HERMITE-BIRKHOFF TRIGONOMETRIC INTERPOLATION IN THE (0, 1, 2, M) CASE

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1. Introduction

Following four important papers on Birkhoff interpolation by Turán and his associates ([2], [3], [4], [14]), Kis ([8], [9]) proved the following theorems.

THEOREM 1. (Kis). *Let $f(z)$ be analytic in $|z| < 1$ and continuous on $|z| \leqq 1$. Let $R_n(z)$ be the unique interpolation polynomial of degree $\leqq 2n-1$ in z such that*

(1.1) $$R_n(z_{kn}) = f(z_{kn}),\ f(z_{kn}),\ R''_n(z_{kn}) = \beta_{kn},\ k = 1,2,\cdots,n$$

(1.2) $$z_{kn} = \exp \frac{2\pi k i}{n},\ k = 1,2,\cdots,n$$

(1.3) $$\beta_{kn} = o\left(\frac{n^2}{\log n}\right),\ k = 1,2,\cdots,n$$

(1.4) $$\underset{\delta \to 0+}{\omega(\delta)\log\delta} = 0.$$

Then $R_n(z)$ converges uniformly to $f(z)$ in $|z| \leqq 1$.

Here $\omega(\delta)$ denotes the modulus of continuity of $f(z)$. Theorem 1 is best possible in the sense that the freedom of β_{kn} cannot be improved. The above theorem is surprising, in view of the arbitrariness of the numbers β_{kn} which satisfy only the order relation (1.3).

THEOREM 2. (Kis). *Let $f(x)$ be a 2π periodic continuous function satisfying Zygmund's condition*

(1.5) $$f(x+h) - 2f(x) + f(x-h) = o(h).$$

Let $R_n(x)$ be the unique trigonometric polynomial of order n satisfying the conditions

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(1.6) $$R_n(x_{kn}) = f(x_{kn}),\ R_n''(x_{kn}) = \delta_{kn},\ k = 1, 2, \cdots, n,$$

$$\left[R_n(x) = a_0 + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) + a_n \cos nx\right]$$

(1.7) $$x_{kn} = \frac{2k\pi}{n},\ k = 0, 1, \cdots, n-1.$$

Then $R_n(x)$ will converge uniformly to $f(x)$ on the real axis provided that

(1.8) $$\delta_{kn} = o(n).$$

Theorem 2 is best possible in the sense that Zygmund's condition cannot be replaced by

(1.9) $$f(x+h) - 2f(x) + f(x-h) = O(h).$$

For this result we refer to the work of Vertesi [20].

Theorem 2 of Kis has received the following generalization. Sharma and the author [11] have considered the problem of $(0, M)$ interpolation. Here the interpolation trigonometric polynomial $R_n(x)$ of order n is given by

(1.10) $$R_n(x_{kn}) = f(x_{kn}),\ R_n^{(M)}(x_{kn}) = \beta_{kn},\ x_{kn} = \frac{2k\pi}{n},\ k = 0, 1, \cdots, n-1$$

(M being a fixed positive integer $\geqq 1$). The trigonometric polynomial $R_n(x)$ given by (1.10) has the following form:

(1.11) $$R_n(x) = a_0 + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) + b_n \sin nx\ (M\text{-odd}) + a_n \cos nx\ (M\text{-even}).$$

The main theorem of the above paper is as follows.

THEOREM 3. (Sharma and Varma). *Let $f(x)$ be a 2π periodic continuous function. Let M be any fixed odd positive integer. Suppose β_{kn} as stated in* (1.10) *satisfy*

(1.12) $$\beta_{kn} = O\left(\frac{n^M}{\log n}\right), \qquad k = 0, 1, \cdots, n-1.$$

Then $R_n(x)$ as defined by (1.10) *and* (1.11) *converges uniformly to $f(x)$ on the real line.*

For the case M-even, let $f(x)$ satisfy the Zygmund condition (1.5) *and*

(1.13) $$\beta_{kn} = o(n^{M-1}), \qquad k = 0, 1, \cdots, n-1.$$

Then $R_n(x)$ will converge uniformly to $f(x)$ on the real line.

