformal map f from $\mathbb{D}$ onto Ω satisfying $f(0) = 0$ and $|f'(0)| = 1$.

PROOF. First, notice from (2.4) that, if $z \in f^{-1}(Q_0)$, then

$$|f(z)| \geq \mathrm{dist}(Q_0, \partial(A_0 \cap \Sigma_\varepsilon \cap \Omega)) \geq \frac{\varepsilon}{8(1+\varepsilon)}. \tag{2.6}$$

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We also observe that, if $w = f(z) \in Q_0$, then

$$\delta_\Omega(w) = \mathrm{dist}(w, \partial\Omega) \geq \mathrm{dist}(Q_0, \partial(A_0 \cap \Sigma_\varepsilon)) \geq \varepsilon/10.$$

Consequently, $\delta_\Omega(s) \geq \varepsilon/10$ for any $s \in [0, w]$, the ray from 0 to w. Now, by the comparison between the hyperbolic metric and the quasihyperbolic metric in (2.2), we get

$$\beta_\Omega(0, f(z)) \leq 2k_\Omega(0, w) \leq 2\int_{[0,w]} \frac{|ds|}{\delta_\Omega(s)} \leq \frac{20}{\varepsilon} \quad \forall z \in f^{-1}(Q_0). \tag{2.7}$$

Thus, from (2.1) we can find a constant $C(\varepsilon, \alpha) > 0$ such that

$$(1 - |z|^2)^\alpha \geq C(\varepsilon, \alpha) \quad \forall z \in f^{-1}(Q_0). \tag{2.8}$$

On the other hand, since $f(0) = 0$ and $\delta_\Omega(0) = 1$, the estimate (v) in Theorem B and (2.7) yield

$$|f'(z)| \leq 4e^{3\beta_\Omega(0, f(z))} \leq 4e^{60/\varepsilon} \quad \forall z \in f^{-1}(Q_0). \tag{2.9}$$

Finally, from estimates (2.8), (2.6) and (2.9) we obtain

$$\begin{aligned}\int_{f^{-1}(Q_0)} \varphi(|f|)\, dA_\alpha &\geq \frac{1}{16} C(\varepsilon, \alpha) e^{-120/\varepsilon} \varphi\left(\frac{\varepsilon}{8(1+\varepsilon)}\right) \int_{f^{-1}(Q_0)} |f'|^2\, dA \\ &= \frac{1}{16} C(\varepsilon, \alpha) e^{-120/\varepsilon} \varphi\left(\frac{\varepsilon}{8(1+\varepsilon)}\right) \mathrm{Area}(Q_0),\end{aligned}$$

and, since $\mathrm{diam}(Q_0) \geq \varepsilon/[4(1+\varepsilon)]$ (see (2.4)), we conclude that there exists a constant $C(\varepsilon, \alpha, \varphi) > 0$ such that

$$\int_{f^{-1}(Q_0)} \varphi(|f|)\, dA_\alpha \geq C(\varepsilon, \alpha, \varphi),$$

as we wished to show. □

3. Proof of Theorem 1.1

To prove Theorem 1.1 we will need some prior results. First, we give a simple technical lemma.

LEMMA 3.1. *Suppose that ψ is a function as in Theorem 1.1 and let $\alpha > -1$. Then there exists a constant $C_1(\alpha) > 0$, depending only on α, such that*

$$\int_{\mathbb{D}} \psi(|f|)\, dA_\alpha \leq C_1(\alpha) \int_{\{z \in \mathbb{D} : |z| > \frac{1}{2}\}} \psi(|f|)\, dA_\alpha,$$

for any function $f \in A^\psi_\alpha$.

