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BOUNDARY SCHWARZ LEMMA FOR SOLUTIONS TO NONHOMOGENEOUS BIHARMONIC EQUATIONS

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Abstract

We establish a boundary Schwarz lemma for solutions to nonhomogeneous biharmonic equations.

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1. Introduction and main result

The classical Schwarz lemma says that an analytic function f from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ into itself with $f(0) = 0$ must map each smaller disk $\{z \in \mathbb{C} : |z| < r < 1\}$ into itself. Further, $|f'(0)| \leq 1$ and $|f'(0)| = 1$ if and only if f is a rotation of $\mathbb{D}$. This is a very powerful tool in complex analysis. An elementary consequence of the Schwarz lemma is that if f extends continuously to some boundary point α, then $|f(\alpha)| = 1$ and, if f is differentiable at α, then $|f'(\alpha)| \geq 1$ (see, for example, [8, 14]).

There are many versions of the Schwarz lemma and boundary Schwarz lemma. Burns and Krantz [4] obtained a Schwarz lemma at the boundary for holomorphic mappings defined on $\mathbb{D}$ as well as on balls in $\mathbb{C}^n$. They also obtained similar results for holomorphic mappings on strongly convex and strongly pseudoconvex domains in $\mathbb{C}^n$. Liu and Tang [10] obtained the boundary Schwarz lemma for holomorphic mappings defined on the unit ball in $\mathbb{C}^n$. We refer to the survey article by Krantz [9] for a brief history of the Schwarz lemma at the boundary.

The Schwarz lemma at the boundary plays an important role in complex analysis. For example, by using the Schwarz lemma at the boundary, Bonk [3] improved the previously known lower bound for the Bloch constant. The boundary Schwarz lemma is also a fundamental tool in the study of the geometric properties of functions of several complex variables (see [10–12]). In this paper, we establish a boundary

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Schwarz lemma for functions which satisfy certain partial differential equations, namely, nonhomogeneous biharmonic equations.

We now give some notation and preliminary observations which are required to state our result. We denote by $\mathbb{T} = \partial\mathbb{D}$ the boundary of $\mathbb{D}$ and by $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$ the closure of $\mathbb{D}$. For any subset Ω of $\mathbb{C}$ and $m \in \mathbb{N} \cup \{0\}$, we denote by $C^m(\Omega)$ the set of all complex-valued m-times continuously differentiable functions from Ω into $\mathbb{C}$. In particular, $C(\Omega) := C^0(\Omega)$ denotes the set of all continuous functions in Ω.

For a real 2×2 matrix A, we use the matrix norm

$$\|A\| = \sup\{|Az| : z \in \mathbb{T}\}$$

and the matrix function

$$\lambda(A) = \inf\{|Az| : z \in \mathbb{T}\}.$$

For $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$, the formal derivative of a complex-valued function $f = u + iv$ is given by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

so that

$$\|D_f\| = |f_z| + |f_{\bar{z}}| \quad \text{and} \quad \lambda(D_f) = \big||f_z| - |f_{\bar{z}}|\big|,$$

where

$$f_z = \tfrac{1}{2}(f_x - i f_y) \quad \text{and} \quad f_{\bar{z}} = \tfrac{1}{2}(f_x + i f_y).$$

The *Jacobian* of f is

$$J_f := \det D_f = |f_z|^2 - |f_{\bar{z}}|^2.$$

Let $f^*, g \in C(\overline{\mathbb{D}})$, $\varphi \in C(\mathbb{T})$ and $f \in C^4(\mathbb{D})$. We consider the *nonhomogeneous biharmonic equation* defined in $\mathbb{D}$:

$$\Delta(\Delta f) = g, \tag{1.1}$$

with the Dirichlet boundary values

$$\begin{cases} f_{\bar{z}} = \varphi & \text{on } \mathbb{T}, \\ f = f^* & \text{on } \mathbb{T}, \end{cases} \tag{1.2}$$

where

$$\Delta f = f_{xx} + f_{yy} = 4 f_{z\bar{z}}$$

is the *Laplacian* of f.