Motivated by the above theorem on trigonometric interpolation, the author has considered the problem of $(0,1,M)$ interpolation on the nodes $x_{kn} = 2k\pi/n$, $k = 0, 1, \cdots, n-1$. By $(0,1,M)$ interpolation we mean the problem of finding interpolatory polynomials of suitable form for which the values, first derivative and Mth derivative, are prescribed at n distinct points. It turns out that these interpolation polynomials exist uniquely only when M is an even integer. In this case we proved that the interpolation polynomials converge uniformly to $f(x)$, provided $f(x) \in \text{Lip}\ \alpha\ 0 < \alpha \leqq 1$. For details we refer to Theorem 2.2 in [15]. Thus, by prescribing also the first derivative of the interpolation polynomials, one obtains a convergence theorem for a much wider class of functions than in $(0,M)$ interpolation (for M-even). But, by doing so, we have increased the order of the trigonometric polynomials. It is also interesting to compare the results of $(0,2,3)$ interpolation [12] with $(0,3)$ case as well. We know from Theorem 3 that $R_n(x)$, obtained from the consideration of $(0,3)$ interpolation, converges uniformly to $f(x)$ for just 2π periodic continuous functions, whereas, in the case of $(0,2,3)$ interpolation we need at least $f(x) \in \text{Lip}\ \alpha$, $0 < \alpha < 1$. Thus, by prescribing also the second derivative of the interpolation polynomials one obtains convergence theorems for much narrower class than in $(0,3)$ interpolation.

2. Statement of results

The object of this paper is to consider the following problem: Let M be a fixed odd positive integer $\geqq 3$. Let $R_n(x)$ be a trigonometric sum of order $2n$ (of the form)

$$(2.1)\qquad d_0 + \sum_{j=1}^{2n-1} (d_j \cos jx + e_j \sin jx) + e_{2n} \sin 2nx.$$

We ask the following question; *Does there exist a unique trigonometric sum of order $2n$ which satisfies* (2.1) *and*

$$(2.2)\qquad \begin{aligned} &R_n(x_{in}) = f(x_{in}),\ R_n'(x_{kn}) = \alpha_{in}, \\ &R_n''(x_{in}) = \beta_{in},\ R_n^{(M)}(x_{in}) = \delta_{in}, \end{aligned} \qquad i = 0, \cdots, n-1\ ?$$

Here x_{in} are given by (1.7). It turns out that the answer to the above question is in the affirmative. We call it $(0,1,2,M)$ trigonometric interpolation. We will show that under suitable restrictions on α_{in}, β_{in}, δ_{in} and $f(x) \in c_{2\pi}$, $R_n(x)$ will converge uniformly to $f(x)$ on the real line. We will also prove some inequalities on trigonometric polynomials analogous to Fejer [7]. For the case when M is even, the results are analogous to the case $(0,1,2,4)$ trigonometric interpolation, which has been dealt with already in my earlier work [17]. Now, we state the main theorem of this paper.

THEOREM 4. *Let $f(x) \in C_{2\pi}$. Then $R_n(f)$ defined by* (2.1) *and* (2.2) *converges uniformly to $f(x)$ on the real line provided that*

$$(2.3)\qquad \alpha_{in} = o\left(\frac{n}{\log n}\right),\ \beta_{in} = o(n^2),\ \delta_{in} = o\left(\frac{n^M}{\log n}\right)\ i = 0, 1, \cdots, n-1.$$

The freedom of these numbers is best possible.

The main interest of the above theorem lies in the fact that as far as the freedom of β_{kn} is concerned, we need only $\beta_{kn} = o(n^2)$. See also the remarks at the end of the paper.

THEOREM 5. *Let $\phi_n(x)$ be any trigonometric polynomial of an order $\leqq 2n$ and satisfying* (2.1). *Let further*

$$(2.4)\qquad |\phi_n^{(j)}(x_{in})| \leqq a_j,\ j = 0, 1, 2, M,\ i = 0, 1, \cdots, n-1.$$

Then we have for $0 \leqq x \leqq 2\pi$,

$$(2.5)\qquad |\phi_n(x)| \leqq c_0\left(a_0 + a_1 \frac{\log n}{n} + \frac{a_2}{n^2} + \frac{a_M}{n^M}\log n\right).$$

Here c_0 is a definite constant independent of n and x. (2.5) *is best possible in the sense that there exists a trigonometric polynomial $g_n(x)$ of the order $2n$ satisfying* (2.1) *and $|g_n^j(x_{in})| = a_j$, $j = 0, 1, 2, M$, $i = 0, 1, \cdots, n-1$, and for which*

$$(2.6)\qquad |g_n(\pi)| > c_1\left(a_0 + a_1\frac{\log n}{n} + \frac{a_2}{n^2} + \frac{a_M}{n^M}\log n\right).$$