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PROOF. Take a function $f \in A^{\psi}_{\alpha}$. Then $|f|$ is subharmonic in $\mathbb{D}$ and, since ψ is an increasing convex function, $\psi(|f|)$ is also subharmonic in $\mathbb{D}$. Now, since the averages of subharmonic functions increase with radius,

$$\frac{16}{9}\int_{D(0,\frac{3}{4})} \psi(|f|)\, dA \geq 4\int_{D(0,\frac{1}{2})} \psi(|f|)\, dA. \tag{3.1}$$

Write $D(0, \frac{3}{4}) = D(0, \frac{1}{2}) \cup \{\frac{1}{2} \leq |z| < \frac{3}{4}\}$. It follows from (3.1) that

$$\int_{\{\frac{1}{2}<|z|<\frac{3}{4}\}} \psi(|f|)\, dA \geq \frac{5}{4}\int_{D(0,\frac{1}{2})} \psi(|f|)\, dA.$$

Let $K_1(\alpha)$ and $K_2(\alpha)$ be two positive constants, depending on α, such that $(1-|z|^2)^{\alpha} \geq K_1(\alpha)$, $|z| < \frac{3}{4}$, and $(1-|z|^2)^{\alpha} \leq K_2(\alpha)$, $|z| < \frac{1}{2}$. Substituting into (3.1) we get

$$\frac{1}{K_1(\alpha)}\int_{\{\frac{1}{2}<|z|<\frac{3}{4}\}} \psi(|f|)\, dA_{\alpha} \geq \frac{5}{4K_2(\alpha)}\int_{D(0,\frac{1}{2})} \psi(|f|)\, dA_{\alpha},$$

and since $\{\frac{1}{2} < |z| < \frac{3}{4}\} \subset \{|z| > \frac{1}{2}\}$ the conclusion follows. □

Next, we will adapt arguments from [1, 13] to prove the following theorem, which could have independent interest.

THEOREM 3.2. *Let ψ be a function as in Theorem 1.1 and let $\varepsilon > 0$. Then there exists $\delta > 0$, depending on α and ψ, such that*

$$\int_{f^{-1}(\Sigma_{\varepsilon})} \psi(|f(z)|)\, dA_{\alpha}(z) \geq \delta |f'(0)|^2 \psi(|f'(0)|)\varepsilon,$$

for any function $f \in A^{\psi}_{\alpha}$ with $f(0) = 0$ and $\|f\|_{\alpha,\psi} \leq 1$.

PROOF. Let $f \in A^{\psi}_{\alpha}$ satisfy $f(0) = 0$ and $\|f\|_{\alpha,\psi} \leq 1$. We will assume that $f'(0) \neq 0$; otherwise we are done. By definition of the Luxemburg norm there exists a positive constant $k \leq 1$ such that $M_{\alpha,\psi}(f/k) \leq 1$, and since ψ is an increasing function,

$$M_{\alpha,\psi}(f) \leq 1.$$

Now, by the Cauchy integral formula

$$|f'(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(re^{i\theta})|}{|re^{i\theta} - z|^2} r\, d\theta,$$

for all $|z| < \frac{1}{2}$ and any $\frac{1}{2} < r < 1$. Integrating from $r = \frac{3}{4}$ to $r = \frac{7}{8}$ we get

$$|f'(z)| \leq 64\int_{\left\{\frac{3}{4}<|s|<\frac{7}{8}\right\}} |f(s)|\, dA(s),$$

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for any $|z| < \frac{1}{2}$. Using the monotonicity of ψ, the Δ_2 condition and Jensen's inequality, we see that there exists a universal constant $K_1 > 0$ such that

$$\psi(|f'(z)|) \le K_1 \int_{\{\frac{3}{4}<|s|<\frac{7}{8}\}} \psi(|f(s)|)\, dA(s), \tag{3.2}$$

for all z such that $|z| < \frac{1}{2}$. Moreover, if $K_3(\alpha)$ is a positive constant such that $(1-|s|^2)^\alpha \ge K_3(\alpha)$ for all $\frac{3}{4} < |s| < \frac{7}{8}$, it follows from (3.2) that

$$\psi(|f'(z)|) \le \frac{K_1}{K_3(\alpha)} \int_{\{\frac{3}{4}<|s|<\frac{7}{8}\}} \psi(|f(s)|)\, dA_\alpha(s) \quad \forall z \in D(0, \tfrac{1}{2}).$$