In particular, if $g \equiv 0$, then any solution to (1.1) is *biharmonic*. For the properties of biharmonic mappings, see [5, 15]. Chen *et al.* [7] discussed the Schwarz-type lemma, Landau-type theorems and bi-Lipschitz properties for the solutions of nonhomogeneous biharmonic equations (1.1) satisfying (1.2). The solvability of the nonhomogeneous biharmonic equations has also been studied in [13].

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We introduce the *biharmonic Green function* and *(harmonic) Poisson kernel* in $\mathbb{D}$,

$$G(z,w) = |z-w|^2 \log\left|\frac{1-z\overline{w}}{z-w}\right|^2 - (1-|z|^2)(1-|w|^2)$$

and

$$P(z,e^{i\theta}) = \frac{1-|z|^2}{|1-ze^{-i\theta}|^2} \quad (\theta\in[0,2\pi]).$$

It follows from [2, Theorem 2] that all the solutions to the Equation (1.1) satisfying the boundary conditions (1.2) are given by

$$f(z) = \mathcal{P}_{f^*}(z) + \frac{1}{2\pi}(1-|z|^2)\int_0^{2\pi} f^*(e^{it})\frac{\overline{z}e^{it}}{(1-\overline{z}e^{it})^2}\,dt - (1-|z|^2)\mathcal{P}_{\varphi_1}(z) - \frac{1}{8}G[g](z), \tag{1.3}$$

where

$$\mathcal{P}_{f^*}(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,e^{it})f(e^{it})\,dt, \quad \mathcal{P}_{\varphi_1}(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,e^{it})\varphi_1(e^{it})\,dt, \tag{1.4}$$

$$\varphi_1(e^{it}) = \varphi(e^{it})e^{-it} \quad \text{and} \quad G[g](z) = \frac{1}{2\pi}\int_{\mathbb{D}} g(w)G(z,w)\,dA(w). \tag{1.5}$$

Here $dA(w)$ denotes the Lebesgue area measure in $\mathbb{D}$.

Let us recall the following version of the boundary Schwarz lemma for analytic functions proved in [10].

THEOREM 1.1 [10, Theorem 1.1′]. *Suppose that f is an analytic function from $\mathbb{D}$ into itself. If $f(0) = 0$ and f is analytic at $z = \alpha \in \mathbb{T}$ with $f(\alpha) = \beta \in \mathbb{T}$, then:*

(1) $\overline{\beta} f'(\alpha)\alpha \geq 1$;
(2) $\overline{\beta} f'(\alpha)\alpha = 1$ *if and only if* $f(z) \equiv e^{i\theta} z$, *where* $e^{i\theta} = \beta\alpha^{-1}$ *and* $\theta \in \mathbb{R}$.

This result has attracted much attention and has been generalised in various ways (see, for example, [6, 17]). Recently, Wang and Zhu [16] obtained a boundary Schwarz lemma for the solutions to Poisson's equation. By analogy with the studies in [16], we derive the following boundary Schwarz lemma for functions with the form (1.3). A different form of the boundary Schwarz lemma for functions with the form (1.3) was proved in [7].

THEOREM 1.2. *Suppose that $f \in C^4(\mathbb{D})$ and $g \in C(\overline{\mathbb{D}})$ satisfy*

$$\begin{cases} \Delta(\Delta f) = g & \text{in } \mathbb{D}, \\ f_{\overline{z}} = \varphi & \text{on } \mathbb{T}, \\ f = f^* & \text{on } \mathbb{T}, \end{cases}$$

where $\varphi \in C(\mathbb{T})$, $f^* \in C(\overline{\mathbb{D}})$, f^* is analytic in $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{D}$. If f is differentiable at $z = \alpha \in \mathbb{T}$, $f(\alpha) = \beta \in \mathbb{T}$ and $f(0) = 0$, then

$$\mathrm{Re}[\overline{\beta}(f_z(\alpha)\alpha + f_{\overline{z}}(\alpha)\overline{\alpha})] \geq \frac{2}{\pi} - 3\|\mathcal{P}_{\varphi_1}\|_\infty - \frac{1}{64}\|g\|_\infty, \tag{1.6}$$

where $\mathcal{P}_{\varphi_1}$ and φ_1 are defined in (1.4) and (1.5), respectively.

