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DISPLAYED EQUATIONS FOR GALOIS REPRESENTATIONS

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Abstract. The Galois representation associated to a p-divisible group over a normal complete noetherian local ring with perfect residue field is described in terms of its Dieudonné display. As a consequence, the Kisin module associated to a commutative finite flat p-group scheme via Dieudonné displays is related to its Galois representation in the expected way.

Introduction

Let R be a normal complete noetherian local ring with perfect residue field k of positive characteristic p and with fraction field K of characteristic zero. For a p-divisible group G over R, the Tate module $T_p(G)$ is a free $\mathbb{Z}_p$-module of finite rank with a continuous action of the absolute Galois group $\mathcal{G}_K$. We want to describe the Tate module in terms of the Dieudonné display $\mathscr{P}=(P,Q,F,F_1)$ associated to G in [Zi2, La3], and relate this to other descriptions of the Tate module when R is a discrete valuation ring.

Let us recall the notion of a Dieudonné display. The Zink ring $\mathbb{W}(R)$ is a certain subring of the ring of Witt vectors $W(R)$ which is stable under the Frobenius endomorphism f of $W(R)$. The components of $\mathscr{P}$ are $\mathbb{W}(R)$-modules $Q\subseteq P$ where P is finite free and P/Q is a free R-module, and f-linear maps $F:P\to P$ and $F_1:Q\to P$ such that $F_1(Q)$ generates P and $F_1(v(u_0a)x)=aF(x)$ for $x\in P$ and $a\in\mathbb{W}(R)$. Here v is the Verschiebung of $W(R)$, and $u_0\in W(R)$ is the unit defined by $u_0=1$ if $p\geqslant 3$ and by $v(u_0)=2-[2]$ if $p=2$. The twist by u_0 is necessary since v does not stabilize $\mathbb{W}(R)$ when $p=2$.

To state the main result we need the following scalar extension of $\mathscr{P}$. Let $\hat{R}^{\mathrm{nr}}$ be the completion of the strict Henselization of R, let $\tilde{K}$ be an algebraic closure of the fraction field $\hat{K}^{\mathrm{nr}}$ of $\hat{R}^{\mathrm{nr}}$, and let $\tilde{R}\subset\tilde{K}$ be the integral closure of $\hat{R}^{\mathrm{nr}}$. We define

$$\mathbb{W}(\tilde{R})=\varinjlim_{E}\mathbb{W}(R_E)$$

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where E runs through the finite extensions of $\hat{K}^{\mathrm{nr}}$ in $\tilde{K}$ and where $R_E = \tilde{R} \cap E$. Let $\tilde{R}^\wedge$ and $\hat{\mathbb{W}}(\tilde{R})$ be the p-adic completions of $\tilde{R}$ and $\mathbb{W}(\tilde{R})$. We define

$$\hat{P}_{\tilde{R}} = \hat{\mathbb{W}}(\tilde{R}) \otimes_{\mathbb{W}(R)} P$$

and

$$\hat{Q}_{\tilde{R}} = \mathrm{Ker}(\hat{P}_{\tilde{R}} \to \tilde{R}^\wedge \otimes_R P/Q).$$

Let $\bar{K}$ be the algebraic closure of K in $\tilde{K}$ and let $\tilde{\mathcal{G}}_K$ be the group of automorphisms of $\tilde{K}$ whose restriction to $\bar{K}\hat{K}^{\mathrm{nr}}$ is induced by an element of $\mathcal{G}_K$. The natural map $\tilde{\mathcal{G}}_K \to \mathcal{G}_K$ is surjective, and bijective when R is one-dimensional since then $\tilde{K} = \bar{K}\hat{K}^{\mathrm{nr}}$. The following is the main result of this note; see Proposition 4.1.

THEOREM A. *There is an exact sequence of $\tilde{\mathcal{G}}_K$-modules*

$$0 \longrightarrow T_p(G) \longrightarrow \hat{Q}_{\tilde{R}} \xrightarrow{F_1 - 1} \hat{P}_{\tilde{R}} \longrightarrow 0.$$

Here F_1 is a natural extension of $F_1 : Q \to P$. If G is connected, a similar description of $T_p(G)$ in terms of the nilpotent display of G is part of Zink's theory of displays. In this case k need not be perfect; see [Me2, Proposition 4.4]. The proof is recalled in Proposition 2.1 below.

The one-dimensional case

Assume now in addition that R is a discrete valuation ring. Then Theorem A can be related to the descriptions of $T_p(G)$ in terms of p-adic Hodge theory and in terms of Breuil–Kisin modules as follows.

Relation with the crystalline period homomorphism

Let M_{cris} be the value of the covariant Dieudonné crystal of G over $A_{\mathrm{cris}}(\bar{R})$. It carries a filtration and a Frobenius, and by [Fa] there is a period homomorphism

$$T_p(G) \to \mathrm{Fil}^1 M_{\mathrm{cris}}^{F=p}$$

which is bijective if $p \geqslant 3$, and injective with cokernel annihilated by p if $p = 2$. The v-stabilized Zink ring $\mathbb{W}^+(R) = \mathbb{W}(R)[v(1)]$ studied in [La3] induces an extension $\hat{\mathbb{W}}^+(\tilde{R})$ of the ring $\hat{\mathbb{W}}(\tilde{R})$ defined above, which is the trivial extension when $p \geqslant 3$. The universal property of $A_{\mathrm{cris}}(\bar{R})$ gives a ring homomorphism

$$\varkappa_{\mathrm{cris}} : A_{\mathrm{cris}}(\bar{R}) \to \hat{\mathbb{W}}^+(\tilde{R}).$$

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Using the crystalline description of Dieudonné displays of [La3], one obtains an $A_{\mathrm{cris}}(\bar{R})$-linear map

$$\tau : M_{\mathrm{cris}} \to \hat{\mathbb{W}}^+(\tilde{R}) \otimes_{\hat{\mathbb{W}}(\tilde{R})} \hat{P}_{\tilde{R}}$$

compatible with Frobenius and filtration. We will show that τ induces the identity on $T_p(G)$, viewed as a submodule of $\mathrm{Fil}^1 M_{\mathrm{cris}}$ by the period homomorphism and as a submodule of $\hat{Q}_{\tilde{R}} \subseteq \hat{P}_{\tilde{R}}$ by Theorem A; see Proposition 6.2.

Relation with Breuil–Kisin modules

Let $\pi \in R$ generate the maximal ideal. Let $\mathfrak{S} = W(k)[[t]]$ and let $\sigma : \mathfrak{S} \to \mathfrak{S}$ extend the Frobenius automorphism of $W(k)$ by $t \mapsto t^p$; see below for the case of more general Frobenius lifts. We consider pairs $M = (M, \phi)$ where M is an $\mathfrak{S}$-module of finite type and where $\phi : M \to M^{(\sigma)} = \mathfrak{S} \otimes_{\sigma,\mathfrak{S}} M$ is an $\mathfrak{S}$-linear map with cokernel annihilated by the minimal polynomial of π over $W(k)$. Following [VZ], M is called a Breuil window if M is free over $\mathfrak{S}$, and M is called a Breuil module if M is a p-power torsion $\mathfrak{S}$-module of projective dimension at most one. These notions are dual to the classical Breuil–Kisin modules.

It is known that p-divisible groups over R are equivalent to Breuil windows. This was conjectured by Breuil [Br] and proved by Kisin [Ki1, Ki2] if $p \geqslant 3$, and for connected groups if $p = 2$. The general case is proved in [La3] by showing that Breuil windows are equivalent to Dieudonné displays. (This equivalence holds when R is regular of arbitrary dimension, with appropriate definition of $\mathfrak{S}$. For $p \geqslant 3$ this equivalence is already proved in [VZ] for some regular rings, including all discrete valuation rings.) As a corollary, commutative finite flat p-group schemes over R are equivalent to Breuil modules. Other proofs for $p = 2$, more closely related to Kisin's methods, were obtained independently by Kim [K] and Liu [Li].

Let K_∞ be the extension of K generated by a chosen system of successive pth roots of π. For a p-divisible group G over R let $T(G)$ be its Tate module, and for a commutative finite flat p-group scheme G over R let $T(G) = G(\bar{K})$. The results of Kisin, Liu, and Kim include a description of $T(G)$ as a $\mathcal{G}_{K_\infty}$-module in terms of the Breuil window or module (M, ϕ) corresponding to G. In the covariant theory used here it takes the form of an isomorphism of $\mathcal{G}_{K_\infty}$-modules $T(G) \cong T^{\mathrm{nr}}(M)$ where

$$T^{\mathrm{nr}}(M) = \{x \in M^{\mathrm{nr}} \mid \phi(x) = 1 \otimes x \text{ in } \mathfrak{S}^{\mathrm{nr}} \otimes_{\sigma,\mathfrak{S}^{\mathrm{nr}}} M^{\mathrm{nr}}\}$$

with $M^{\mathrm{nr}} = \mathfrak{S}^{\mathrm{nr}} \otimes_{\mathfrak{S}} M$; the ring $\mathfrak{S}^{\mathrm{nr}}$ is recalled in Section 7.

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To complete the approach via Dieudonné displays, we will show how the isomorphism $T(G) \cong T^{\mathrm{nr}}(M)$ can be deduced from Theorem A; see Corollary 8.6. It suffices to consider the case where G is a p-divisible group. The equivalence between Breuil windows and Dieudonné displays over R is induced by a ring homomorphism $\varkappa : \mathfrak{S} \to \mathbb{W}(R)$, which extends to a ring homomorphism $\varkappa^{\mathrm{nr}} : \mathfrak{S}^{\mathrm{nr}} \to \hat{\mathbb{W}}(\tilde{R})$. Using Theorem A, this allows to define a homomorphism of $\mathcal{G}_{K_\infty}$-modules

$$\tau : T^{\mathrm{nr}}(M) \to T(G),$$

and we show in Proposition 8.5 that τ is bijective. The verification is easy if G is étale, and the general case follows quite formally using a duality argument.

Other lifts of Frobenius

The equivalence between Breuil windows and p-divisible groups requires only a Frobenius lift $\sigma : \mathfrak{S} \to \mathfrak{S}$ which stabilizes the ideal $t\mathfrak{S}$ such that p^2 divides the linear term of the power series $\sigma(t)$. In this case, let K_∞ be the extension of K generated by a system $\pi^{(n)} \in \bar{K}$ of successive $\sigma(t)$-roots of π, which means that $\pi^{(0)} = \pi$ and $\sigma(t)(\pi^{(n+1)}) = \pi^{(n)}$. Then we obtain an isomorphism of $\mathcal{G}_{K_\infty}$-modules $T(G) \cong T^{\mathrm{nr}}(M)$ as before; here the ring $\mathfrak{S}^{\mathrm{nr}}$ depends on σ as well.

§1. Notation

All rings are commutative and unitary unless the contrary is stated. For the convenience of the reader we recall the notion of frames, windows, and displays.

A frame $\mathscr{F} = (S, I, R, \sigma, \sigma_1)$ in the sense of [La2] consists of a pair of rings S and $R = S/I$ with $I + pS \subseteq \mathrm{Rad}(S)$, a ring endomorphism $\sigma : S \to S$ that lifts the Frobenius of S/pS, and a σ-linear map $\sigma_1 : I \to S$ with $\sigma_1(I)S = S$.

We assume that S is a local ring. Then an $\mathscr{F}$-window $\mathscr{P} = (P, Q, F, F_1)$ consists of a finite free S-module P, a submodule $Q \subseteq P$ with $IP \subseteq Q$ such that P/Q is free over R, and a pair of σ-linear maps $F : P \to P$ and $F_1 : Q \to P$ with $F_1(ax) = \sigma_1(a)F(x)$ for $a \in I$ and $x \in P$, such that $F_1(Q)$ generates P. Then there is a unique S-linear map $V^\sharp : P \to S \otimes_{\sigma,S} P = P^{(\sigma)}$ with $V^\sharp(F_1(x)) = 1 \otimes x$ for $x \in Q$. A sequence $0 \to \mathscr{P} \to \mathscr{P}' \to \mathscr{P}'' \to 0$ of $\mathscr{F}$-windows will be called exact if the resulting sequences of P's and of Q's are exact.