Hence $\psi(|f'(z)|) \le K_1/K_3(\alpha)$ because $M_{\alpha,\psi}(f) \le 1$. This implies that there is a constant $K_4(\alpha, \psi) > 0$ such that

$$|f'(z)| \le K_4(\alpha, \psi) \quad \forall z \in D(0, \tfrac{1}{2}). \tag{3.3}$$

On the other hand, an application of Schwarz's lemma yields

$$|f'(z) - f'(0)| \le 4K_4(\alpha, \psi)|z| \quad \forall |z| < \tfrac{1}{2}, \tag{3.4}$$

and, if we put

$$R = \frac{1}{8K_4(\alpha, \psi)}|f'(0)|,$$

then, $R < \frac{1}{8}$ by (3.3), and from (3.4) we deduce that $|f'(z) - f'(0)| < \frac{1}{2}|f'(0)|$, $|z| < R$. This implies that f is one-to-one on the disk $D(0, R)$. We can use a standard modification of the Koebe one-quarter theorem to get $D(0, \sigma) \subset f(D(0, R))$, where $\sigma = |f'(0)|^2/[32K_4(\alpha, \psi)]$.

Therefore

$$\begin{aligned}
\int_{f^{-1}(\Sigma_\varepsilon)} \psi(|f(z)|)\, dA_\alpha(z) &\ge \int_{f^{-1}(\Sigma_\varepsilon \cap \{R/2<|w|<R\}) \cap D(0,R)} \psi(|f(z)|)\, dA_\alpha(z) \\
&\ge K_5(\alpha)\psi\left(\frac{R}{2}\right) \int_{f^{-1}(\Sigma_\varepsilon \cap \{R/2<|w|<R\}) \cap D(0,R)} dA(z) \\
&\ge \frac{K_5(\alpha)}{K_4^2(\alpha, \psi)} \psi\left(\frac{R}{2}\right) \\
&\quad \times \int_{f^{-1}(\Sigma_\varepsilon \cap \{R/2<|w|<R\}) \cap D(0,R)} |f'(z)|^2\, dA(z) \\
&= K_6(\alpha, \psi)\psi(|f'(0)|)|f'(0)|^2 \varepsilon,
\end{aligned}$$

where we have used the fact that $|z| < R < \frac{1}{8}$, condition (3.3), the Δ_2 condition and the fact that f is injective on the disk $D(0, R)$. This finishes the proof. □

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Corollary 3.3. *Let ψ be a function as in Theorem 1.1 and let $\varepsilon > 0$. Then there exists $\delta > 0$, depending on α and ψ, such that*

$$\int_{f^{-1}(\Sigma_\varepsilon)} \psi(|f(z)|)\, dA_\alpha(z) \geq \delta \frac{|f'(0)|^2}{\|f\|_{\alpha,\psi}} \psi\left(\frac{|f'(0)|}{\|f\|_{\alpha,\psi}}\right)\varepsilon,$$

for any function $f \in A_\alpha^\psi$ with $f(0) = 0$ and $\|f\|_{\alpha,\psi} > 1$.

Proof. If f is a function in A_α^ψ satisfying $f(0) = 0$ and $\|f\|_{\alpha,\psi} > 1$, we set $g(z) = (1/\|f\|_{\alpha,\psi}) f(z)$, for all $z \in \mathbb{D}$. Then, applying Theorem 3.2 to the function g yields

$$\int_{f^{-1}(\Sigma_\varepsilon)} \psi\left(\frac{|f(z)|}{\|f\|_{\alpha,\psi}}\right) dA_\alpha(z) \geq K_6(\alpha, \psi)\psi\left(\frac{|f'(0)|}{\|f\|_{\alpha,\psi}}\right)\frac{|f'(0)|^2}{\|f\|_{\alpha,\psi}^2}\varepsilon.$$

Since $\psi(rt) \leq r\psi(t)$ for all $r \in (0, 1)$ and $t \geq 0$, the proof is complete. □