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In particular, when $\|\mathcal{P}_{\varphi_1}\|_\infty = \|g\|_\infty = 0$, *the following inequality is sharp:*

$$\mathrm{Re}[\overline{\beta}(f_z(\alpha)\alpha + f_{\overline{z}}(\alpha)\overline{\alpha})] \geq \frac{2}{\pi}. \tag{1.7}$$

Remark 1.3. For analytic functions, the value of $\overline{\beta}f'(\alpha)\alpha$ in Theorem 1.1 is a real number. However, this is not true for the case of the solutions to the equation (1.1) (see Example 3.1 below). Hence, in Theorem 1.2, we consider the real part of the quantity $\overline{\beta}(f_z(\alpha)\alpha + f_{\overline{z}}(\alpha)\overline{\alpha})$.

The lower bound for the quantity $\mathrm{Re}[\overline{\beta}(f_z(\alpha)\alpha + f_{\overline{z}}(\alpha)\overline{\alpha})]$ in (1.6) is always positive for all φ_1 and g with $(\|\mathcal{P}_{\varphi_1}\|_\infty, \|g\|_\infty) \in \{(x, y) : x \geq 0, y \geq 0, 3x + y/64 < 2/\pi\}$.

2. Proof of Theorem 1.2

We start with the following lemma.

Lemma 2.1. *Suppose that* $\mathfrak{g} \in C(\overline{\mathbb{D}})$ *and* $h \in C^4(\mathbb{D})$ *satisfy*

$$\begin{cases} \Delta(\Delta h) = \mathfrak{g} & \text{in } \mathbb{D}, \\ h_{\overline{z}} = \psi & \text{on } \mathbb{T}, \\ h = h^* & \text{on } \mathbb{T}, \end{cases}$$

where $\psi \in C(\mathbb{T})$, $h^* \in C(\overline{\mathbb{D}})$, h^* *is analytic in* $\mathbb{D}$ *and* $h(\mathbb{D}) \subset \mathbb{D}$. *If* h *is differentiable at* $z = 1$, $h(1) = 1$ *and* $h(0) = 0$, *then*

$$\mathrm{Re}[h_z(1) + h_{\overline{z}}(1)] \geq \frac{2}{\pi} - 3\|\mathcal{P}_{\psi_1}\|_\infty - \frac{1}{64}\|\mathfrak{g}\|_\infty,$$

where $\psi_1(e^{it}) = \psi(e^{it})e^{-it}$.

In particular, when $\|\mathcal{P}_{\psi_1}\|_\infty = \|\mathfrak{g}\|_\infty = 0$, *the following inequality is sharp:*

$$\mathrm{Re}[h_z(1) + h_{\overline{z}}(1)] \geq \frac{2}{\pi}. \tag{2.1}$$

Proof. The assumptions of the lemma ensure that h has the form (1.3), that is,

$$h(z) = \mathcal{P}_{h^*}(z) + \frac{1}{2\pi}(1 - |z|^2)\int_0^{2\pi} h^*(e^{it})\frac{\overline{z}e^{it}}{(1 - \overline{z}e^{it})^2}dt - (1 - |z|^2)\mathcal{P}_{\psi_1}(z) - \frac{1}{8}G[\mathfrak{g}](z).$$