THEOREM 6. *Let $f(x) \in c_{2\pi}$ have $\omega(\delta)$ as its module of continuity. Then under the assumption $\alpha_{in} = \beta_{in} = \delta_{in} = 0$,*

$$(2.7)\qquad |R_n(x) - f(x)| \leqq c_2\omega\left(\frac{1}{\sqrt{n}}\right).$$

THEOREM 7. *The explicit representation of $R_n(x)$ is given by*

$$(2.8)\qquad \begin{aligned} R_n(x) = &\sum_{k=0}^{n-1} f(x_{kn})A(x - x_{kn}) + \sum_{k=0}^{n-1} \alpha_{kn}B(x - x_{kn}) \\ &+ \sum_{k=0}^{n-1} \beta_{kn}C(x - x_{kn}) + \sum_{k=0}^{n-1} \delta_{kn}D(x - x_{kn}), \end{aligned}$$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are defined in (2.15), (2.13), (2.11) *and* (2.9) *respectively.*

Here $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are given by:

$$(2.9)\qquad D(x) = \frac{2(-1)^{(M+1)/2}(1 - \cos nx)}{n}\left[2\sum_{j=1}^{n-1}\frac{\sin jx}{a_{j,M}} + \frac{\sin nx}{a_{n,M}}\right],$$

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where

(2.10) $$a_{j,M} = (2n-j)^M + (n+j)^M - 3\{(n-j)^M + j^M\},$$

(2.11) $$C(x) = \frac{(1-\cos nx)}{n^3}\left[1 + 2\sum_{j=1}^{n-1} \frac{b_{j,M}}{a_{j,M}}\cos jx\right],$$

where

(2.12) $$b_{j,M} = (2n-j)^M - 2(n-j)^M - j^M,$$

(2.13) $$B(x) = G(x) + \frac{(1-\cos nx)}{n^3}\left[2\sum_{j=1}^{n-1} \frac{c_{j,M}}{a_{j,M}}\sin jx + \frac{c_{n,M}}{a_{n,M}}\sin nx\right],$$

where

(2.14) $$c_{j,M} = (3n-2j)j^M + 4(n-j)^{M+1} - (n-2j)(2n-j)^M,$$

(2.15) $$A(x) = F(x) + \frac{2(1-\cos nx)}{n^3}\sum_{j=1}^{n-1} \frac{d_{j,M}}{a_{j,M}}\cos jx,$$

where

(2.16) $$d_{j,M} = j(n-j)(2n-j)\{(2n-j)^{M-1} - 2(n-j)^{M-1} + j^{M-1}\}.$$

Here $F(x)$ and $G(x)$ are fundamental polynomials of Hermite interpolation (see [11]) and they are given by

(2.17) $$F(x) = \frac{1}{n}\left[1 + \frac{2}{n}\sum_{j=1}^{n-1}(n-j)\cos jx\right],$$

(2.18) $$G(x) = \frac{1}{n^2}\left[2\sum_{j=1}^{n-1}\sin jx + \sin nx\right].$$

REMARK 1. It is interesting to mention that for $M = 3$. the fundamental polynomials $A(x)$ and $C(x)$ are nonnegative, but for the cases when $M > 3$, this property, in general, breaks down. Indeed for $M = 3$ we have

(2.19) $$D(x) = \frac{(1-\cos nx)G(x)}{3n^2},$$

(2.20) $$C(x) = \frac{(1-\cos nx)F(x)}{n^2},$$

(2.21) $$B(x) = F(x)G(x) - C(x) + \frac{(n^2-1)}{2}D(x)$$

and

(2.22) $$A(x) = F^2(x) + \frac{(n^2-1)}{3}C(x).$$

3. Preliminaries

Here we state those results which we shall require in the proof of theorems stated in Article 2.