Let us recall that it is a standard fact that if ψ is a function as in Theorem 1.1 and $t_0 > 0$ is fixed, then (see [5, p. 49])

$$\psi(t) \leq \frac{\psi(t_0)}{t_o^{K_\Delta}} t^{K_\Delta} \quad \forall t \geq t_0. \tag{3.5}$$

Proof of Theorem 1.1. Let f be a univalent function in A_α^ψ satisfying $f(0) = 0$ and $|f'(0)| = 1$. Note that if $\|f\|_{\alpha,\psi} \leq 1$ then Theorem 1.1 is a simple consequence of Lemma 2.1. So, we can assume that $\|f\|_{\alpha,\psi} > 1$. It follows from the distortion in (i) of Theorem B that

$$|f'(0)| \geq \tfrac{1}{4}(1 - |z|^2)^2 |f(z)| \quad \forall z \in \mathbb{D}.$$

Thus, $(1 - |z|^2)^{2K_\Delta} |f(z)|^{K_\Delta} \leq 4^{K_\Delta}$, for all $z \in \mathbb{D}$, and integrating against the measure dA_β, $\beta = \alpha - 2K_\Delta > -1$, we obtain

$$\|f\|_{\alpha, K_\Delta} \leq C_2, \tag{3.6}$$

C_2 being a constant depending on α and K_Δ. Moreover, since ψ is continuous and $\psi(0) = 0$, we can choose $t_0 > 0$ such that

$$\psi(t_0) < \frac{1}{C_1(\alpha)}\left(\frac{1}{8C_2}\right)^{K_\Delta}, \tag{3.7}$$

where $C_1(\alpha)$ is the constant that we found in Lemma 3.1. Using again (i) in Theorem B

$$|f(z)| \geq \frac{|z|}{(1 + |z|)^2} \quad \forall z \in \mathbb{D}.$$

In particular, $|f(z)| \geq \frac{1}{8}$, whenever $|z| > \frac{1}{2}$. Putting $M = 1/(8t_0)$, we can write

$$\frac{|f(z)|}{M} \geq t_0 \quad \forall |z| > \frac{1}{2}.$$

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Now, estimates (3.5) and (3.7) yield

$$\psi\left(\frac{|f(z)|}{M}\right) \le \frac{\psi(t_0)}{t_0^{K_\Delta}} \frac{|f(z)|^{K_\Delta}}{M^{K_\Delta}} \le \frac{|f(z)|^{K_\Delta}}{C_2^{K_\Delta} C_1(\alpha)},$$

for any $|z| > \frac{1}{2}$. Thus,

$$\int_{\{|z|>\frac{1}{2}\}} \psi\left(\frac{|f|}{M}\right) dA_\alpha \le \frac{(1/C_2)^{K_\Delta}}{C_1(\alpha)} \int_{\mathbb{D}} |f|^{K_\Delta} \, dA_\alpha \le \frac{1}{C_1(\alpha)},$$

by (3.6). Now, from Lemma 3.1 we get

$$\int_{\mathbb{D}} \psi\left(\frac{|f|}{M}\right) dA_\alpha \le 1,$$

which, by definition of the Luxemburg norm, means that

$$\|f\|_{\alpha,\psi} \le M.$$

This inequality and Corollary 3.3 lead to

$$\int_{f^{-1}(\Sigma_\varepsilon)} \psi(|f(z)|) \, dA_\alpha(z) \ge \frac{\delta}{M^2} \psi\left(\frac{1}{M}\right) \varepsilon \|f\|_{\alpha,\psi},$$

and the proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.2

Let f be a conformal map on $\mathbb{D}$ such that $f(0) = 0 = 1 - |f'(0)|$ and let $f(\mathbb{D}) = \Omega$. In this section we will assume that Q_n, Ω_n, z_n, and so on are as in the construction of the covering of the domain Ω in Section 2. The proof of Theorem 1.2 will based upon the following lemma.