Since the analyticity of h^* in $\mathbb{D}$ gives

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{z}e^{it}h^*(e^{it})\frac{1 - |z|^2}{(1 - \overline{z}e^{it})^2}\,dt = 0, \tag{2.2}$$

we obtain

$$\begin{aligned} |h(z)| &= \left|\mathcal{P}_{h^*}(z) - (1 - |z|^2)\mathcal{P}_{\psi_1}(z) - \frac{1}{8}G[\mathfrak{g}](z)\right| \\ &\leq \left|\mathcal{P}_{h^*}(z) - \frac{1 - |z|^2}{1 + |z|^2}\mathcal{P}_{h^*}(0)\right| + (1 - |z|^2)\left|\mathcal{P}_{\psi_1}(z) - \frac{1 - |z|^2}{1 + |z|^2}\mathcal{P}_{\psi_1}(0)\right| \\ &\quad + \frac{1 - |z|^2}{1 + |z|^2}(|\mathcal{P}_{h^*}(0) - \mathcal{P}_{\psi_1}(0)| + |z|^2|\mathcal{P}_{\psi_1}(0)|) + \left|\frac{1}{8}G[\mathfrak{g}](z)\right|. \end{aligned} \tag{2.3}$$

By the proof of Theorem 1.1 in [7], we have the following estimates:

$$\left|\mathcal{P}_{h^*}(z) - \frac{1-|z|^2}{1+|z|^2}\mathcal{P}_{h^*}(0)\right| \le \frac{4}{\pi}\|\mathcal{P}_{h^*}\|_\infty \arctan|z|,$$
$$\left|\mathcal{P}_{\psi_1}(z) - \frac{1-|z|^2}{1+|z|^2}\mathcal{P}_{\psi_1}(0)\right| \le \frac{4}{\pi}\|\mathcal{P}_{\psi_1}\|_\infty \arctan|z|$$

and

$$|G[\mathfrak{g}](z)| \le \tfrac{1}{8}\|\mathfrak{g}\|_\infty(1-|z|^2)^2.$$

Moreover, it follows from the assumption $h(0) = 0$ that

$$\mathcal{P}_{h^*}(0) - \mathcal{P}_{\psi_1}(0) = \tfrac{1}{8}G[\mathfrak{g}](0)$$

and so

$$|\mathcal{P}_{h^*}(0) - \mathcal{P}_{\psi_1}(0)| \le \tfrac{1}{64}\|\mathfrak{g}\|_\infty.$$

Based on these estimates, together with $\|\mathcal{P}_{h^*}\|_\infty \le 1$, the inequality (2.3) takes the form

$$|h(z)| \le \frac{4}{\pi}\arctan|z| + \frac{1-|z|^2}{1+|z|^2}\left(\frac{1}{64}\|\mathfrak{g}\|_\infty + |z|^2\|\mathcal{P}_{\psi_1}\|_\infty\right)$$
$$+ \frac{4}{\pi}\|\mathcal{P}_{\psi_1}\|_\infty(1-|z|^2)\arctan|z| + \frac{1}{64}\|\mathfrak{g}\|_\infty(1-|z|^2)^2 =: M(|z|). \tag{2.4}$$

Since h is differentiable at $z = 1$,

$$h(z) = 1 + h_z(1)(z-1) + h_{\bar{z}}(1)(\bar{z}-1) + o(|z-1|),$$

where $o(x)$ means a function with $\lim_{x\to 0} o(x)/x = 0$. Then we deduce from (2.4) that

$$2\mathrm{Re}[h_z(1)(1-z) + h_{\bar{z}}(1)(1-\bar{z})] \ge 1 - M^2(|z|) - o(|z-1|).$$

Letting $z = r \in (0,1)$ and $r \to 1^-$,

$$\mathrm{Re}[h_z(1) + h_{\bar{z}}(1)] \ge \lim_{r\to 1^-} M'(r) = \frac{2}{\pi} - 3\|\mathcal{P}_{\psi_1}\|_\infty - \frac{1}{64}\|\mathfrak{g}\|_\infty.$$