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A frame homomorphism $\alpha : \mathscr{F} \to \mathscr{F}' = (S', I', R', \sigma', \sigma'_1)$ is a ring homomorphism $\alpha : S \to S'$ with $\alpha(I) \subseteq I'$ such that $\sigma'\alpha = \alpha\sigma$ and $\sigma'_1\alpha = u \cdot \alpha\sigma_1$ for a unit $u \in S'$, which then is unique. If $u = 1$ then α is called strict. There is a base change functor

$$\alpha_* : (\mathscr{F}\text{-windows}) \to (\mathscr{F}'\text{-windows})$$

where $\alpha_*(\mathscr{P}) = (P', Q', F', F'_1)$ is determined by $P' = S' \otimes_S P$ and $P'/Q' = (P/Q) \otimes_R R'$ with $F'(1 \otimes x) = 1 \otimes F(x)$ for $x \in P$ and $F'_1(1 \otimes x) = u \otimes F_1(x)$ for $x \in Q$.

For a not necessarily unitary ring R let $W(R)$ be the ring of p-typical Witt vectors. If R is p-adic and unitary, we have a frame

$$\mathscr{W}(R) = (W(R), I_R, R, f, f_1)$$

where I_R is the image of the Verschiebung $v : W(R) \to W(R)$, where f is the Frobenius, and f_1 is the inverse of v. Windows over $\mathscr{W}(R)$ are the displays over R of [Zi1]. A display is called V-nilpotent if the map $V^\sharp$ becomes nilpotent over R/pR. A homomorphism $R \to R'$ gives a strict frame homomorphism $\mathscr{W}(R) \to \mathscr{W}(R')$, and we write $\mathscr{P} \mapsto \mathscr{P} \otimes_R R'$ for the resulting base change of displays.

If N is a nilpotent nonunitary ring, $\hat{W}(N) \subseteq W(N)$ denotes the subgroup of all Witt vectors with only finitely many nonzero coefficients. If A is a local Artin ring with perfect residue field $k = A/\mathfrak{m}$ of characteristic p, there is a unique ring homomorphism $s : W(k) \to W(A)$ that lifts the projection $W(A) \to W(k)$, and the Zink ring $\mathbb{W}(A) = \hat{W}(\mathfrak{m}) \oplus s(W(k))$ is a subring of $W(A)$. There is a frame $\mathscr{D}_A = (\mathbb{W}(A), \mathbb{I}(A), A, f, \mathbb{f}_1)$ with an injective frame homomorphism $\mathscr{D}_A \to \mathscr{W}_A$, which is strict when $p \geqslant 3$; see [La3, Section 2.C]. Windows over $\mathscr{D}_A$ are called Dieudonné displays over A.

§2. The case of connected p-divisible groups

Let R be a normal complete noetherian local ring with (not necessarily perfect) residue field k of positive characteristic p, with fraction field K of characteristic zero, and with maximal ideal $\mathfrak{m}$. In this section, we recall how the Tate module of a connected p-divisible group over R is expressed in terms of its nilpotent display.

We fix an algebraic closure $\bar{K}$ of K and write $\mathcal{G}_K = \mathrm{Gal}(\bar{K}/K)$. Let $\bar{R} \subset \bar{K}$ be the integral closure of R, and for a finite extension E/K in $\bar{K}$ let $R_E = \bar{R} \cap E$. Then R_E is finite over R, and R_E is a complete noetherian

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local ring. Thus $\bar{R}$ is a local ring. Let $\bar{\mathfrak{m}} \subset \bar{R}$ and $\mathfrak{m}_E \subset R_E$ be the maximal ideals. We write

$$\hat{W}(\mathfrak{m}_E) = \varprojlim_n \hat{W}(\mathfrak{m}_E/\mathfrak{m}_E^n); \qquad \hat{W}(\bar{\mathfrak{m}}) = \varinjlim_E \hat{W}(\mathfrak{m}_E).$$

Let $\bar{W}(\bar{\mathfrak{m}})$ be the p-adic completion of $\hat{W}(\bar{\mathfrak{m}})$ and let $\bar{\mathfrak{m}}^\wedge$ be the p-adic completion of $\bar{\mathfrak{m}}$. The natural map $\bar{W}(\bar{\mathfrak{m}}) \to \bar{\mathfrak{m}}^\wedge$ is surjective. For a display $\mathscr{P} = (P, Q, F, F_1)$ over R let

$$\bar{P}_{\bar{\mathfrak{m}}} = \bar{W}(\bar{\mathfrak{m}}) \otimes_{W(R)} P; \qquad \bar{Q}_{\bar{\mathfrak{m}}} = \operatorname{Ker}(\bar{P}_{\bar{\mathfrak{m}}} \to \bar{\mathfrak{m}}^\wedge \otimes_R P/Q).$$

We call $\mathscr{P}$ nilpotent if the reduction $\mathscr{P} \otimes_R k$ is V-nilpotent in the usual sense, or equivalently if $\mathscr{P} \otimes_R R/\mathfrak{m}_R^n$ is V-nilpotent for all n; cf. [Zi1, Definition 13]. The functor BT of [Zi1] induces an equivalence of categories between nilpotent displays over R and connected p-divisible groups over R; this follows from [Zi1, Theorem 9] applied to the rings $R/\mathfrak{m}_R^n$, using that V-nilpotent displays and p-divisible groups over R are equivalent to compatible systems of such objects over $R/\mathfrak{m}_R^n$ for all n. A variant of the following result is stated in [Me2, Proposition 4.4].

PROPOSITION 2.1. (Zink) *Let $\mathscr{P}$ be a nilpotent display over R and let $G = \mathrm{BT}(\mathscr{P})$ be the associated connected p-divisible group over R. There is a natural exact sequence of $\mathcal{G}_K$-modules*

$$0 \longrightarrow T_p(G) \longrightarrow \bar{Q}_{\bar{\mathfrak{m}}} \xrightarrow{F_1 - 1} \bar{P}_{\bar{\mathfrak{m}}} \longrightarrow 0.$$

Here $T_p(G) = \operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G(\bar{K}))$ is the Tate module of G, and $\mathcal{G}_K$ acts on $\bar{P}_{\bar{\mathfrak{m}}}$ and $\bar{Q}_{\bar{\mathfrak{m}}}$ by its natural action on $\bar{W}(\bar{\mathfrak{m}})$.

The proof of Proposition 2.1 uses the following standard facts.

LEMMA 2.2. *Let A be an abelian group.*

(i) *If A has no p-torsion then* $\operatorname{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, A) = \varprojlim A/p^n A$.

(ii) *If $pA = A$ then* $\operatorname{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, A)$ *is zero.*

Proof. The group $\operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, A)$ is isomorphic to $\varprojlim \operatorname{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A)$ with transition maps induced by $p : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z}$. If the abelian group A is injective, the projective system $\operatorname{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A)$ has surjective transition maps and thus its $\varprojlim^1$ vanishes. Hence there is a Grothendieck spectral sequence for the functor $A \mapsto \operatorname{Hom}(\mathbb{Z}/p^n, A)_n$ from abelian groups

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to projective systems of abelian groups, composed with the functor $\varprojlim$,

$$\varprojlim^i(\mathrm{Ext}^j(\mathbb{Z}/p^n, A)) \Rightarrow \mathrm{Ext}^{i+j}(\mathbb{Q}_p/\mathbb{Z}_p, A). \tag{2.1}$$

The projective system of groups $\mathrm{Ext}^1(\mathbb{Z}/p^n\mathbb{Z}, A)$ is isomorphic to the system A/p^nA with transition maps induced by id_A. Thus the exact sequence of low degree terms (see for example, [We, Theorem 5.8.3]) associated to (2.1) gives an exact sequence

$$0 \to \varprojlim^1 \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A) \to \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, A) \to \varprojlim A/p^nA \to 0.$$

If A has no p-torsion then $\mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A) = 0$, and (i) follows. If $pA = A$ then the projective system $\mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A)$ has surjective transition maps, thus its $\varprojlim^1$ is zero, moreover $A/p^nA = 0$. This proves (ii). ∎

For a p-divisible group G over R and for E as above we write

$$\hat{G}(R_E) = \varprojlim_n G(R_E/\mathfrak{m}_E^n); \qquad \hat{G}(\bar{R}) = \varinjlim_E \hat{G}(R_E).$$

Lemma 2.3. *Multiplication by p is surjective on $\hat{G}(\bar{R})$.*

Proof. Let $x \in \hat{G}(R_E)$ be given. The inverse image of x under the multiplication map $p : G \to G$ is a compatible system of $G[p]$-torsors Y_n over $R_E/\mathfrak{m}_E^n$. Let $Y_n = \mathrm{Spec}\, A_n$ and $A = \varprojlim A_n$. Then $Y = \mathrm{Spec}\, A$ is a $G[p]$-torsor over R_E. For some finite extension F of E the set $Y(F) = Y(R_F)$ is nonempty, and x becomes divisible by p in $\hat{G}(R_F)$. ∎

Lemma 2.4. *There is an isomorphism $G(\bar{K})[p^r] \cong \hat{G}(\bar{R})[p^r]$ of $\mathcal{G}_K$-modules.*

Proof. Let $G_r = G[p^r]$. Then $\hat{G}(R_E)[p^r] = \varprojlim_n G_r(R_E/\mathfrak{m}_E^n) \cong G_r(R_E)$ since R_E is complete. Hence $\hat{G}(\bar{R})[p^r] \cong G_r(\bar{R}) = G_r(\bar{K}) = G(\bar{K})[p^r]$. ∎

Proof of Proposition 2.1. For a finite Galois extension E/K in $\bar{K}$ we write

$$\hat{P}_{E,n} = \hat{W}(\mathfrak{m}_E/\mathfrak{m}_E^n) \otimes_{W(R)} P$$

and define $\hat{Q}_{E,n}$ by the exact sequence of $\mathcal{G}_K$-modules

$$0 \to \hat{Q}_{E,n} \to \hat{P}_{E,n} \to \mathfrak{m}_E/\mathfrak{m}_E^n \otimes_R P/Q \to 0.$$

The definition of the functor BT in [Zi1, Theorem 81] gives an exact sequence of $\mathcal{G}_K$-modules

$$0 \longrightarrow \hat{Q}_{E,n} \xrightarrow{F_1-1} \hat{P}_{E,n} \longrightarrow G(R_E/\mathfrak{m}_E^n) \longrightarrow 0;$$

note that in [Zi1] a formal group G is viewed as a functor G' on nilpotent algebras, and $G(R_E/\mathfrak{m}_E^n) = G'(\mathfrak{m}_E/\mathfrak{m}_E^n)$ under this identification. The modules $\hat{Q}_{E,n}$ form a projective system with respect to n with surjective transition maps. Indeed, using a normal decomposition of $\mathscr{P}$ as in the paragraph before [Zi1, Theorem 81], this is reduced to the assertion that $\hat{W}(\mathfrak{m}_E/\mathfrak{m}_E^{n+1}) \to \hat{W}(\mathfrak{m}_E/\mathfrak{m}_E^n)$ is surjective, which is clear. Thus taking $\varinjlim_E \varprojlim_n$ of the preceding two sequences gives exact sequences of $\mathcal{G}_K$-modules

$$0 \to \hat{Q}_{\bar{\mathfrak{m}}} \to \hat{P}_{\bar{\mathfrak{m}}} \to \bar{\mathfrak{m}} \otimes_R P/Q \to 0 \tag{2.2}$$

and

$$0 \longrightarrow \hat{Q}_{\bar{\mathfrak{m}}} \xrightarrow{F_1-1} \hat{P}_{\bar{\mathfrak{m}}} \longrightarrow \hat{G}(\bar{R}) \longrightarrow 0 \tag{2.3}$$

with $\hat{Q}_{\bar{\mathfrak{m}}} = \varinjlim_E \varprojlim_n \hat{Q}_{E,n}$ and $\hat{P}_{\bar{\mathfrak{m}}} = \hat{W}(\bar{\mathfrak{m}}) \otimes_{W(R)} P$. Since $\bar{\mathfrak{m}} \otimes_R P/Q$ has no p-torsion, the p-adic completion of (2.2) remains exact, moreover the p-adic completion of the second and third terms are $\bar{P}_{\bar{\mathfrak{m}}}$ and $\bar{\mathfrak{m}}^\wedge \otimes_R P/Q$. Thus the p-adic completion of $\hat{Q}_{\bar{\mathfrak{m}}}$ is $\bar{Q}_{\bar{\mathfrak{m}}}$. Moreover $\hat{P}_{\bar{\mathfrak{m}}}$ has no p-torsion since $\hat{W}(\bar{\mathfrak{m}})$ is contained in the $\mathbb{Q}$-algebra $W(\bar{K})$. Using Lemmas 2.3 and 2.2, the Ext-sequence of $\mathbb{Q}_p/\mathbb{Z}_p$ with (2.3) reduces to the short exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \hat{G}(\bar{R})) \longrightarrow \bar{Q}_{\bar{\mathfrak{m}}} \xrightarrow{F_1-1} \bar{P}_{\bar{\mathfrak{m}}} \longrightarrow 0.$$

Lemma 2.4 gives an isomorphism $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \hat{G}(\bar{R})) \cong T_p(G)$ of $\mathcal{G}_K$-modules. ∎

§3. Module of invariants

Before we proceed we introduce a formal definition. Let $\mathscr{F} = (S, I, R, \sigma, \sigma_1)$ be a frame in the sense of [La2] such that S is a $\mathbb{Z}_p$-algebra and σ is $\mathbb{Z}_p$-linear; see Section 1. For an $\mathscr{F}$-window $\mathscr{P} = (P, Q, F, F_1)$ we consider the *module of invariants*

$$T(\mathscr{P}) = \{x \in Q \mid F_1(x) = x\};$$

this is a $\mathbb{Z}_p$-module. Let us record some of its formal properties.