Lemma 4.1. *Let φ be as in Theorem 1.2 and let $\alpha \ge 0$ and $\varepsilon > 0$. Then, there exists a constant $C = C(\alpha, \varphi, \varepsilon) > 0$ such that, for any $n \in \mathbb{N} \cup \{0\}$,*

$$\int_{f^{-1}(Q_n)} \varphi(|f(z)|) \, dA_\alpha(z) \ge C \int_{f^{-1}(\Omega_n)} \varphi(|f(z)|) \, dA_\alpha(z),$$

for every univalent function f in $\mathbb{D}$ satisfying $f(0) = 0$ and $|f'(0)| = 1$.

Proof. We first notice that, for any $\alpha \ge 0$ and $n \in \mathbb{N} \cup \{0\}$, there exists a constant $C(\alpha, \varepsilon) > 0$ such that

$$(1 - |z|^2)^\alpha \ge C(\alpha, \varepsilon)(1 - |z_n|^2)^\alpha, \tag{4.1}$$

for every $z \in f^{-1}(Q_n)$. When $n = 0$ we replace z_n by 0 and, analogously there exists a constant $C(\alpha, \varepsilon) > 0$ such that

$$(1 - |z|^2)^\alpha \le C(\alpha, \varepsilon)(1 - |z_n|^2)^\alpha, \tag{4.2}$$

for any $z \in f^{-1}(\Omega_n)$ (see [10]).

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Thus, from (4.1) and (4.2) we see that it will be sufficient to show that

$$\int_{f^{-1}(Q_n)} \varphi(|f(z)|)\, dA(z) \geq C \int_{f^{-1}(\Omega_n)} \varphi(|f(z)|)\, dA(z), \tag{4.3}$$

for all $n \in \mathbb{N} \cup \{0\}$. With this aim in mind, let us consider first the case $n = 0$. Since φ is an increasing function,

$$\begin{aligned}
\int_{f^{-1}(\Omega_0)} \varphi(|f|)\, dA &\leq \sum_{j=0}^{\infty} \varphi((1+\varepsilon)^j) \operatorname{Area}(f^{-1}(\Omega_0 \cap A_j)) \\
&\leq C(\varepsilon) \sum_{j=0}^{\infty} \varphi((1+\varepsilon)^j) e^{-\beta_\Omega(0, \Omega_n \cap A_j)} \omega_\Omega(0,\, \Omega_n \cap A_j) \\
&\leq C(\varepsilon) \sum_{j=0}^{\infty} \varphi((1+\varepsilon)^j)(1+\varepsilon)^{-j(1+C\varepsilon)},
\end{aligned}$$

where we have used part (ii) of Proposition D in the second inequality and Proposition C in the last one. Set

$$S(\varepsilon) = \sum_{j=0}^{\infty} (1+\varepsilon)^{-j(1+C\varepsilon)},$$

where C is the constant given in Proposition C. Since φ is concave, we can write

$$\int_{f^{-1}(\Omega_0)} \varphi(|f|)\, dA \leq C(\varepsilon) S(\varepsilon) \varphi\bigg(\frac{1}{S(\varepsilon)} \sum_{j=0}^{\infty} (1+\varepsilon)^{-jC\varepsilon}\bigg),$$

and thus there is a constant $C(\varepsilon, \varphi) > 0$, depending only on ε and φ, such that

$$\int_{f^{-1}(\Omega_0)} \varphi(|f|)\, dA \leq C(\varepsilon, \varphi). \tag{4.4}$$

Then (4.4) and Lemma 2.1 show that (4.3) holds for $n = 0$.

Suppose now that $n > 0$. Since φ is an increasing function, assertions (ii)–(iv) in Proposition D and Proposition C imply that

$$\begin{aligned}
\int_{f^{-1}(\Omega_n)} \varphi(|f|)\, dA &\leq C(\varepsilon) \sum_{j=0}^{\infty} \varphi((1+\varepsilon)^j) e^{-\beta_\Omega(0, \Omega_n \cap A_j)} \omega_\Omega(0,\, \Omega_n \cap A_j) \\
&\leq C(\varepsilon) \sum_{j=0}^{\infty} \varphi((1+\varepsilon)^j) e^{-\beta_\Omega(0, Q_n)} \omega_\Omega(0,\, Q_n) e^{-\beta_\Omega(w_n, \Omega_n \cap A_j)} \\
&\qquad \times \omega_\Omega(w_n,\, \Omega_n \cap A_j) \\
&\leq C(\varepsilon) \sum_{j=0}^{\infty} \varphi((1+\varepsilon)^j) e^{-\beta_\Omega(0, Q_n)} \\
&\qquad \times \omega_\Omega(0,\, Q_n)(1+\varepsilon)^{-|j-n|(1+C\varepsilon)}.
\end{aligned}$$