Following identities are easy to obtain from (2.8):

$$\sum_{k=0}^{n-1} A(x-x_{kn}) \equiv 1, \tag{3.1}$$

$$\sum_{k=0}^{n-1} C(x-x_{kn}) = \frac{1-\cos nx}{n^2}. \tag{3.2}$$

From (see Zygmund [21]) the known results due to Jackson we have:

$$\sum_{k=0}^{n-1} F(x-x_{kn}) \equiv 1, \quad \sum_{k=0}^{n-1} \left|G(x-x_{kn})\right| \lneqq \frac{2}{n}\log n, \tag{3.3}$$

$$\left|G(x-x_{kn})\right| \leqq \frac{2}{n}, \; k=0,1,\cdots,n-1. \tag{3.4}$$

Following the arguments given in Jackson [5] we have for $x \neq x_{kn}$:

$$\sum_{k=0}^{n-1} \max_{1\leqq p\leqq n} \left|\sum_{j=1}^{p} \sin j(x-x_{kn})\right| \leqq 4n\log n. \tag{3.5}$$

Let $0 \leqq \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_p$ then we have

$$\left|\sum_{j=1}^{p} \alpha_j \sin jx\right| \leqq 2\alpha_p \max_{1\leqq v\leqq p} \left|\sum_{j=1}^{v} \sin jx\right|. \tag{3.6}$$

Similarly, if $\alpha_1 \geqq \alpha_2 \geqq \alpha_3 \geqq \cdots \geqq \alpha_p$ then we have

$$\left|\sum_{j=1}^{p} \alpha_j \sin jx\right| \leqq 2\alpha_1 \max_{1\leqq v\leqq p} \left|\sum_{j=1}^{v} \sin jx\right|. \tag{3.7}$$

Proof of (3.6 and (3.7) follows easily from Abel's Lemma.

We denote the Fejér-kernel by

$$\tau_{j,k}(x) = 1 + \frac{2}{j}\sum_{i=1}^{j-1} (j-i)\cos i(x-x_{kn}). \tag{3.8}$$

It is easy to verify the following properties of Fejér-kernel:

$$\sum_{k=0}^{n-1} \tau_{j,k}(x) = n, \quad \tau_{j,k}(x) = \frac{1}{j}\left[\frac{\sin\frac{j(x-x_{kn})}{2}}{\sin\frac{x-x_{kn}}{2}}\right]^2 \tag{3.9}$$

and

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(3.10) $\quad (j+1)\tau_{j+1,k}(x) - 2j\tau_{j,k}(x) + (j-1)\tau^{(x)}_{j-1,k} = 2\cos j(x - x_{kn}).$

Let $a_{j,M}$ be defined as given in (2.10). Denote $a''_{j,M}$, the second derivative of $a_{j,M}$, with respect to j. By using

(3.11) $$(2n-j)^M \geqq 2^M(n-j)^M + j^M, \ (n+j)^M \geqq (n-j)^M + 2^M j^M,$$

it follows that

(3.12) $$a_{j,M} > 0, \ a''_{j,M} > 0 \text{ for } M \geqq 3.$$

By using (3.12), we conclude that $a'_{j,M}$ is an increasing function of j, for $j = 0, 1, \cdots, n$. But $a'_{m,M} \leqq 0$ $(n = 2m+1)$, or $(n = 2m)$. On account of

$$a_{j,M} = a_{n-j,M} \text{ for } j = 1, 2, \cdots, m,$$

we finally obtain that $a_{j,M}$ is a decreasing function of j, for $j = 0, 1, \cdots, m$ and an increasing function of j for $j = m+1, \cdots, n$. From these observations we easily conclude that:

(3.13) $$n(3m)^{M-1} < a_{j,M} < (2n)^M,$$

(3.14) $$|a'_{j,M}| < 2M(2n)^{M-1},$$

(3.15) $$|a''_{j,M}| < M(M-1)(2n)^{M-2},$$

(3.16) $$\frac{|a_{j+1,M} - a_{j,M}|}{|a_{j,M}|\,|a_{j+1,M}|} \leqq \frac{M}{n^{M+1}}.$$

4. Upper estimates of the fundamental polynomials

Here, we shall prove the following result:

LEMMA 4.1. *The following estimates are valid:*

(4.1) $$\sum_{k=0}^{n-1} |D(x - x_{kn})| \leqq \frac{25 \log n}{n^M},$$

(4.2) $$\sum_{k=0}^{n-1} |C(x - x_{kn})| \leqq \frac{f_1}{n^2},$$

(4.3) $$\sum_{k=0}^{n-1} |B(x - x_{kn})| \leqq \frac{f_2 \log n}{n},$$

(4.4) $$\sum_{k=0}^{n-1} |A(x - x_{kn})| \leqq f_3.$$

Here f_1, f_2, f_3 are positive constants independent of n and x.