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Functoriality in $\mathscr{F}$

Let $\alpha : \mathscr{F} \to \mathscr{F}' = (S', I', R', \sigma', \sigma_1')$ be a u-homomorphism of frames; see Section 1. Assume that a unit $c \in S'$ with $c\sigma'(c)^{-1} = u$ is given. For an $\mathscr{F}$-window $\mathscr{P}$ as above, one verifies that the S-linear map $P \to S' \otimes_S P$, $x \mapsto c \otimes x$ induces a $\mathbb{Z}_p$-linear map

$$\tau(\mathscr{P}) = \tau_c(\mathscr{P}) : T(\mathscr{P}) \to T(\alpha_* \mathscr{P}). \tag{3.1}$$

Duality

A bilinear form of $\mathscr{F}$-windows $\gamma : \mathscr{P} \times \mathscr{P}' \to \mathscr{P}''$ is a bilinear map of S-modules $\gamma : P \times P' \to P''$ that restricts to $Q \times Q' \to Q''$ such that for $x \in Q$ and $x' \in Q'$ we have

$$\gamma(F_1(x), F_1'(x')) = F_1''(\gamma(x, x')); \tag{3.2}$$

see [La2, Section 2]. It induces a bilinear map of $\mathbb{Z}_p$-modules $T(\mathscr{P}) \times T(\mathscr{P}') \to T(\mathscr{P}'')$ and a $\mathbb{Z}_p$-linear map $T(\mathscr{P}) \to \mathrm{Hom}(\mathscr{P}', \mathscr{P}'')$. Let us denote the $\mathscr{F}$-window (S, I, σ, σ_1) by $\mathscr{F}$ again. For each $\mathscr{F}$-window $\mathscr{P}$ there is a well-defined dual $\mathscr{F}$-window $\mathscr{P}^t = (P^t, Q^t, F^t, F_1^t)$ with a perfect bilinear form $\mathscr{P} \times \mathscr{P}^t \to \mathscr{F}$; see [La2, Section 2]. Explicitly, $P^t = \mathrm{Hom}_S(P, S)$ and $Q^t = \{g \in P^t \mid g(Q) \subseteq I\}$; the maps F_1^t and F^t are determined by (3.2) and the window axioms. The resulting homomorphism

$$T(\mathscr{P}) \to \mathrm{Hom}(\mathscr{P}^t, \mathscr{F}) \tag{3.3}$$

is bijective, which can be verified as follows: We have $\mathscr{F}^t = (S, S, \sigma_{-1}, \sigma)$ for some σ-linear map σ_{-1},[1] thus $T(\mathscr{P}) \cong \mathrm{Hom}(\mathscr{F}^t, \mathscr{P})$, which identifies (3.3) with the duality isomorphism $\mathrm{Hom}(\mathscr{F}^t, \mathscr{P}) \cong \mathrm{Hom}(\mathscr{P}^t, \mathscr{F})$.

Functoriality of duality

Let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a u-homomorphism of frames, and let c be as above. For a bilinear form of $\mathscr{F}$-windows $\gamma : \mathscr{P} \times \mathscr{P}' \to \mathscr{P}''$, the base change of γ multiplied by c^{-1} is a bilinear form of $\mathscr{F}'$-windows $\alpha_* \mathscr{P} \times \alpha_* \mathscr{P}' \to \alpha_* \mathscr{P}''$, which we denote by $\alpha_*(\gamma)$; see [La2, Lemma 2.14] and its proof. By passing

[1] Actually $\sigma_{-1} = \theta\sigma$ for θ as in [La2, Lemma 2.2].

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to the modules of invariants we obtain a commutative diagram

$$\begin{array}{ccc} T(\mathscr{P}) \times T(\mathscr{P}') & \xrightarrow{\gamma} & T(\mathscr{P}'') \\ {\scriptstyle \tau(\mathscr{P})\times\tau(\mathscr{P}')}\downarrow & & \downarrow{\scriptstyle \tau(\mathscr{P}'')} \\ T(\alpha_*\mathscr{P}) \times T(\alpha_*\mathscr{P}') & \xrightarrow{\alpha_*(\gamma)} & T(\alpha_*\mathscr{P}''). \end{array}$$

This will be applied to the bilinear form $\mathscr{P} \times \mathscr{P}^t \to \mathscr{F}$.

§4. The case of perfect residue fields

Let R, K, k, $\mathfrak{m}$ be as in Section 2, and assume in addition that the residue field k is perfect. As in [La3, Sections 2.C and 2.G] we consider the frame

$$\mathscr{D}_R = \varprojlim_n \mathscr{D}_{R/\mathfrak{m}^n} = (\mathbb{W}(R), \mathbb{I}_R, R, f, \mathbb{f}_1).$$

Windows over $\mathscr{D}_R$, called Dieudonné displays over R, are equivalent to p-divisible groups G over R by [Zi2] if $p \geqslant 3$ and by [La3, Theorem A] in general. The Tate module $T_p(G)$ can be expressed in terms of the Dieudonné display of G by a variant of Proposition 2.1 as follows.

Let R^{nr} be the strict Henselization of R. This is a normal local domain, which is excellent by [Gre, Corollary 5.6] or [Se], and thus its completion $\hat{R}^{\mathrm{nr}}$ is a normal complete noetherian local ring; see EGA IV, (7.8.3.1). Let $K^{\mathrm{nr}} \subset \hat{K}^{\mathrm{nr}}$ be the fraction fields of the rings $R^{\mathrm{nr}} \subset \hat{R}^{\mathrm{nr}}$, let $\tilde{K}$ be an algebraic closure of $\hat{K}^{\mathrm{nr}}$, and let $\tilde{R}$ be the integral closure of $\hat{R}^{\mathrm{nr}}$ in $\tilde{K}$. For each finite extension $E/\hat{K}^{\mathrm{nr}}$ in $\tilde{K}$ the ring $R_E = \tilde{R} \cap E$ is finite over $\hat{R}^{\mathrm{nr}}$, and R_E is a normal complete noetherian local ring. We define a frame

$$\mathscr{D}_{\tilde{R}} = \varinjlim_E \mathscr{D}_{R_E} = (\mathbb{W}(\tilde{R}), \mathbb{I}_{\tilde{R}}, \tilde{R}, f, \mathbb{f}_1)$$

where the direct limit is taken componentwise. Here $\mathbb{W}(\tilde{R})$ is a local ring since all $\mathbb{W}(R_E)$ are local with local homomorphisms in between. Since $\tilde{R}$ has no p-torsion, the componentwise p-adic completion of $\mathscr{D}_{\tilde{R}}$ is a frame

$$\hat{\mathscr{D}}_{\tilde{R}} = (\hat{\mathbb{W}}(\tilde{R}), \hat{\mathbb{I}}_{\tilde{R}}, \tilde{R}^\wedge, f, \mathbb{f}_1).$$

There are natural strict frame homomorphisms $\mathscr{D}_R \to \mathscr{D}_{\tilde{R}} \to \hat{\mathscr{D}}_{\tilde{R}}$.

Let $\bar{K}$ be the algebraic closure of K in $\tilde{K}$ and let $\mathcal{G}_K = \mathrm{Gal}(\bar{K}/K)$. The tensor product $\bar{K} \otimes_{K^{\mathrm{nr}}} \hat{K}^{\mathrm{nr}}$ is a subfield of $\tilde{K}$. Indeed, this ring is algebraic

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over $\hat{K}^{\mathrm{nr}}$, and it is a localization of $\bar{K} \otimes_{R^{\mathrm{nr}}} \hat{R}^{\mathrm{nr}}$, which is an integral domain by [Ra, Chapitre XI, Théorème 3]. If R is one-dimensional, then $\bar{K} \otimes_{K^{\mathrm{nr}}} \hat{K}^{\mathrm{nr}} = \tilde{K}$ because for every R, the étale coverings of the complements of the maximal ideals in Spec R^{nr} and Spec $\hat{R}^{\mathrm{nr}}$ coincide by [Ar, Part II, Theorem 2.1] or by [El, Théorème 5]. Let $\tilde{\mathcal{G}}_K$ be the group of automorphisms of $\tilde{K}$ whose restriction to $\bar{K}\hat{K}^{\mathrm{nr}}$ is induced by an element of $\mathcal{G}_K$. This group acts naturally on $\mathscr{D}_{\tilde{R}}$ and on $\hat{\mathscr{D}}_{\tilde{R}}$. By the above, the projection $\tilde{\mathcal{G}}_K \to \mathcal{G}_K$ is surjective, and bijective if R is one-dimensional.

Proposition 4.1. *Let G be a p-divisible group over R and let $\mathscr{P} = \Phi_R(G)$ be the Dieudonné display over R associated to G in [La3]. Let $\hat{\mathscr{P}}_{\tilde{R}} = (\hat{P}_{\tilde{R}}, \hat{Q}_{\tilde{R}}, F, F_1)$ be the base change of $\mathscr{P}$ to $\hat{\mathscr{D}}_{\tilde{R}}$. There is a natural exact sequence of $\tilde{\mathcal{G}}_K$-modules*

$$0 \longrightarrow T_p(G) \longrightarrow \hat{Q}_{\tilde{R}} \xrightarrow{F_1 - 1} \hat{P}_{\tilde{R}} \longrightarrow 0.$$

In particular, we have an isomorphism of $\mathcal{G}_K$-modules

$$\mathrm{per}_G : T_p(G) \xrightarrow{\sim} T(\hat{\mathscr{P}}_{\tilde{R}})$$

which we call the period isomorphism in display theory.

Proof of Proposition 4.1. For a finite extension $E/\hat{K}^{\mathrm{nr}}$ in $\tilde{K}$ let $\mathfrak{m}_E$ be the maximal ideal of R_E. For a p-divisible group G over R we set

$$\hat{G}(R_E) = \varprojlim_n G(R_E/\mathfrak{m}_E^n); \qquad \hat{G}(\tilde{R}) = \varinjlim_E \hat{G}(R_E).$$

The group $\tilde{\mathcal{G}}_K$ acts on the system $\hat{G}(R_E)$ for varying E and thus on $\hat{G}(\tilde{R})$. The latter can be described using [La3, Section 9] as follows.