Also, from assertion (i) in Proposition D we get

$$\int_{f^{-1}(Q_n)} \varphi(|f|)\,dA \geq C(\varepsilon)\varphi((1+\varepsilon)^{n-1})e^{-\beta_\Omega(0,Q_n)}\omega_\Omega(0,\ Q_n).$$

Hence, there is a constant $C(\varepsilon) > 0$ such that

$$\begin{aligned}&\int_{f^{-1}(\Omega_n)} \varphi(|f|)\,dA\\ &\quad\leq C(\varepsilon)\sum_{j=0}^{\infty}\varphi((1+\varepsilon)^j)\frac{(1+\varepsilon)^{-|j-n|(1+C\varepsilon)}}{\varphi((1+\varepsilon)^{n-1})}\int_{f^{-1}(Q_n)} \varphi(|f|)\,dA.\end{aligned}$$

We claim that there exists a constant $C(\varepsilon) > 0$ (not depending on n) such that

$$\sum_{j=0}^{\infty}\varphi((1+\varepsilon)^j)(1+\varepsilon)^{-|j-n|(1+C\varepsilon)} \leq C(\varepsilon)\varphi((1+\varepsilon)^{n-1}). \tag{4.5}$$

Indeed, for each $n > 0$ we set

$$S_n(\varepsilon) = \sum_{j=0}^{\infty}(1+\varepsilon)^{-|j-n|(1+C\varepsilon)}.$$

Due to the concavity of φ we can write

$$\begin{aligned}&\sum_{j=0}^{\infty}\varphi((1+\varepsilon)^j)(1+\varepsilon)^{-|j-n|(1+C\varepsilon)}\\ &\qquad\leq S_n(\varepsilon)\varphi\bigg(\frac{1}{S_n(\varepsilon)}\sum_{j=0}^{\infty}(1+\varepsilon)^{j-|j-n|(1+C\varepsilon)}\bigg)\\ &\qquad= S_n(\varepsilon)\varphi\bigg(\frac{(1+\varepsilon)^n}{S_n(\varepsilon)}\sum_{j=0}^{\infty}(1+\varepsilon)^{j-n-|j-n|(1+C\varepsilon)}\bigg),\end{aligned}$$

and, by the definition of $S_n(\varepsilon)$,

$$\begin{aligned}S_n(\varepsilon) &= \sum_{j=0}^{n-1}(1+\varepsilon)^{-(n-j)\,(1+C\varepsilon)} + \sum_{j=n}^{\infty}(1+\varepsilon)^{-(j-n)\,(1+C\varepsilon)}\\ &\geq \sum_{k=0}^{\infty}(1+\varepsilon)^{-k(1+C\varepsilon)} = K_1(\varepsilon).\end{aligned}$$

Notice that we also have $S_n(\varepsilon) \leq 2K_1(\varepsilon)$, and since

$$\begin{aligned}\sum_{j=0}^{\infty}(1+\varepsilon)^{j-n-|j-n|(1+C\varepsilon)} &= \sum_{j=0}^{n-1}(1+\varepsilon)^{-(n-j)\,(2+C\varepsilon)} + \sum_{j=n}^{\infty}(1+\varepsilon)^{-(j-n)C\varepsilon}\\ &\leq \sum_{k=0}^{\infty}(1+\varepsilon)^{-k(2+C\varepsilon)} + \sum_{k=0}^{\infty}(1+\varepsilon)^{-kC\varepsilon} = K_2(\varepsilon),\end{aligned}$$