PROOF. We note that for $M = 3$, we have more precise constants. In this case, we have:

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(4.1a) $$\sum_{k=0}^{n-1} |D(x-x_{kn})| \leqq \frac{2\log n}{n^3},$$

(4.2a) $$\sum_{k=0}^{n-1} |C(x-x_{kn})| \leqq \frac{2}{n^2},$$

(4.3a) $$\sum_{k=0}^{n-1} |B(x-x_{kn})| \leqq \frac{2\log n}{n},$$

(4.4a) $$\sum_{k=0}^{n-1} |A(x-x_{kn})| \leqq 1.$$

This follows immediately by using (2.19)–(2.22), (3.3) and (3.4).

First, we prove (4.1). We note that (4.1)–(4.4) are valid for $x = x_{in}$, $i = 0, 1, \cdots, n-1$. Let $x \neq x_{in}$ and let $n = 2m$ (Proof for $n = 2m+1$ is similar). From (2.9) we have

$$|D(x-x_{kn})| \leqq \frac{4}{n}\left[2\left|\sum_{j=1}^{m}\frac{\sin j(x-x_{kn})}{a_{j,M}}\right| + \frac{1}{(2^M-2)n^M} + 2\left|\sum_{j=m+1}^{n-1}\frac{\sin j(x-x_{kn})}{a_{j,M}}\right|\right].$$

Since $a_{j,M}$ is a decreasing function of j for $j = 0, 1, \cdots,$ and increasing function of j for $j = m+1, \cdots, n$ (see Art 3), by using (3.6) and (3.7) we obtain

$$|D(x-x_{kn})| \leqq \frac{4}{n}\left[\frac{4}{m^M(3^M-3)}\max_{1\leqq v\leqq n-1}\left|\sum_{j=1}^{v}\sin j(x-x_{kn})\right| + \frac{1}{(2^M-2)n^M}\right].$$

Now, we note that for $M \geqq 3$ we have

$$\frac{1}{3^M-3} < 3.2^{-M-3}.$$

Therefore, by using (3.5) and the above estimates we obtain

$$\sum_{k=0}^{n-1}{}' |D(x-x_{kn})| \leqq \frac{4}{n}\left[\frac{3}{2n^M}4n\log n + \frac{1}{4n^{M-1}}\right] \leqq \frac{25\log n}{n^M},$$

which proves (4.1). To prove (4.2) we need some estimates of the coefficients involved in $C(x-x_{kn})$. First we observe from (2.12) and (3.11) that

$$b_{j,M} = (2n-j)^M - 2(n-j)^M - j^M \geqq 0.$$

Next, we note that

$$b''_{j,M} = M(M-1)b_{j,M-2} \geqq 0.$$

From this it follows that $b'_{j,M}$ is a monotonic increasing function of j. It is easy to check that $b'_{j,M} \leqq 0$ for $j = 0, 1, \cdots, n$. Therefore, $b_{j,M}$ is a monotonic decreasing function of j for $j = 0, 1, \cdots, n$. Thus, we obtain the following estimates:

$$0 \leqq b_{j,M} \leqq 2n^M, \quad |b'_{j,M}| < M(2n)^{M-1}.$$

By using the estimates of $a_{j,M}$, $a'_{j,M}$, $a''_{j,M}$, as given in (3.13)–(3.15), and the above estimates of $b_{j\,M}$, we finally obtain

$$\left| \left[\frac{b_{j,M}}{a_{j,M}} \right]'' \right| \leqq \frac{f_4(M)}{n^2}.$$

With the help of (3.10) and (3.8), $C(x)$ (as stated in (2.11)) can be rewritten in the form

$$C(x - x_{kn}) = \frac{(1 - \cos nx)}{n^3} \left[1 + \sum_{j=1}^{n-1} \frac{b_{j,M}}{a_{j,M}} \{(j+1)\tau_{j+1,k}(x) - 2j\,\tau_{j,k}(x) + (j-1)\tau_{j-1,k}(x)\} \right].$$

Let us write:

$$p(j,M) = \frac{b_{j,M}}{a_{j,M}}, \quad g(j,M) = p(j+1,M) - 2p(j,M) + p(j-1,M) \tag{4.6}$$

so that $p(o,M) = 1$ and $p(n,M) = 0$. Thus, $C(x - x_{kn})$ can be expressed in the form

$$C(x - x_{kn}) = \frac{(1 - \cos nx)}{n^3} \left[\sum_{j=1}^{n-1} g(j,M) j\,\tau_{j,k}(x) + n\,\tau_{n,k}(x)\, p(n-1,M) \right]. \tag{4.7}$$