To finish the proof of the lemma, it remains to check the sharpness of the inequality (2.1). For this, we borrow the following function from [1, page 127]:

$$\mathfrak{h}(z) = \begin{cases} \dfrac{2}{\pi}\arctan\dfrac{z+\bar{z}}{1-|z|^2} & \text{if } z \in \mathbb{D}, \\ 1 & \text{if } z \in \mathbb{T}. \end{cases} \tag{2.5}$$

It can be seen that $\mathfrak{h}$ is harmonic in $\mathbb{D}$ with $\mathfrak{h}(0) = 0$ and $\mathfrak{h}(1) = 1$. Since

$$\mathfrak{h}_z(z) = \frac{2}{\pi}\frac{1+\bar{z}^2}{(1-|z|^2)^2 + (z+\bar{z})^2} \quad\text{and}\quad \mathfrak{h}_{\bar{z}}(z) = \frac{2}{\pi}\frac{1+z^2}{(1-|z|^2)^2 + (z+\bar{z})^2}, \tag{2.6}$$

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both $\mathfrak{h}_z$ and $\mathfrak{h}_{\bar{z}}$ are continuous at $z = 1$. This guarantees the differentiability of $\mathfrak{h}$ at this point. Let

$$\mathfrak{h}^*(z) = 1$$

in $\overline{\mathbb{D}}$. It is clear that $\mathfrak{h}^*$ is analytic in $\mathbb{D}$ and $\mathfrak{h} = \mathfrak{h}^*$ on $\mathbb{T}$. Further, the harmonicity of $\mathfrak{h}$ in $\mathbb{D}$, together with [7, (1.5)] and (2.2), ensures that

$$\mathcal{P}_{\psi_1} = 0.$$

Since (2.6) leads to

$$\mathrm{Re}[\mathfrak{h}_z(1) + \mathfrak{h}_{\bar{z}}(1)] = \frac{2}{\pi},$$

we see that $\mathfrak{h}$ is an extremal function for the sharpness of (2.1). The proof of the lemma is complete. □

Proof of Theorem 1.2. Let

$$\begin{aligned} h(z) &= \overline{\beta} f(\alpha z) \quad \text{in } \mathbb{D}, \\ \mathfrak{g}(z) &= \overline{\beta} g(\alpha z) \quad \text{in } \mathbb{D}, \\ \psi(\xi) &= \overline{\beta}\overline{\alpha}\varphi(\alpha \xi) \quad \text{on } \mathbb{T}, \\ h^*(z) &= \overline{\beta} f^*(\alpha z) \quad \text{in } \overline{\mathbb{D}}. \end{aligned}$$

From Lemma 2.1,

$$\mathrm{Re}[h_z(1) + h_{\bar{z}}(1)] \geq \frac{2}{\pi} - 3\|\mathcal{P}_{\varphi_1}\|_\infty - \frac{1}{64}\|g\|_\infty,$$

from which the inequality (1.6) in Theorem 1.2 follows since

$$\mathrm{Re}[\overline{\beta}(f_z(\alpha)\alpha + f_{\bar{z}}(\alpha)\overline{\alpha})] = \mathrm{Re}[h_z(1) + h_{\bar{z}}(1)].$$

The inequality (1.7) is obvious. For its sharpness, let

$$\mathfrak{f}(z) = \frac{2\beta}{\pi} \arctan \frac{\overline{\alpha} z + \alpha \bar{z}}{1 - |z|^2}$$

in $\mathbb{D}$. Then

$$\mathfrak{f}(z) = \beta \mathfrak{h}(\overline{\alpha} z),$$

where the function $\mathfrak{h}$ is defined in (2.5). By the discussions on the sharpness of the inequality (2.1) in the proof of Lemma 2.1, we see that $\mathfrak{f}$ demonstrates the sharpness of the inequality (1.7). The theorem is proved. □

3. An example

In this section, we construct an example to show that it is reasonable to consider the real part of the quantity $\overline{\beta}(f_z(\alpha)\alpha + f_{\bar{z}}(\alpha)\overline{\alpha})$ in Theorem 1.2.