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we obtain that there are positive constants $K_1(\varepsilon)$ and $K_3(\varepsilon)$, not depending on n, such that

$$\sum_{j=0}^{\infty} \varphi((1+\varepsilon)^j)(1+\varepsilon)^{-|j-n|(1+C\varepsilon)} \leq K_1(\varepsilon)\varphi(K_3(\varepsilon)\,(1+\varepsilon)^n). \tag{4.6}$$

On the other hand, if $N(\varepsilon)$ is a positive integer, depending only on ε, such that $K_3(\varepsilon) \leq N(\varepsilon)$, then subadditivity of the function φ allows us to write

$$\begin{aligned}\varphi(K_3(\varepsilon)\,(1+\varepsilon)^n) &\leq N(\varepsilon)\varphi((1+\varepsilon)^n)\\ &= N(\varepsilon)\varphi((1+\varepsilon)^{n-1} + \varepsilon(1+\varepsilon)^{n-1})\\ &\leq 2N(\varepsilon)\varphi((1+\varepsilon)^{n-1}),\end{aligned}$$

where we have used the fact that φ is increasing. The last estimate and (4.6) give (4.5), from which (4.3) follows as wished. The proof of Lemma 4.1 is complete. □

PROOF OF THEOREM 1.2. We will see that Theorem 1.2 is, in fact, a corollary of Lemma 4.1. Indeed, since $\bigcup Q_n \subset \Sigma_\varepsilon$ and $\bigcup f^{-1}(\Omega_n) = \mathbb{D}$, we can write

$$\begin{aligned}\int_{f^{-1}(\Sigma_\varepsilon)} \varphi(|f(z)|)\,dA_\alpha(z) &\geq \int_{f^{-1}(\cup Q_n)} \varphi(|f(z)|)\,dA_\alpha(z)\\ &= \sum_{n=0}^{\infty} \int_{f^{-1}(Q_n)} \varphi(|f(z)|)\,dA_\alpha(z)\\ &\geq C(\alpha,\varphi,\varepsilon) \sum_{n=0}^{\infty} \int_{f^{-1}(\Omega_n)} \varphi(|f(z)|)\,dA_\alpha(z)\\ &\geq C(\alpha,\varphi,\varepsilon) \int_{f^{-1}(\cup\Omega_n)} \varphi(|f(z)|)\,dA_\alpha(z)\\ &= C(\alpha,\varphi,\varepsilon) \int_{\mathbb{D}} \varphi(|f(z)|)\,dA_\alpha(z).\end{aligned}$$

This finishes the proof. □

REMARK 4.2. In general, Theorem 1.2 is not true for $\alpha \in (-1, 0)$. In fact, in [10] we gave a counterexample with $\alpha < 0$ and $\varphi(x) = x$.

As an immediate consequence of Theorem 1.2 we can state the following result.

COROLLARY 4.3. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a concave modulus function, $\alpha \geq 0$ and $\varepsilon > 0$. Then there exists a constant $\delta > 0$, depending on α, φ and ε, such that:*

(a) *if $f \in A_\alpha^\varphi$ is a conformal map with $f(0) = 0$ and $|f'(0)| \leq 1$ then*

$$\int_{f^{-1}(\Sigma_\varepsilon)} \varphi(|f(z)|)\,dA_\alpha(z) \geq \delta|f'(0)|\|f\|_{\alpha,\varphi};$$

(b) *if* $f \in A^{\varphi}_{\alpha}$ *is a conformal map with* $f(0) = 0$ *and* $|f'(0)| > 1$ *then*

$$\int_{f^{-1}(\Sigma_\varepsilon)} \varphi(|f(z)|)\, dA_\alpha(z) \geq \frac{\delta}{|f'(0)|} \|f\|_{\alpha,\varphi}.$$

PROOF. Let us consider the function $g(z) = 1/(|f'(0)|) f(z)$, and use Theorem 1.2, bearing in mind that $\varphi(\lambda x) \geq \lambda\varphi(x)$ for every $\lambda \in (0, 1)$ and any $x \geq 0$. □

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