From (4.5) it follows that

$$|g(j,M)| = |(\xi_2 - \xi_1)\, p''(\xi,M)| < \frac{2f_4(M)}{n^2}, \tag{4.8}$$

where $j - 1 < \xi_2 < j < \xi_1 < j + 1$. Further, it is easy to verify that

$$|p(n-1,M)| \leqq \frac{f_5(M)}{n^2}. \tag{4.9}$$

On using (3.9) and (4.7)–(4.9), we obtain

$$\sum_{k=0}^{n-1} |C(x - x_{kn})| \leqq \frac{2}{n^3}[(n-1)f_4(M) + f_M(M)],$$

$$\leqq \frac{f_1(M)}{n^2}.$$

This proves (4.2). The proof for (4.3) is similar to (4.1) and the proof for (4.4) is similar to (4.2). We omit the proof.

5. Lower estimates of the fundamental polynomials

The inequalities of Lemma 4.1 are, in a sense, best possible as is shown by the following lemma.

LEMMA 5.1. *There exist positive constants $f_7(M)$ and $f_8(M)$ for which the following inequalities hold true for $n = 2m + 1$:*

$$\sum_{k=0}^{n-1} |A(\pi - x_{kn})| \geqq 1, \tag{5.1}$$

$$\sum_{k=0}^{n-1} |B(\pi - x_{kn})| \geqq \frac{f_7(M)\log n}{n}, \tag{5.2}$$

$$\sum^{n-1} |C(\pi - x_{kn})| \geqq \frac{2}{n^2}, \tag{5.3}$$

$$\sum_{k=0}^{n-1} |D(\pi - x_{kn})| \geqq \frac{f_8(M)\log n}{n^M}. \tag{5.4}$$

PROOF. We observe from (3.1) that

$$\sum_{k=0}^{n-1} |A(\pi - x_{kn})| > \sum_{k=0}^{n+1} A(\pi - x_{kn}) = 1$$

which proves (5.1). Similarly from (3.2) we have

$$\sum_{k=0}^{n-1} |C(\pi - x_{kn})| \geqq \sum_{k=0}^{n-1} C(\pi - x_{kn}) = \frac{1 - \cos n\pi}{n^2} = \frac{2}{n^2}$$

which proves (5.3). Proofs for (5.2) and (5.4) are similar. We will only prove (5.4). First, we note that

$$D(\pi - x_{kn}) = \frac{8(-1)^{(M+1)/2}}{n} \sum_{j=1}^{n-1} \frac{\sin j(\pi - x_{kn})}{a_{j,M}}.$$

Therefore, we have

$$\sin\frac{\pi - x_{kn}}{2} D(\pi - x_{kn}) = \frac{4(-1)^{(M+1)/2}}{n} \left\{ \cos\frac{[\pi - x_{kn}/2]}{a_{1,M}} + \sum_{j=1}^{n-1} \frac{(a_{j+1,M} - a_{j,M})}{a_{j,M}a_{j+1,M}} \cos(j + \tfrac{1}{2})(\pi - x_{kn}) \right\}.$$

From (2.10) it follows that

$$a_{1,M} < 2^M n^M. \tag{5.6}$$

It is well known that

$$\sum_{k=0}^{n-1} \left|\cot \frac{\pi - x_{kn}}{2}\right| \geqq f_9 n \log n. \tag{5.7}$$

From (5.5)–(5.7) and (3.16) we obtain

$$\sum_{k=0}^{n-1} \left|D(\pi - x_{kn})\right| \geqq \frac{4f_9 \log n}{(2n)^M} - \frac{4^M}{n^M} \geqq \frac{f_8(m) \log n}{n^M},$$

which proves (5.4) as well.

6. Proof of Theorems

The upper and lower estimates of fundamental polynomials obtained in Articles 4 and 5 lead to the proof of theorems very easily.