Following [La3, Definition 9.1] let $\mathcal{J}_n = \mathcal{J}_{R/\mathfrak{m}^n}$ be the category of all $R/\mathfrak{m}^n$-algebras A such that the nilradical $\mathcal{N}_A$ is bounded nilpotent and such that $A/\mathcal{N}_A$ is a union of finite-dimensional k-algebras. Let $\mathscr{P}_n$ be the base change of $\mathscr{P}$ to $R/\mathfrak{m}^n$, and for $A \in \mathcal{J}_n$ let $\mathscr{P}_A = (P_A, Q_A, F, F_1)$ be the base change of $\mathscr{P}$ to A. As in [La3, (9–2)] we define a complex of presheaves $Z'(\mathscr{P}_n)$ on $\mathcal{J}_n^{\mathrm{op}}$ whose value on A is the complex of abelian groups

$$[Q_A \xrightarrow{F_1 - 1} P_A] \otimes [\mathbb{Z} \to \mathbb{Z}[1/p]]$$

in degrees $-1, 0, 1$. By [La3, Proposition 9.4], $Z'(\mathscr{P}_n)$ is a complex of pro-étale sheaves on $\mathcal{J}_n^{\mathrm{op}}$, which is acyclic outside degree zero, and the

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middle cohomology sheaf $H^0(Z'(\mathscr{P}))$ is represented by a well-defined p-divisible group $\mathrm{BT}(\mathscr{P})$ over R. By [La3, Proposition 9.7] there is a canonical isomorphism $G \cong \mathrm{BT}(\mathscr{P})$.

The ring $R_{E,n} = R_E/\mathfrak{m}_E^n$ is a local Artin ring with residue field $\bar{k}$, and thus it lies in $\mathcal{J}_n$. Every pro-étale covering of Spec $R_{E,n}$ has a section since every étale covering of Spec $R_{E,n}$ has a nonempty finite set of sections, and the projective limit of a projective system of nonempty finite sets is nonempty by [SP, Tag 086J]. Hence evaluating pro-étale sheaves at $R_{E,n}$ is an exact functor. It follows that the complex of abelian groups

$$C_{E,n} = [Q_{R_{E,n}} \xrightarrow{F_1-1} P_{R_{E,n}}] \otimes [\mathbb{Z} \to \mathbb{Z}[1/p]]$$

in degrees $-1, 0, 1$ is acyclic outside degree zero, and there is a canonical isomorphism $G(R_{E,n}) \cong H^0(C_{E,n})$. For varying n and E these are preserved by the action of $\tilde{\mathcal{G}}_K$. Let

$$C_E = \varprojlim_n C_{E,n}; \qquad C = \varinjlim_E C_E,$$

where E runs through the finite extensions of $\hat{K}^{\mathrm{nr}}$ in $\tilde{K}$. The group $\tilde{\mathcal{G}}_K$ acts on the complex C. Since the groups $G(R_{E,n})$ and the components of $C_{E,n}$ form surjective systems with respect to n, the complex C is acyclic outside degree zero, and we have an isomorphism of $\tilde{\mathcal{G}}_K$-modules $\hat{G}(\tilde{R}) \cong H^0(C)$. We will verify the following chain of isomorphisms $\cong$ and quasi-isomorphisms $\simeq$ of complexes of $\tilde{\mathcal{G}}_K$-modules, where Hom, $R\,\mathrm{Hom}$, and Ext^1 are taken in the category of abelian groups using a projective resolution of $\mathbb{Q}_p/\mathbb{Z}_p$, in particular Ext^1 is taken componentwise with respect to the second argument.

$$\begin{aligned} T_p(G) &\underset{(1)}{\cong} \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \hat{G}(\tilde{R})) \underset{(2)}{\simeq} R\,\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \hat{G}(\tilde{R})) \\ &\underset{(3)}{\simeq} R\,\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C) \underset{(4)}{\simeq} \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C[-1]) \underset{(5)}{\cong} [\hat{Q}_{\tilde{R}} \xrightarrow{F_1-1} \hat{P}_{\tilde{R}}] \end{aligned}$$

This will prove the proposition.

The torsion subgroups of $G(\bar{K})$ and of $\hat{G}(\tilde{R})$ coincide by Lemma 2.4 applied over $\hat{R}^{\mathrm{nr}}$, and (1) follows. Multiplication by p is surjective on $\hat{G}(\tilde{R})$ by Lemma 2.3 applied over $\hat{R}^{\mathrm{nr}}$, thus Lemma 2.2 gives $\mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, \hat{G}(\tilde{R})) = 0$, which proves (2). Since the cohomology of C is $\hat{G}(\tilde{R})$ in degree zero and zero otherwise, we obtain (3). To prove (4) we choose an exact sequence of

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abelian groups $0 \to F_1 \to F_0 \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$ with free F_i. This gives an exact sequence of complexes of $\tilde{\mathcal{G}}_K$-modules

$$0 \to \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C) \to \mathrm{Hom}(F_0, C) \xrightarrow{u} \mathrm{Hom}(F_1, C) \to \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C) \to 0.$$

We claim that $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C)$ is zero. Then the complex $\mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C)[-1]$ is quasi-isomorphic to the cone of u, which represents $R\,\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C)$, and (4) follows. Let $(P_{\tilde{R}}, Q_{\tilde{R}}, F, F_1)$ be the base change of $\mathscr{P}$ to $\mathscr{D}_{\tilde{R}}$ and let $P_{\bar{k}} = W(\bar{k}) \otimes_{\mathbb{W}(R)} P$. We have $Q_{R_{E,n}}[1/p] = P_{R_{E,n}}[1/p] = P_{\bar{k}}[1/p]$ because the cokernel of the inclusion $Q_{R_{E,n}} \to P_{R_{E,n}}$ is an $R_{E,n}$-module and thus p-power torsion, and the kernel of the surjective map $P_{R_{E,n}} \to P_{\bar{k}}$ is p-power torsion by [Zi3, Lemma 2.2]. Thus the complex C can be identified with the cone of the map of complexes

$$[Q_{\tilde{R}} \xrightarrow{F_1 - 1} P_{\tilde{R}}] \longrightarrow [P_{\bar{k}}[1/p] \xrightarrow{F_1 - 1} P_{\bar{k}}[1/p]].$$

Since $\tilde{R}$ is a domain of characteristic zero, the ring $W(\tilde{R})$ has no p-torsion. Since $\mathbb{W}(\tilde{R})$ is a subring of $W(\tilde{R})$ the projective $\mathbb{W}(\tilde{R})$-module $P_{\tilde{R}}$ and its submodule $Q_{\tilde{R}}$ have no p-torsion. Clearly $P_{\bar{k}}[1/p]$ has no p-torsion. Hence $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C)$ is zero, and (4) is proved. The p-adic completions of $P_{\tilde{R}}$ and $Q_{\tilde{R}}$ are $\hat{P}_{\tilde{R}}$ and $\hat{Q}_{\tilde{R}}$, while the p-adic completion of $P_{\bar{k}}[1/p]$ is zero. Thus Lemma 2.2 gives (5). □

Remark 4.2. Let $G_0 = \mathbb{Q}_p/\mathbb{Z}_p$. The isomorphisms per_G for all G can be altered by multiplication with a common p-adic unit. This allows to assume that per_{G_0} is the identity of $\mathbb{Z}_p$ in the following sense. Clearly $T_p(G_0) = \mathbb{Z}_p$. The Dieudonné display of μ_{p^∞} is $\mathscr{D}_R = (\mathbb{W}(R), \mathbb{I}_R, f, \mathbb{f}_1)$, and thus the Dieudonné display of G_0 is the dual $\mathscr{D}_R^t = (\mathbb{W}(R), \mathbb{W}(R), pu_0 f, f)$; cf. [La3, Section 2.C]. Then $T(\hat{\mathscr{D}}_{\tilde{R}}^t) = \hat{\mathbb{W}}(\tilde{R})^{f=1} = \mathbb{Z}_p$ by Lemma 4.3 below, and per_{G_0} can be viewed as a $\mathbb{Z}_p$-linear automorphism of $\mathbb{Z}_p$.

We note that the construction in the proof of Proposition 4.1 actually defines per_G only up to multiplication by a common p-adic unit because it uses the isomorphism $\mathrm{BT}(\mathscr{P}) \cong G$ provided by [La3, Proposition 9.7], which relies on [La3, Lemma 8.2], and that takes as an input the choice of such an isomorphism for G_0.

Lemma 4.3. *Let S be a p-adic torsion free ring with a Frobenius lift $\sigma : S \to S$. If $\mathrm{Spec}(S/pS)$ is connected, for example if S is a local ring, then $S^{\sigma=1} = \mathbb{Z}_p$.*

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Proof. It suffices to show that $(S/p^n)^{\sigma=1}=\mathbb{Z}/p^n$. The case $n=1$ holds because the polynomial $X^p-X=\prod_{a\in\mathbb{F}_p}(X-a)$ is separable. The general case follows by induction using the exact sequences $0\to S/p\xrightarrow[p^n]{} S/p^{n+1}\to S/p^n\to 0$. ∎

§5. A variant for the prime 2

We keep the notation and assumptions of Section 4 and assume that $p=2$. One can ask what the preceding constructions give when $\mathbb{W}$ and $\mathscr{D}$ are replaced by their v-stabilized variants $\mathbb{W}^+$ and $\mathscr{D}^+$ defined in [La3, Sections 1.D, 2.E]. This will be used in Section 6. We recall that $\mathbb{W}(R)\subset\mathbb{W}^+(R)\subset W(R)$ where the ring $\mathbb{W}^+(R)$ is stable under v, and we have a frame

$$\mathscr{D}_R^+=\varprojlim\mathscr{D}_{R/\mathfrak{m}^n}^+=(\mathbb{W}^+(R),\mathbb{I}_R^+,R,f,f_1)$$

where f_1 is the inverse of v. As earlier we put

$$\mathscr{D}_{\tilde{R}}^+=\varinjlim_{E}\mathscr{D}_{R_E}^+=(\mathbb{W}^+(\tilde{R}),\mathbb{I}_{\tilde{R}}^+,\tilde{R},f,f_1)$$

where E runs through the finite extensions of $\hat{K}^{\mathrm{nr}}$ in $\tilde{K}$ as in Section 4, and we denote the componentwise 2-adic completion of $\mathscr{D}_{\tilde{R}}^+$ by

$$\hat{\mathscr{D}}_{\tilde{R}}^+=(\hat{\mathbb{W}}^+(\tilde{R}),\hat{\mathbb{I}}_{\tilde{R}}^+,\tilde{R}^\wedge,f,f_1).$$

For a 2-divisible group G over R let G^m and G^u be the multiplicative and unipotent parts of G and define G^+ as a pushout of fppf sheaves in the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^m & \longrightarrow & G & \longrightarrow & G^u & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 2} & & \downarrow & & \| & & \\ 0 & \longrightarrow & G^m & \longrightarrow & G^+ & \longrightarrow & G^u & \longrightarrow & 0 \end{array} \tag{5.1}$$

The rows of (5.1) are exact, so G^+ is a 2-divisible group by [Me1, Chapter I, (2.4.3)]. On the level of Tate modules (5.1) gives a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T_2(G^m) & \longrightarrow & T_2(G) & \longrightarrow & T_2(G^u) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle 2} & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & T_2(G^m) & \longrightarrow & T_2(G^+) & \longrightarrow & T_2(G^u) & \longrightarrow & 0,
\end{array}
\tag{5.2}
$$

which shows that $T_2(G^+)$ is the pushout in the left hand square as a Galois module.