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EXAMPLE 3.1. Assume that

$$g(z) = 32M[2 - 3i(z^2 + \bar{z}^2)] \quad \text{and} \quad f(z) = (1 - M)z^2 + \frac{Mi}{4}(1 - |z|^4)(z^2 + \bar{z}^2) + M|z|^4$$

in $\overline{\mathbb{D}}$, where $0 < M < 2\sqrt{5}(3 - \sqrt{2})/35\pi$. Then:

(1) f and g satisfy the nonhomogeneous biharmonic equation $\Delta^2 f = g$ and all the other assumptions in Theorem 1.2 with $\alpha = \beta = 1$;

(2) $\mathrm{Re}(f_z(1) + f_{\bar{z}}(1)) = 2(1 + M)$, $\mathrm{Im}(f_z(1) + f_{\bar{z}}(1)) = -2M \neq 0$, $\|\mathcal{P}_{\varphi_1}\|_\infty = \sqrt{5}M$ and $\|g\|_\infty = 64\sqrt{10}M$, where $\varphi_1(\zeta) = \frac{1}{2}M(4 - i(\zeta^2 + \bar{\zeta}^2))$ on $\mathbb{T}$.

PROOF. Elementary computations yield

$$f_z(z) = 2(1 - M)z + \frac{Mi}{2}[z(1 - |z|^4) - z\bar{z}^2(z^2 + \bar{z}^2)] + 2Mz\bar{z}^2, \tag{3.1}$$

$$f_{\bar{z}}(z) = \frac{Mi}{2}[\bar{z}(1 - |z|^4) - z^2\bar{z}(z^2 + \bar{z}^2)] + 2Mz^2\bar{z} \tag{3.2}$$

and

$$\Delta^2 f = g.$$

Obviously, $f(0) = 0$ and $f(1) = 1$. Let

$$\varphi(\zeta) = \frac{M}{2}\zeta(4 - i(\zeta^2 + \bar{\zeta}^2)) \text{ on } \mathbb{T} \quad \text{and} \quad f^*(z) = (1 - M)z^2 + M \text{ in } \overline{\mathbb{D}}.$$

Then f^* is analytic in $\mathbb{D}$ and $f_{\bar{z}} = \varphi$ and $f^* = f$ on $\mathbb{T}$. For $z \in \mathbb{D}$,

$$|f(z)| \le |z|^2\Big(1 - \frac{M}{2}(1 - |z|^2)^2\Big) < 1$$

and so $f(\mathbb{D}) \subset \mathbb{D}$. The differentiability of f at $z = 1$ can be seen from the continuity of its partial derivatives (see (3.1) and (3.2)). This gives the first statement of the example.

The equalities

$$\mathrm{Re}(f_z(1) + f_{\bar{z}}(1)) = 2(1 + M) \quad \text{and} \quad \mathrm{Im}[f_z(1) + f_{\bar{z}}(1)] = -2M \neq 0$$

easily follow from (3.1) and (3.2) and elementary computations give

$$\|\mathcal{P}_{\varphi_1}\|_\infty = \max_{z \in \overline{\mathbb{D}}} \Big\{\frac{M}{2}|4 - i(z^2 + \bar{z}^2)|\Big\} = \sqrt{5}M$$

and

$$\|g\|_\infty = \max_{z \in \overline{\mathbb{D}}}\{32M|2 - 3i(z^2 + \bar{z}^2)|\} = 64\sqrt{10}M.$$

This gives the second statement of the example and completes the proof. □

REMARK 3.2. The reason for the condition $0 < M < 2\sqrt{5}(3 - \sqrt{2})/35\pi$ in Example 3.1 is to guarantee that

$$3\|\mathcal{P}_{\varphi_1}\|_\infty + \frac{1}{64}\|g\|_\infty < \frac{2}{\pi},$$

that is, the quantity $2/\pi - 3\|\mathcal{P}_{\varphi_1}\|_\infty - \|g\|_\infty/64$ is positive.

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