Proof of Theorem 4. From (3.1) and (2.8) we obtain

$$\begin{aligned} R_n(x) - f(x) &= \sum_{i=0}^{n-1} [f(x_{in}) - f(x)] A(x - x_{in}) \\ &+ \sum_{i=0}^{n-1} \alpha_{in} B(x - x_{in}) + \sum_{i=0}^{n-1} \beta_{in} C(x - x_{in}) \\ &+ \sum_{i=0}^{n-1} \delta_{in} D(x - x_{in}). \end{aligned} \tag{6.1}$$

Let us denote the expression on the right-hand side by I_1, I_2, I_3 and I_4 respectively.

From (2.3) and (4.1) we have

$$|I_4| = o\left(\frac{n^M}{\log n}\right) \sum_{i=0}^{n-1} |D(x - x_{in})| = o\left(\frac{n^M}{\log n}\right) \frac{25 \log n}{n^M} = o(1). \tag{6.2}$$

From (2.3) and (4.2) we have

$$|I_3| = o(n^2) \sum_{i=0}^{n-1} |C(x - x_{in})| = \frac{o(n^2) f_1(M)}{n^2} = o(1). \tag{6.3}$$

From (2.3) and (4.3) we obtain

$$|I_2| = o\left(\frac{n}{\log n}\right) \sum_{i=0}^{n-1} |B(x - x_{in})| = o\left(\frac{n}{\log n}\right) \frac{f_2(M) \log n}{n} = o(1). \tag{6.4}$$

For the estimation of I_1 we use the fact that $f(x)$ is continuous 2π periodic function. Given $\varepsilon > 0 \,\exists\, \delta$ such that $|f(x) - f(x_{in})| < \varepsilon$ whenever $|x - x_{in}| \leqq \delta = \delta(\varepsilon)$. Put $\max_{0 \leqq x \leqq 2\pi} |f(x)| = B$. From (3.9), it follows that

$$0 \leqq \tau_{j,k}(x) < \frac{1}{j(\sin(\delta/2))^2} \text{ for } |x - x_{in}| > \delta.$$

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By expressing $A(x-x_{in})$ in terms of Fejér-kernel and using the above result, we obtain

$$\sum_{|x-x_{in}|>\delta} |A(x-x_{in})| \leqq \frac{C}{n\sin^2(\delta/2)}. \tag{6.5}$$

Next, we express I_1 as

$$I_1 = \sum_{|x-x_{in}|\leqq\delta} [f(x)-f(x_{in})]\, A(x-x_{in}) + \sum_{|x-x_{in}|>\delta} [f(x)-f(x_{in})]\, A(x-x_{in}).$$

By using (6.5) and (5.4), we obtain

$$\begin{aligned} |I_1| &\leqq \varepsilon \sum_{|x-x_{in}|\leqq\delta} |A(x-x_{in})| + 2B \sum_{|x-x_{in}|>\delta} |A(x-x_{in})| \\ &\leqq \varepsilon \sum_{i=1}^{n} |A(x-x_{in})| + \frac{2BC}{n\sin\ (\delta/2)} \leqq \varepsilon f_3(M) + \frac{2BC}{n\sin^2(\delta/2)}. \end{aligned}$$

Since the second term on the right-hand side can be made as small as we please by choosing n sufficiently large, we have

$$I_1 = o(1). \tag{6.6}$$

From (6.1)–(6.4) and (6.6) we have $R_n[x]-f(x) = o(1)$ which proves Theorem 4.

Proof of Theorem 5. From the uniqueness of $(0,1,2,M)$ trigonometric interpolation it follows that for an arbitrary trigonometric polynomial $\phi_n(x)$ of the order $2n$ (satisfying (2.1.)) we have

$$\begin{aligned} \phi_n(x) = \sum_{i=0}^{n-1} \phi_n(x_{in})A_{in}(x) + \sum_{i=0}^{n-1} \phi_n'(x_{in})B_{in}(x) \\ + \sum_{i=0}^{n-1} \phi_n''(x_{in})C_{in}(x) + \sum_{i=0}^{n-1} \phi_n^{(M)}(x_{in})D_{in}(x). \end{aligned}$$

Let $\phi_n(x)$ satisfy further the condition (2.4). On using (4.1)–(4.4) it follows that

$$|\phi_n(x)| \leqq f_3(M)a_0 + f_2(M)\frac{a_1\log n}{n} + \frac{f_1 a_2(M)}{n^2} + \frac{25\log n}{n^M}a_M.$$

Therefore for $0 \leqq x \leqq 2\pi$, we have

$$|\phi_n(x)| \leqq c_0\left(a_0 + \frac{a_1\log n}{n} + \frac{a_2}{n^2} + \frac{a_M\log n}{n^M}\right),$$

where $c_0 = \max(f_3, f_2, f_1, 25)$.