PROPOSITION 5.1. *Let G be a 2-divisible group over R with associated Dieudonné display $\mathscr{P} = \Phi_R(G)$. Let $\hat{\mathscr{P}}^+_{\tilde{R}} = (\hat{P}^+_{\tilde{R}}, \hat{Q}^+_{\tilde{R}}, F, F_1^+)$ be the base change of $\mathscr{P}$ to $\hat{\mathscr{D}}^+_{\tilde{R}}$. There is a natural exact sequence of $\tilde{\mathcal{G}}_K$-modules*

$$0 \longrightarrow T_2(G^+) \longrightarrow \hat{Q}^+_{\tilde{R}} \xrightarrow{F_1^+ - 1} \hat{P}^+_{\tilde{R}} \longrightarrow 0.$$

In particular, we have an isomorphism of $\mathcal{G}_K$-modules

$$\mathrm{per}^+_G : T_2(G^+) \xrightarrow{\sim} T(\hat{\mathscr{P}}^+_{\tilde{R}}).$$

Proof. Let $\bar{P}_{\bar{k}} = \bar{k} \otimes_{\mathbb{W}(R)} P$. We will construct the following commutative diagram with exact rows, where $\bar{F}$ is induced by F.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{Q}_{\tilde{R}} & \longrightarrow & \hat{Q}^+_{\tilde{R}} & \longrightarrow & \bar{P}_{\bar{k}} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle F_1 - 1} & & \Big\downarrow{\scriptstyle F_1^+ - 1} & & \Big\downarrow{\scriptstyle \bar{F} - 1} & & \\
0 & \longrightarrow & \hat{P}_{\tilde{R}} & \longrightarrow & \hat{P}^+_{\tilde{R}} & \longrightarrow & \bar{P}_{\bar{k}} & \longrightarrow & 0.
\end{array}
\tag{5.3}
$$

Assume that (5.3) is constructed and functorial in G. Since $\bar{P}_{\bar{k}} = \bar{k} \otimes_{\mathbb{W}(R)} P$ is the reduction mod 2 of the covariant Dieudonné module of $G_{\bar{k}}$, the Frobenius-linear endomorphism $\bar{F}$ is nilpotent if G is unipotent, and is given by an invertible matrix if G is of multiplicative type. Thus $\bar{F} - 1$ is surjective with kernel an $\mathbb{F}_2$-vector space of dimension equal to the height of G^m. Hence Proposition 4.1 implies that $F_1^+ - 1$ is surjective and gives an exact sequence

$$0 \to T_2(G) \to T(\hat{\mathscr{P}}^+_{\tilde{R}}) \to \mathrm{Ker}(\bar{F} - 1) \to 0.$$

The ring $W(\tilde{R})$ and its subring $\mathbb{W}^+(\tilde{R})$ are torsion free, which carries over to the 2-adic completion, hence $T(\hat{\mathscr{P}}^+_{\tilde{R}})$ is torsion free. It follows that $T(\hat{\mathscr{P}}^+_{\tilde{R}})=T_2(G)$ if G is unipotent, and multiplication by 2 gives an isomorphism $T(\hat{\mathscr{P}}^+_{\tilde{R}})\to T_2(G)$ if G is multiplicative type. Hence there is a pushout diagram (5.2) with $T(\hat{\mathscr{P}}^+_{\tilde{R}})$ in place of $T_2(G^+)$, which gives an isomorphism between these modules as required.

Let us construct (5.3). [La3, Lemma 1.10] implies that the inclusion map $\mathbb{W}(R_E/\mathfrak{m}_E^n)\to\mathbb{W}^+(R_E/\mathfrak{m}_E^n)$ is bijective when $2\in\mathfrak{m}_E^n$, and its cokernel is $\bar{k}\cdot v(1)$ as a $\mathbb{W}(R_E)$-module when $2\notin\mathfrak{m}_E^n$. It follows that the natural map $\iota:\hat{\mathbb{W}}(\tilde{R})\to\hat{\mathbb{W}}^+(\tilde{R})$ is injective with cokernel

$$\hat{\mathbb{W}}^+(\tilde{R})/\hat{\mathbb{W}}(\tilde{R})=\hat{\mathbb{I}}^+_{\tilde{R}}/\hat{\mathbb{I}}_{\tilde{R}}=\bar{k}\cdot v(1). \tag{5.4}$$

Moreover ι is a u_0-homomorphism of frames $\hat{\mathscr{D}}_{\tilde{R}}\to\hat{\mathscr{D}}^+_{\tilde{R}}$ where the unit $u_0\in\mathbb{W}^+(\mathbb{Z}_2)$ is defined by $v(u_0)=2-[2]$; see [La3, Section 2.E]. Since u_0 maps to 1 in $W(\mathbb{F}_2)$ there is a unique unit c_0 of $\mathbb{W}^+(\mathbb{Z}_2)$ which maps to 1 in $W(\mathbb{F}_2)$ such that $c_0f(c_0^{-1})=u_0$, namely $c_0=u_0f(u_0)f^2(u_0)\cdots$; see the proof of [La2, Proposition 8.7].

We extend the operator $\mathbb{f}_1$ of $\hat{\mathscr{D}}_{\tilde{R}}$ to $\hat{\mathscr{D}}^+_{\tilde{R}}$ by $\mathbb{f}_1=u_0^{-1}f_1$. Then $\mathbb{f}_1$ induces an f-linear endomorphism $\bar{\mathbb{f}}_1$ of $\bar{k}\cdot v(1)$. We claim that $\bar{\mathbb{f}}_1(v(1))=v(1)$. It suffices to prove this formula in $\mathbb{W}^+(\mathbb{Z}_2)/\mathbb{W}(\mathbb{Z}_2)\cong\mathbb{F}_2$, and thus it suffices to show that $\mathbb{f}_1(v(1))\notin\mathbb{W}(\mathbb{Z}_2)$. But $\mathbb{W}(\mathbb{Z}_2)$ is stable under $x\mapsto\mathbb{v}(x)=v(u_0x)$, and the element $\mathbb{v}(\mathbb{f}_1(v(1)))=v(1)$ does not lie in $\mathbb{W}(\mathbb{Z}_2)$. This proves the claim.

Similarly, we extend the operator F_1 of $\hat{\mathscr{P}}_{\tilde{R}}$ to $\hat{\mathscr{P}}^+_{\tilde{R}}$ by $F_1=u_0^{-1}F_1^+$. Then we have $c_0(F_1-1)=(F_1^+-1)c_0$ as homomorphisms $\hat{Q}^+_{\tilde{R}}\to\hat{P}^+_{\tilde{R}}$, and it suffices to construct the desired diagram with F_1 in place of F_1^+. Now (5.4) implies that $\hat{Q}^+_{\tilde{R}}/\hat{Q}_{\tilde{R}}=\hat{P}^+_{\tilde{R}}/\hat{P}_{\tilde{R}}=\bar{P}_{\bar{k}}\cdot v(1)$, which gives the exact rows. Clearly the left hand square of (5.3) commutes. The relation $F_1(ax)=\mathbb{f}_1(a)F(x)$ for $x\in\hat{P}^+_{\tilde{R}}$ and $a\in\hat{\mathbb{I}}^+_{\tilde{R}}$ applied with $a=v(1)$, together with $\bar{\mathbb{f}}_1(v(1))=v(1)$, shows that the right hand square of (5.3) commutes. □

REMARK 5.2. The period isomorphisms per_G and per_G^+ are related by $\mathrm{per}_G^+\circ i=\tau_{c_0}\circ\mathrm{per}_G$ where $i:T_2(G)\to T_2(G^+)$ is the inclusion map and $\tau_{c_0}:T(\hat{\mathscr{P}}_{\tilde{R}})\to T(\hat{\mathscr{P}}^+_{\tilde{R}})$ is the homomorphism defined in (3.1).

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§6. The relation with A_{cris}

Let R be a complete discrete valuation ring with perfect residue field k of characteristic p and fraction field K of characteristic zero. In this case the ring $\tilde{R}^\wedge$ is equal to $\bar{R}^\wedge$, the p-adic completion of the integral closure of R in $\bar{K}$. Let $A_{\mathrm{cris}} = A_{\mathrm{cris}}(\bar{R})$, this is the p-adic completion of the divided power envelope of the kernel of the canonical homomorphism $\theta : A_{\mathrm{inf}} \to \bar{R}^\wedge$, where $A_{\mathrm{inf}} = W(\mathcal{R})$, and where $\mathcal{R}$ is the projective limit of $\bar{R}/p\bar{R}$ under Frobenius. We have a frame

$$\mathcal{A}_{\mathrm{cris}} = (A_{\mathrm{cris}}, \mathrm{Fil}^1 A_{\mathrm{cris}}, \bar{R}^\wedge, \sigma, \sigma_1)$$

with $\sigma_1 = p^{-1}\sigma$. [2]

For a p-divisible group G over R let $\mathbb{D}(G)$ be its covariant Dieudonné crystal. The free A_{cris}-module $M = \mathbb{D}(G_{\bar{R}^\wedge})_{A_{\mathrm{cris}}}$ carries a filtration $\mathrm{Fil}^1 M$ and a σ-linear endomorphism F. The operator $F_1 = p^{-1}F$ is well defined on $\mathrm{Fil}^1 M$, and we get an $\mathcal{A}_{\mathrm{cris}}$-window $\mathcal{M} = (M, \mathrm{Fil}^1 M, F, F_1)$; see [Ki1, Lemma A.2] or [La3, Proposition 3.17]. The window associated to $\mathbb{Q}_p/\mathbb{Z}_p$ in this way is $\mathcal{A}^t_{\mathrm{cris}} = (A_{\mathrm{cris}}, A_{\mathrm{cris}}, p\sigma, \sigma)$.

Following [Fa, Section 6] one defines a period homomorphism

$$\mathrm{per}_{G,\mathrm{cris}} : T_p(G) \to T(\mathcal{M})$$

as follows. An element of $T_p(G)$ corresponds to a homomorphism $\mathbb{Q}_p/\mathbb{Z}_p \to G$ over $\bar{R}^\wedge$, and the resulting map of $\mathcal{A}_{\mathrm{cris}}$-windows $\mathcal{A}^t_{\mathrm{cris}} \to \mathcal{M}$ corresponds to an element of $T(\mathcal{M})$.[3] By [Fa, Theorem 7], $\mathrm{per}_{G,\mathrm{cris}}$ is bijective when $p \geqslant 3$, and injective with cokernel annihilated by p when $p = 2$. More precisely, for $p = 2$ the cokernel of $\mathrm{per}_{G,\mathrm{cris}}$ is zero if G is unipotent by [Ki2, Proposition 1.1.10], but the cokernel is an $\mathbb{F}_2$-vector space of dimension equal to the height of G if G is of multiplicative type; this can be verified for the multiplicative group $G = \mu_{p^\infty}$ and then follows from the fact that Fontaine's element $t \in A_{\mathrm{cris}}$ satisfies $t^{p-1} \in pA_{\mathrm{cris}}$; see [Fo2, (2.3.4)]. As in the proof of Proposition 5.1 it follows that for $p = 2$, the homomorphism $\mathrm{per}_{G,\mathrm{cris}}$ extends to an isomorphism $T_p(G^+) \cong T(\mathcal{M})$ with G^+ as in Section 5.

[2] The frame axioms require that $\sigma_1(\mathrm{Fil}^1 A_{\mathrm{cris}})$ generates A_{cris}. But $\xi = p - [\underline{p}]$ lies in $\mathrm{Fil}^1 A_{\mathrm{cris}}$, and $\sigma_1(\xi) = 1 - [\underline{p}]^p/p$ is a unit because $[\underline{p}]$ lies in the divided power ideal $\mathrm{Fil}^1 A_{\mathrm{cris}} + pA_{\mathrm{cris}}$.

[3] Actually [Fa] uses the contravariant Dieudonné crystal, which gives rise to the dual window $\mathcal{M}^t$ and the dual homomorphism $\mathcal{M}^t \to \mathcal{A}_{\mathrm{cris}}$. In the following this makes no difference since $\mathrm{Hom}(\mathcal{M}^t, \mathcal{A}_{\mathrm{cris}}) \cong \mathrm{Hom}(\mathcal{A}^t_{cris}, \mathcal{M}) \cong T(\mathcal{M})$; see (3.3).

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We want to relate this with the period isomorphisms of Sections 4 and 5. For the sake of uniformity, for $p \geqslant 3$ we write $\mathbb{W}^+ = \mathbb{W}$ etc. Then $\hat{\mathbb{W}}^+(\tilde{R}) \to \bar{R}^\wedge$ is a divided power thickening of p-adic rings for every p.