To prove (2.6), we denote $q_n(x)$ to be the trigonometric polynomial

$$\begin{aligned} q_n(x) &= \sum_{i=0}^{n-1} a_0 A(x-x_{in}) \operatorname{sign} A(\pi - x_{in}) \\ &+ \sum_{i=0}^{n-1} a_1 B(x-x_{in}) \operatorname{sign} B(\pi - x_{in}) \\ &+ \sum_{i=0}^{n-1} a_2 C(x-x_{in}) \operatorname{sign} C(\pi - x_{in}) \\ &+ \sum_{i=0}^{n-1} a_M D(x-x_{in}) \operatorname{sign} D(\pi - x_{in}). \end{aligned}$$

Let $x = \pi$. By using (5.1)–(5.4) we can deduce (2.6) and this proves Theorem 5.

PROOF OF THEOREM 6. Let $\alpha_{in} = \beta_{in} = \delta_{in} = 0$. Then (2.8) reduces to

$$R_n(x) = \sum_{i=0}^{n-1} f(x_{in}) A(x - x_{in}).$$

By using (3.1) we have

$$f(x) - R_n(x) = \sum_{i=0}^{n-1} [f(x) - f(x_{in})] A(x - x_{in}). \tag{6.7}$$

Let us denote $\omega(\delta)$ as modulus of continuity of $f(x)$. From the result of Shisha and Mond [12], we have for any $\delta > 0$ and all x, y

$$|f(x) - f(y)| \leqq \left(1 + \frac{\pi^2}{\delta^2} \sin^2 \frac{x-y}{2}\right) \omega(\delta). \tag{6.8}$$

By using (6.7) and (6.8) we obtain

$$|f(x) - R_n(x)| \leqq \sum_{i=0}^{n-1} \omega(\delta) \left(1 + \frac{\pi^2}{\delta^2} \sin^2 \frac{(x-x_{in})}{2}\right) |A(x - x_{in})|. \tag{6.9}$$

Following [17], it can be shown that

$$\sum_{i=0}^{n-1} \sin^2 \frac{x - x_{in}}{2} |A(x - x_{in})| \leqq \frac{f_{10}(M)}{n}. \tag{6.1}$$

Let $\delta = 1/\sqrt{n}$ in (6.9) and use (6.10) and (4.4) to obtain

$$|f(x) - R_n(x)| = O(\omega_\delta(1/\sqrt{n})).$$

This proves Theorem 6 as well.

Remark. Let M be an even positive integer. Let $S_n(x)$ be the unique trigonometric polynomial determined by $(0, M)$ interpolation. From Theorem 3 we know that $S_n(x)$ converges uniformly to $f(x)$ (satisfying (1.5)) provided that the freedom of $S_n^{(M)}(x)$ at the points x_k's is given by

$$S_n^{(M)}(x_n) = o(n^{M-1}),\ k = 0, 1, \cdots, n-1.$$

This is best possible. In [15] the author has considered the problem of $(0, 1, M)$ (M-even) interpolation. One of the main features of this result is that by prescribing the first derivative, the freedom of $S_n^{(M)}(x_k)$ has considerably increased.

$$S_n^{(M)}(x_k) = o\left(\frac{n^M}{\log n}\right),\ k = 0, 1, \cdots, n-1.$$

It may be noted that for M-odd $(0, 1, M)$ trigonometric interpolation does not exist uniquely. This is shown also in [15].

Let $S_n(x)$ be the unique trigonometric polynomial determined by $(0, M)$ interpolation (M-odd). From Theorem 3 we know that $S_n(x)$ converges uniformly to $f(x)$ ($f(x) \in c_{2\pi}$) provided the freedom of $S_n^{(M)}(x)$ at the points x_k's is given by

$$S_n^{(M)}(x_k) = o\left(\frac{n^M}{\log n}\right).$$

Further, this is best possible. Let $R_n(x)$ be the unique trigonometric polynomial determined by $(0, 1, 2, M)$ interpolation (M-odd). Here we prescribed $R_n(x)$ and $R_n''(x)$ at $x = x_k$ as well. One of the main features of Theorem 4 is that even with these new restrictions, the freedom of $R_n^{(M)}(x)$ at $x = x_k$'s can not be improved.

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