LEMMA 6.1. *There are unique homomorphisms $\varkappa_{\inf}$ and $\varkappa_{\mathrm{cris}}$ of thickenings of $\bar{R}^\wedge$ as below. They commute with Frobenius, and the diagram commutes.*

$$\begin{array}{ccc} A_{\inf} & \xrightarrow{\varkappa_{\inf}} & \hat{\mathbb{W}}(\tilde{R}) \\ \downarrow & & \downarrow \\ A_{\mathrm{cris}} & \xrightarrow{\varkappa_{\mathrm{cris}}} & \hat{\mathbb{W}}^+(\tilde{R}) \end{array}$$

Proof. Briefly said, the universal property of A_{cris} gives $\varkappa_{\mathrm{cris}}$, and the lemma explicates its construction. Namely, each x in the kernel of $\hat{\mathbb{W}}^+(\tilde{R})/p^n \to \bar{R}^\wedge/p$ satisfies $x^{p^n} = 0$ due to the divided powers on this ideal. Since the cokernel of the inclusion $\hat{\mathbb{W}}(\tilde{R}) \to \hat{\mathbb{W}}^+(\tilde{R})$ is the $\bar{k}$-vector space with basis $v(1)$ by (5.4), the kernel of $\hat{\mathbb{W}}(\tilde{R})/p^n \to \hat{\mathbb{W}}^+(\tilde{R})/p^n$ is the $\bar{k}$-vector space with basis $p^n v(1)$. Since $v(1)^2 = pv(1)$ this kernel has square zero. Thus for each x in the kernel of $\hat{\mathbb{W}}(\tilde{R})/p^n \to \bar{R}^\wedge/p$ we have $x^{p^{n+1}} = 0$, and the universality of the Witt vectors (see for example [La3, Lemma 1.4]) gives a unique homomorphism $\varkappa_{\inf}$ of extensions of $\bar{R}^\wedge/p$. The universality also implies that $\varkappa_{\inf}$ commutes with the Frobenius and with the projections to $\bar{R}^\wedge$. Since $\hat{\mathbb{W}}^+(\tilde{R}) \to \bar{R}^\wedge$ is a divided power extension of p-adic rings, $\varkappa_{\inf}$ extends uniquely to a homomorphism $\varkappa_{\mathrm{cris}}$, and $\varkappa_{\mathrm{cris}}$ commutes with the Frobenius because this holds for $\varkappa_{\inf}$. ∎

Since $\hat{\mathbb{W}}^+(\tilde{R})$ has no p-torsion it follows that $\varkappa_{\mathrm{cris}}$ is a $\tilde{\mathcal{G}}_K$-equivariant strict frame homomorphism

$$\varkappa_{\mathrm{cris}} : \mathcal{A}_{\mathrm{cris}} \to \hat{\mathscr{D}}^+_{\tilde{R}}.$$

For G and $\mathcal{M}$ as above let $\mathscr{P} = \Phi_R(G)$ be the Dieudonné display associated to G and let $\Phi^+_R(G)$ be its base change under the inclusion $\iota : \mathscr{D}_R \to \mathscr{D}^+_R$, which is the identity when $p \geqslant 3$. The $\mathscr{D}^+_R$-window $\Phi^+_R(G)$ can be defined by evaluating the crystal $\mathbb{D}(G)$ at $\mathbb{W}^+(R)$; see [La3, Theorem 3.19] if $p \geqslant 3$ and [La3, Proposition 3.24 & Theorem 4.9] if $p = 2$. By the functoriality of $\mathbb{D}(G)$ we get an isomorphism $\hat{\mathscr{P}}^+_{\tilde{R}} \cong \varkappa_{\mathrm{cris}\,*}(\mathcal{M})$ of $\hat{\mathscr{D}}^+_{\tilde{R}}$-

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windows, which induces a homomorphism of $\mathcal{G}_K$-modules

$$\tau : T(\mathcal{M}) \to T(\hat{\mathscr{P}}^+_{\tilde{R}})$$

as defined in (3.1) with $c = 1$.

PROPOSITION 6.2. *The following diagram of $\mathcal{G}_K$-modules commutes, and τ is an isomorphism.*

$$\begin{array}{ccc} T_p(G) & \xrightarrow{\mathrm{per}_{G,\mathrm{cris}}} & T(\mathcal{M}) \\ {\scriptstyle \mathrm{per}_G}\downarrow & & \downarrow{\scriptstyle \tau} \\ T(\hat{\mathscr{P}}_{\tilde{R}}) & \xrightarrow{\tau_{c_0}} & T(\hat{\mathscr{P}}^+_{\tilde{R}}). \end{array}$$

Proof of Proposition 6.2. The composition $\tau_{c_0} \circ \mathrm{per}_G$ extends to an isomorphism $T_p(G^+) \cong T(\hat{\mathscr{P}}^+_{\tilde{R}})$ by Proposition 5.1 and Remark 5.2. Thus if the diagram commutes, by the properties of $\mathrm{per}_{G,\mathrm{cris}}$ recalled above it follows that τ is an isomorphism. Let us prove that the diagram commutes.

We start with the case $G = \mathbb{Q}_p/\mathbb{Z}_p$. Then $T_p(G) = \mathbb{Z}_p$. By Remark 4.2, the associated windows can be identified as $\mathscr{P} = (\mathbb{W}(R), \mathbb{W}(R), pu_0 f, f)$ and $\mathscr{P}^+ = (\mathbb{W}^+(R), \mathbb{W}^+(R), pf, f)$ and $\mathcal{M} = (A_{\mathrm{cris}}, A_{\mathrm{cris}}, p\sigma, \sigma)$. The three modules $T(\mathcal{M}) = A_{\mathrm{cris}}^{\sigma=1}$ and $T(\hat{\mathscr{P}}_{\tilde{R}}) = \hat{\mathbb{W}}(\tilde{R})^{f=1}$ and $T(\hat{\mathscr{P}}^+_{\tilde{R}}) = \hat{\mathbb{W}}^+(\tilde{R})^{f=1}$ are then all identified with $\mathbb{Z}_p$; see Lemma 4.3. Under these identifications, the three arrows τ and $\mathrm{per}_{G,\mathrm{cris}}$ and per_G are the identity of $\mathbb{Z}_p$; see Remark 4.2. The base change $\iota_*(\mathscr{P})$ is equal to $(\mathbb{W}^+(R), \mathbb{W}^+(R), pu_0 f, u_0 f)$, and the implicit isomorphism $\iota_*(\mathscr{P}) \cong \mathscr{P}^+$ is necessarily given by multiplication with the unique unit $c \in \mathbb{W}^+(\mathbb{Z}_p)$ with $cu_0 = f(c)$ which maps to 1 in $W(\mathbb{F}_p)$, namely $c = c_0^{-1}$. Thus under the chosen identifications, $\tau_{c_0} = c_0 c_0^{-1}$ is the identity as well, and the diagram commutes for $\mathbb{Q}_p/\mathbb{Z}_p$.

Let now G be arbitrary. Since the map $\tau_{c_0} \circ \mathrm{per}_G = \mathrm{per}^+_G$ is injective with cokernel annihilated by p, the composition $\gamma = p \cdot (\mathrm{per}^+_G)^{-1} \circ \tau \circ \mathrm{per}_{G,\mathrm{cris}}$ is a well-defined functorial endomorphism of T_pG. We have to show that $\gamma = p$. By [Ta, Corollary 1], γ comes from an endomorphism γ_G of G; moreover γ_G is functorial in G and compatible with normal finite extensions of the base ring R inside $\bar{K}$. The endomorphisms γ_G induce a functorial endomorphism γ_H of each commutative finite flat p-group scheme H over a normal finite extension R' of R inside $\bar{K}$ because H can be embedded into a p-divisible group G by Raynaud [BBM, Theorem 3.1.1], and the quotient G/H is a

p-divisible group, so γ_G induces γ_H; cf. the proof of [Ki1, Theorem 2.3.5] or [La3, Proposition 8.1]. Assume that H is annihilated by p^r and let $H_0 = \mathbb{Z}/p^r\mathbb{Z}$. There is a normal finite extension R'' of R' inside $\bar{K}$ such that $H(\bar{K}) = H(R'') = \mathrm{Hom}_{R''}(H_0, H)$. Since $\gamma_{H_0} = p$ it follows that $\gamma_H = p$, and thus $\gamma_G = p$ for all G. ∎

§7. The ring $\mathfrak{S}^{\mathrm{nr}}$

Let us recall the ring $\mathfrak{S}^{\mathrm{nr}}$ of [Ki1], which is denoted by A_S^+ in [Fo1]. One starts with a two-dimensional complete regular local ring $\mathfrak{S}$ of characteristic zero with perfect residue field k of characteristic p equipped with a Frobenius lift $\sigma : \mathfrak{S} \to \mathfrak{S}$.

There is a unique ring homomorphism $\Delta : \mathfrak{S} \to W(\mathfrak{S})$ with $w_n \circ \Delta = \sigma^n$ where w_n is the nth Witt polynomial, and then $\Delta \circ \sigma = f \circ \Delta$; see [Laz, Chapter VII, Proposition 4.12]. The composition $\mathfrak{S} \to W(\mathfrak{S}) \to W(k)$ is surjective, which implies that $p \notin \mathfrak{m}_{\mathfrak{S}}^2$. Let $t \in \mathfrak{m}_{\mathfrak{S}} \setminus \mathfrak{m}_{\mathfrak{S}}^2$ map to zero in $W(k)$. Then $\mathfrak{S} = W(k)[[t]]$ and t generates the kernel of $\mathfrak{S} \to W(k)$, in particular $\sigma(t) \in t\mathfrak{S}$.

Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[t^{-1}]$ and let $\mathbb{E} = k((t))$ be its residue field. Fix a maximal unramified extension $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}}$ of $\mathcal{O}_{\mathcal{E}}$ and let $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ be its p-adic completion. Let $\mathbb{E}^{\mathrm{sep}}$ be the residue field of $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}}$, let $\bar{\mathbb{E}}$ be an algebraic closure of $\mathbb{E}^{\mathrm{sep}}$, let $\mathcal{O}_{\mathbb{E}} = \mathfrak{S}/p\mathfrak{S} = k[[t]]$, and let $\mathcal{O}_{\bar{\mathbb{E}}} \subset \bar{\mathbb{E}}$ be its integral closure. The Frobenius lift σ on $\mathfrak{S}$ extends uniquely to $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ and induces a homomorphism

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \xrightarrow{\Delta} W(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}) \to W(\bar{\mathbb{E}}) \tag{7.1}$$

with Δ as above. (7.1) is injective since both sides are discrete valuation rings with prime element p, and the reduction modulo p is injective. One defines $\mathfrak{S}^{\mathrm{nr}} = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \cap W(\mathcal{O}_{\bar{\mathbb{E}}})$ inside $W(\bar{\mathbb{E}})$. This ring is stable under σ, and $\mathfrak{S}^{\mathrm{nr}} = \varprojlim \mathfrak{S}_n^{\mathrm{nr}}$ with $\mathfrak{S}_n^{\mathrm{nr}} = (\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}}/p^n\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}}) \cap W_n(\mathcal{O}_{\bar{\mathbb{E}}})$ inside $\mathbb{W}_n(\bar{\mathbb{E}})$. By [Fo1, B 1.8.3] we have $\mathfrak{S}_n^{\mathrm{nr}} = \mathfrak{S}^{\mathrm{nr}}/p^n\mathfrak{S}^{\mathrm{nr}}$, in particular $\mathfrak{S}^{\mathrm{nr}}$ is p-adically complete.

§8. Breuil–Kisin modules

Let R be a complete discrete valuation ring with perfect residue field k of characteristic p and fraction field K of characteristic zero. We recall briefly the classification of commutative finite flat p-group schemes over R following [La3]; see the introduction for a brief discussion of the history of this result.

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Let $\mathfrak{S} = W(k)[[t]]$ and let $\sigma : \mathfrak{S} \to \mathfrak{S}$ be a Frobenius lift that stabilizes the ideal $t\mathfrak{S}$. We choose a presentation $R = \mathfrak{S}/E\mathfrak{S}$ where E has constant term p. Let $\pi \in R$ be the image of t, so π generates the maximal ideal of R.

For an $\mathfrak{S}$-module M let $M^{(\sigma)} = \mathfrak{S} \otimes_{\sigma,\mathfrak{S}} M$. We consider pairs (M, ϕ) where M is an $\mathfrak{S}$-module of finite type and where $\phi : M \to M^{(\sigma)}$ is an $\mathfrak{S}$-linear map with cokernel annihilated by E. Following the [VZ] terminology, (M, ϕ) is called a Breuil window (respectively a Breuil module) relative to $\mathfrak{S} \to R$ if the $\mathfrak{S}$-module M is free (respectively annihilated by a power of p and of projective dimension at most one).

We have a frame in the sense of [La2]

$$\mathscr{B} = (\mathfrak{S}, E\mathfrak{S}, R, \sigma, \sigma_1)$$

with $\sigma_1(Ex) = \sigma(x)$ for $x \in \mathfrak{S}$. Windows $\mathscr{P} = (P, Q, F, F_1)$ over $\mathscr{B}$ are equivalent to Breuil windows relative to $\mathfrak{S} \to R$ by the functor $\mathscr{P} \mapsto (Q, \phi)$ where $\phi : Q \to Q^{(\sigma)}$ is the composition of the inclusion $Q \to P$ with the inverse of the isomorphism $Q^{(\sigma)} \cong P$ defined by $a \otimes x \mapsto aF_1(x)$; the inverse functor maps (Q, ϕ) to (P, Q, F, F_1) with $P = Q^{(\sigma)}$ such that the inclusion $Q \to P$ is ϕ and $F_1 : Q \to P$ is $x \mapsto 1 \otimes x$, which also gives $F(x) = F_1(Ex)$; see [La2, Lemma 8.2].

As in [La3, Section 6] let $\varkappa$ be the ring homomorphism

$$\varkappa : \mathfrak{S} \xrightarrow{\Delta} W(\mathfrak{S}) \to W(R).$$

Its image lies in $\mathbb{W}(R)$ if and only if the endomorphism of $t\mathfrak{S}/t^2\mathfrak{S}$ induced by σ is divisible by p^2. In this case, $\varkappa : \mathfrak{S} \to \mathbb{W}(R)$ is a $\mathbb{u}$-homomorphism of frames $\mathscr{B} \to \mathscr{D}_R$ for the unit $\mathbb{u} = \mathbb{f}_1(\varkappa(E))$ of $\mathbb{W}(R)$, and $\varkappa$ induces an equivalence between $\mathscr{B}$-windows and $\mathscr{D}_R$-windows, which are equivalent to p-divisible groups over R. As a consequence, Breuil modules relative to $\mathfrak{S} \to R$ are equivalent to commutative finite flat p-group schemes over R; see [La3, Corollary 6.8].

Since $\mathbb{u}$ maps to 1 under $\mathbb{W}(R) \to W(k)$, there is a unique invertible element $\mathbb{c} \in \mathbb{W}(R)$ which maps to 1 in $W(k)$ with $\mathbb{c}\sigma(\mathbb{c}^{-1}) = \mathbb{u}$. It is given by $\mathbb{c} = \mathbb{u}\sigma(\mathbb{u})\sigma^2(\mathbb{u}) \cdots$; see the proof of [La2, Proposition 8.7].

8.1 Modules of invariants

For a Breuil module or Breuil window (M, ϕ) relative to $\mathfrak{S} \to R$ we write $M^{\mathrm{nr}} = \mathfrak{S}^{\mathrm{nr}} \otimes_{\mathfrak{S}} M$ and $M^{\mathrm{nr}}_{\mathcal{E}} = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}} M$ and define

$$T^{\mathrm{nr}}(M, \phi) = \{x \in M^{\mathrm{nr}} \mid \phi(x) = 1 \otimes x \text{ in } \mathfrak{S}^{\mathrm{nr}} \otimes_{\sigma,\mathfrak{S}^{\mathrm{nr}}} M^{\mathrm{nr}}\},$$

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$$T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi)=\{x\in M^{\mathrm{nr}}_{\mathcal{E}}\mid \phi(x)=1\otimes x \text{ in } \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}\otimes_{\sigma,\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}} M^{\mathrm{nr}}_{\mathcal{E}}\}.$$

For reference we record the following consequence of some results of [Fo1].

LEMMA 8.1. *The $\mathbb{Z}_p$-module $T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi)$ is finitely generated, and the natural map*

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}\otimes_{\mathbb{Z}_p} T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi)\to M^{\mathrm{nr}}_{\mathcal{E}} \tag{8.1}$$

is bijective. The natural map

$$T^{\mathrm{nr}}(M,\phi)\to T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi) \tag{8.2}$$

is bijective as well.

Proof. The homomorphism $\phi: M^{\mathrm{nr}}_{\mathcal{E}}\to (M^{\mathrm{nr}}_{\mathcal{E}})^{(\sigma)}$ is bijective. If $\psi: M^{\mathrm{nr}}_{\mathcal{E}}\to M^{\mathrm{nr}}_{\mathcal{E}}$ is the σ-linear map whose linearization is the inverse of ϕ, then $T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi)$ is equal to $\{x\in M^{\mathrm{nr}}_{\mathcal{E}}\mid \psi(x)=x\}$, and [Fo1, A 1.2.6] gives the first part of the lemma.

It remains to show that (8.2) is bijective. Assume first that (M,ϕ) is a Breuil window, let $M^*=\mathrm{Hom}_{\mathfrak{S}}(M,\mathfrak{S})$, and let $\psi: M^*\to M^*$ be the σ-linear map whose linearization is the dual of ϕ. Then (M^*,ψ) is a Kisin module as considered in [Ki1, (2.1.3)], and $T^{\mathrm{nr}}(M,\phi)$ can be identified with the module of $\mathfrak{S}$-linear maps $\lambda: M^*\to\mathfrak{S}^{\mathrm{nr}}$ with $\sigma\lambda=\lambda\psi$, and similarly for $T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi)$. Thus (8.2) is bijective by [Ki1, Corollary 2.1.4], which builds on [Fo1, B 1.8.4].

Assume now that (M,ϕ) is a Breuil module. Using that M is annihilated by a power of p and of projective dimension $\leqslant 1$ and that $C=\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}/\mathfrak{S}^{\mathrm{nr}}$ has no p-torsion, we see that $\mathrm{Tor}_1^{\mathfrak{S}}(C,M)$ is zero. It follows that $M^{\mathrm{nr}}\to M^{\mathrm{nr}}_{\mathcal{E}}$ is injective, and thus (8.2) is injective. One can find a Breuil window (M',ϕ') and a surjective map $(M',\phi')\to(M,\phi)$; see (b) in the proof of [La2, Theorem 8.5]. Then $T^{\mathrm{nr}}(M',\phi')\cong T^{\mathrm{nr}}_{\mathcal{E}}(M',\phi')\to T^{\mathrm{nr}}_{\mathcal{E}}(M,\phi)$ is surjective, and thus (8.2) is surjective. □

8.2 The choice of K_∞

Let $\bar{\mathfrak{m}}^\wedge$ be the maximal ideal of $\bar{R}^\wedge$. The power series $\sigma(t)$ defines a map $\sigma(t):\bar{\mathfrak{m}}^\wedge\to\bar{\mathfrak{m}}^\wedge$. This map is surjective, and the inverse images of algebraic elements are algebraic by the Weierstrass preparation theorem. Choose a system of elements $(\pi^{(n)})_{n\geqslant 0}$ of $\bar{K}$ with $\pi^{(0)}=\pi$ and $\sigma(t)(\pi^{(n+1)})=\pi^{(n)}$, and let K_∞ be the extension of K generated by all $\pi^{(n)}$. The system $(\pi^{(n)})$

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corresponds to an element $\underline{\pi} \in \mathcal{R} = \varprojlim \bar{R}/p\bar{R}$, the limit taken with respect to Frobenius.

We embed $\mathcal{O}_{\mathbb{E}} = k[[t]]$ into $\mathcal{R}$ by $t \mapsto \underline{\pi}$, and identify $\mathbb{E}^{\mathrm{sep}}$ and $\bar{\mathbb{E}}$ with subfields of $\operatorname{Frac} \mathcal{R}$; thus $W(\bar{\mathbb{E}}) \subset W(\operatorname{Frac} \mathcal{R})$. Then $\mathfrak{S}^{\mathrm{nr}} = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \cap W(\mathcal{R})$, and the unique ring homomorphism $\theta : W(\mathcal{R}) \to \bar{R}^{\wedge}$ which lifts the projection $W(\mathcal{R}) \to \bar{R}/p\bar{R}$ induces a homomorphism

$$pr^{\mathrm{nr}} : \mathfrak{S}^{\mathrm{nr}} \to \bar{R}^{\wedge}.$$

Let us verify that its restriction to $\mathfrak{S}$ is the given projection $\mathfrak{S} \to R$.

LEMMA 8.2. *We have $pr^{\mathrm{nr}}(t) = \pi$.*

Proof. The lemma is easy if $\sigma(t) = t^p$ since then $\Delta(t) = [t]$ in $W(\mathfrak{S})$, which maps to $[\underline{\pi}]$ in $W(\mathcal{R})$, and $\theta([\underline{\pi}]) = \pi$ in this case. In general let $\Delta(t) = (g_0, g_1, \ldots)$ with $g_i \in \mathfrak{S}$; these power series are determined by the relations

$$g_0^{p^n} + p g_1^{p^{n-1}} + \cdots + p^n g_n = \sigma^n(t)$$

for $n \geqslant 0$. Let $x = (x_0, x_1, \ldots) \in W(\mathcal{R})$ be the image of t, thus $x_i = g_i(\underline{\pi})$. Write $x_i = (x_{i,0}, x_{i,1}, \ldots)$ with $x_{i,n} = g_i(\underline{\pi})_n \in \bar{R}/p\bar{R}$. If $\tilde{x}_{i,n} \in \bar{R}^{\wedge}$ lifts $x_{i,n}$ we have

$$pr^{\mathrm{nr}}(t) = \theta(x) = \lim_{n \to \infty} ((\tilde{x}_{0,n})^{p^n} + p(\tilde{x}_{1,n})^{p^{n-1}} + \cdots + p^n \tilde{x}_{n,n}).$$

If we choose $\tilde{x}_{i,n} = g_i(\pi^{(n)})$, the sum in the limit becomes $\sigma^n(t)(\pi^{(n)}) = \pi$, and the lemma is proved. □

The natural action of $\mathcal{G}_{K_\infty} = \operatorname{Gal}(\bar{K}/K_\infty)$ on $W(\operatorname{Frac} \mathcal{R})$ is trivial on $\mathcal{O}_{\mathcal{E}}$, and therefore it stabilizes $\mathcal{O}_{\hat{\mathcal{E}}^{\mathrm{nr}}}$ and $\mathfrak{S}^{\mathrm{nr}}$ with trivial action on $\mathfrak{S}$. Thus $\mathcal{G}_{K_\infty}$ acts on $T^{\mathrm{nr}}(M, \phi)$ for each Breuil window or Breuil module (M, ϕ).

8.3 From $\mathfrak{S}^{\mathrm{nr}}$ to Zink rings

The composition of the inclusion $\mathfrak{S}^{\mathrm{nr}} \to W(\mathcal{R})$ chosen above with the homomorphism $\varkappa_{\mathrm{inf}} : W(\mathcal{R}) \to \hat{\mathbb{W}}(\tilde{R})$ from Lemma 6.1 is a ring homomorphism

$$\varkappa^{\mathrm{nr}} : \mathfrak{S}^{\mathrm{nr}} \to \hat{\mathbb{W}}(\tilde{R})$$

that commutes with Frobenius and with the projections to $\bar{R}^{\wedge}$.

LEMMA 8.3. *If the image of $\varkappa : \mathfrak{S} \to W(R)$ lies in $\mathbb{W}(R)$, then the following diagram of rings commutes, where the vertical maps are the* *obvious inclusions.*

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{\varkappa} & \mathbb{W}(R) \\ \downarrow & & \downarrow \iota \\ \mathfrak{S}^{\mathrm{nr}} & \xrightarrow{\varkappa^{\mathrm{nr}}} & \hat{\mathbb{W}}(\tilde{R}) \end{array}$$

Proof. The assumption $\varkappa(\mathfrak{S}) \subset \mathbb{W}(R)$ is equivalent to $\Delta(\mathfrak{S}) \subset \mathbb{W}(\mathfrak{S})$; see [La3, Proposition 6.2]. As in the proof of Lemma 8.2 we write $\Delta(t) = (g_0, g_1, \ldots)$ with $g_i \in \mathfrak{S}$. Note that $g_0 = t$. We have to show that

$$\varkappa_{\inf}((g_0(\underline{\pi}), g_1(\underline{\pi}), \ldots)) = \iota((g_0(\pi), g_1(\pi), \ldots))$$

in $\hat{\mathbb{W}}(\tilde{R})$. Again, if $y_{i,n} \in \hat{\mathbb{W}}(\tilde{R})$ is a lift of $x_{i,n} = g_i(\underline{\pi})_n \in \bar{R}/p\bar{R}$, the left hand side of this equation is equal to

$$\lim_{n\to\infty} \big((y_{0,n})^{p^n} + p(y_{1,n})^{p^{n-1}} + \cdots + p^n y_{n,n}\big).$$

We will choose $y_{i,n} \in \mathbb{W}(\tilde{R})$ (no p-adic completion) such that the sum in the limit is equal to $(g_0(\pi), g_1(\pi), \ldots)$ in $\mathbb{W}(\tilde{R})$; this will prove the lemma. In the special case $\sigma(t) = t^p$ we have $g_i = 0$ for $i \geqslant 1$, and we can take $y_{0,n} = [\pi^{(n)}]$ and $y_{i,n} = 0$ for $i \geqslant 1$; then the calculation is trivial. In general, let $\Delta(g_i) = (h_{i,0}, h_{i,1}, \ldots)$ in $\mathbb{W}(\mathfrak{S})$, so the power series $h_{i,j}$ are determined by the equations

$$h_{i,0}^{p^m} + p h_{i,1}^{p^{m-1}} + \cdots + p^m h_{i,m} = \sigma^m(g_i) = g_i(\sigma^m(t))$$

for $m \geqslant 0$, and put $y_{i,n} = (h_{i,0}(\pi^{(n)}), h_{i,1}(\pi^{(n)}), \ldots) \in \mathbb{W}(\tilde{R})$. Since the Witt polynomials $w_m(X_0, \ldots, X_m) = X_0^{p^m} + \cdots + p^m X_m$ for $m \geqslant 0$ define an injective map $\mathbb{W}(\tilde{R}) \subset W(\tilde{R}) \to \tilde{R}^\infty$, we have to show that for $n, m \geqslant 0$ the following holds.

$$w_m((y_{0,n})^{p^n} + p(y_{1,n})^{p^{n-1}} + \cdots + p^n y_{n,n}) = w_m((g_0(\pi), g_1(\pi), \ldots))$$

The right hand side is equal to $\sigma^m(t)(\pi)$. Since w_m is a ring homomorphism and since $w_m(y_{i,n}) = g_i(\sigma^m(t)(\pi^{(n)}))$, the left hand side is equal to $\sigma^n(t)(\sigma^m(t)(\pi^{(n)})) = \sigma^{n+m}(t)(\pi^{(n)}) = \sigma^m(t)(\pi)$ too. ∎

We define a frame

$$\mathscr{B}^{\mathrm{nr}} = (\mathfrak{S}^{\mathrm{nr}}, E\mathfrak{S}^{\mathrm{nr}}, \mathfrak{S}^{\mathrm{nr}}/E\mathfrak{S}^{\mathrm{nr}}, \sigma, \sigma_1)$$

with $\sigma_1(Ex) = \sigma(x)$ for $x \in \mathfrak{S}^{\mathrm{nr}}$.

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LEMMA 8.4. *The element* $\mathbb{u}' = \mathbb{f}_1(\varkappa^{\mathrm{nr}}(E)) \in \hat{\mathbb{W}}(\tilde{R})$ *is a unit, and the ring homomorphism* $\varkappa^{\mathrm{nr}} : \mathfrak{S}^{\mathrm{nr}} \to \hat{\mathbb{W}}(\tilde{R})$ *is a* $\mathbb{u}'$*-homomorphism of frames* $\varkappa^{\mathrm{nr}} : \mathscr{B}^{\mathrm{nr}} \to \hat{\mathscr{D}}_{\tilde{R}}$.

Proof. Clearly $\varkappa^{\mathrm{nr}}$ commutes with the projections to $\bar{R}^{\wedge}$ and with the Frobenius. Lemma 8.2 implies that $pr^{\mathrm{nr}}(E) = 0$, thus $\varkappa^{\mathrm{nr}}(E) \in \hat{\mathbb{I}}_{\tilde{R}}$. For $x \in \mathfrak{S}^{\mathrm{nr}}$ we compute $\mathbb{f}_1(\varkappa^{\mathrm{nr}}(Ex)) = \mathbb{f}_1(\varkappa^{\mathrm{nr}}(E)) \cdot f(\varkappa^{\mathrm{nr}}(x)) = \mathbb{u}' \cdot \varkappa^{\mathrm{nr}}(\sigma_1(Ex))$ as required. It remains to show that $\mathbb{u}'$ is a unit. The projection $\tilde{R} \to \bar{k}$ induces a local homomorphism of local rings $\hat{\mathbb{W}}(\tilde{R}) \to W(\bar{k})$ that commutes with f and $\mathbb{f}_1$. The composition $\mathfrak{S} \to \mathfrak{S}^{\mathrm{nr}} \to \hat{\mathbb{W}}(\tilde{R}) \to W(\bar{k})$ commutes with Frobenius and is thus equal to the homomorphism $t \mapsto 0$. Thus E maps to p in $W(\bar{k})$, so $\mathbb{u}'$ maps to $\mathbb{f}_1(p) = v^{-1}(p) = 1$ in $W(\bar{k})$, and it follows that $\mathbb{u}'$ is a unit. □

From now on we assume that the image of $\varkappa$ lies in $\mathbb{W}(R)$, so that Lemma 8.3 applies. Then $\mathbb{u}'$ is the image of $\mathbb{u} \in \mathbb{W}(R)$, and we get a commutative square of frames where the horizontal arrows are $\mathbb{u}$-homomorphisms and the vertical arrows are strict:

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\ \varkappa\ } & \mathscr{D}_R \\ \downarrow & & \downarrow \\ \mathscr{B}^{\mathrm{nr}} & \xrightarrow{\ \varkappa^{\mathrm{nr}}\ } & \hat{\mathscr{D}}_{\tilde{R}} \end{array}$$

Here $\mathcal{G}_K$ acts on $\hat{\mathscr{D}}_{\tilde{R}}$ and $\mathcal{G}_{K_\infty}$ acts on $\mathscr{B}^{\mathrm{nr}}$, and $\varkappa^{\mathrm{nr}}$ is $\mathcal{G}_{K_\infty}$-equivariant.

8.4 Identification of modules of invariants

Now we can state the main result of this section. Let (M, ϕ) be a Breuil window relative to $\mathfrak{S} \to R$ with associated $\mathscr{B}$-window $\mathscr{P}$, and let $\mathscr{P}^{\mathrm{nr}}$ be the base change of $\mathscr{P}$ to $\mathscr{B}^{\mathrm{nr}}$. By definition we have $T^{\mathrm{nr}}(M, \phi) = T(\mathscr{P}^{\mathrm{nr}})$ as $\mathcal{G}_{K_\infty}$-modules. Let $\mathscr{P}_{\mathscr{D}}$ be the base change of $\mathscr{P}$ to $\mathscr{D}_R$ and let $\hat{\mathscr{P}}_{\tilde{R}}$ be the common base change of $\mathscr{P}^{\mathrm{nr}}$ and $\mathscr{P}_{\mathscr{D}}$ to $\hat{\mathscr{D}}_{\tilde{R}}$. As in (3.1), multiplication by $\mathbb{c}$ induces a $\mathcal{G}_{K_\infty}$-invariant homomorphism

$$\tau(\mathscr{P}^{\mathrm{nr}}) : T(\mathscr{P}^{\mathrm{nr}}) \to T(\hat{\mathscr{P}}_{\tilde{R}}).$$

We recall that the $\mathcal{G}_K$-module $T(\hat{\mathscr{P}}_{\tilde{R}})$ is canonically isomorphic to the Tate module of the p-divisible group associated to (M, ϕ); see Proposition 4.1.

PROPOSITION 8.5. *The homomorphism* $\tau(\mathscr{P}^{\mathrm{nr}})$ *is bijective.*

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Proof. Let h be the $\mathfrak{S}$-rank of M. The source and target of $\tau(\mathscr{P}^{\mathrm{nr}})$ are free $\mathbb{Z}_p$-modules of rank h which are exact functors of $\mathscr{P}$. Indeed, for $T(\mathscr{P}^{\mathrm{nr}})=T^{\mathrm{nr}}(M,\phi)$ this follows from Lemma 8.1, and it holds for $T(\hat{\mathscr{P}}_{\tilde{R}})$ by Proposition 4.1, using that the height of a p-divisible group is equal to the rank of its Dieudonné display; this can be verified over perfect fields, and then the Dieudonné display is the classical Dieudonné module.

Consider first the case where the p-divisible group associated to $\mathscr{P}$ is étale, which means that $\mathscr{P}=(P,Q,F,F_1)$ has $P=Q$, and $F_1:Q\to P$ is a σ-linear isomorphism. Then (P,F_1) is an étale σ-module over $\mathfrak{S}$. Since $\mathfrak{S}^{\mathrm{nr}}$ is p-adically complete with $\mathfrak{S}^{\mathrm{nr}}/p=\mathcal{O}_{\mathbb{E}^{\mathrm{sep}}}$, a $\mathbb{Z}_p$-basis of $T(\mathscr{P}^{\mathrm{nr}})$ is an $\mathfrak{S}^{\mathrm{nr}}$-basis of P^{nr}. Using Lemma 4.3 it follows that a $\mathbb{Z}_p$-basis of $T(\hat{\mathscr{P}}_{\tilde{R}})$ is a $\hat{\mathbb{W}}(\tilde{R})$-basis of $\hat{P}_{\tilde{R}}=\hat{\mathbb{W}}(\tilde{R})\otimes_{\mathfrak{S}^{\mathrm{nr}}}P^{\mathrm{nr}}$. Thus the homomorphism of $\mathbb{Z}_p$-modules $\tau(\mathscr{P}^{\mathrm{nr}})$ becomes an isomorphism over $\hat{\mathbb{W}}(\tilde{R})$. Since $\mathbb{Z}_p\to\hat{\mathbb{W}}(\tilde{R})$ is a local homomorphism it follows that $\tau(\mathscr{P}^{\mathrm{nr}})$ is bijective.

Consider next the case $\mathscr{P}=\mathscr{B}$, which corresponds to the p-divisible group μ_{p^∞}. Assume that the proposition does not hold for $\mathscr{B}$, i.e., that $\tau(\mathscr{B}^{\mathrm{nr}})$ is divisible by p. For a perfect extension k' of k let $\mathfrak{S}'=W(k')[[t]]$ and $R'=\mathfrak{S}'/E\mathfrak{S}'$, and let $\mathscr{B}'$ be the corresponding analogue of the frame $\mathscr{B}$; note that the Frobenius lift σ of $\mathfrak{S}$ extends uniquely to $\mathfrak{S}'$. The natural homomorphism $T(\mathscr{B}^{\mathrm{nr}})\to T(\mathscr{B}'^{\,\mathrm{nr}})$ is bijective because it becomes bijective over $\mathcal{O}_{\widehat{\mathcal{E}'^{\mathrm{nr}}}}$ by Lemma 8.1. The natural homomorphism $T(\hat{\mathscr{D}}_{\tilde{R}})\to T(\hat{\mathscr{D}}_{\tilde{R}'})$ is bijective since the equivalence between p-divisible groups and Dieudonné displays is compatible with arbitrary base change by [La3, Lemma 9.6]. Hence the homomorphism $\tau(\mathscr{B}^{\mathrm{nr}})$ can be identified with $\tau(\mathscr{B}'^{\,\mathrm{nr}})$, so k can be replaced by k', which allows to assume that k is uncountable. Let $\mathscr{P}_0$ be the étale $\mathscr{B}$-window that corresponds to $\mathbb{Q}_p/\mathbb{Z}_p$. We consider extensions of $\mathscr{B}$-windows $0\to\mathscr{B}\to\mathscr{P}_1\to\mathscr{P}_0\to 0$, which correspond to extensions in $\mathrm{Ext}^1_R(\mathbb{Q}_p/\mathbb{Z}_p,\mu_{p^\infty})$. Since $\tau(\mathscr{P}_0^{\mathrm{nr}})$ is bijective and $\tau(\mathscr{B}^{\mathrm{nr}})$ is divisible by p, the image of $\tau(\mathscr{P}_1^{\mathrm{nr}})$ provides a splitting of the reduction modulo p of the exact sequence of $\mathcal{G}_{K_\infty}$-modules

$$0\to T(\hat{\mathscr{D}}_{\tilde{R}})\to T((\hat{\mathscr{P}}_1)_{\tilde{R}})\to T((\hat{\mathscr{P}}_0)_{\tilde{R}})\to 0.$$

Hence the composite homomorphism

$$\mathrm{Ext}^1_R(\mathbb{Q}_p/\mathbb{Z}_p,\mu_{p^\infty})\to\mathrm{Ext}^1_{\mathbb{F}_p[\mathcal{G}_K]}(\mathbb{Z}/p\mathbb{Z},\mu_p)\to\mathrm{Ext}^1_{\mathbb{F}_p[\mathcal{G}_{K_\infty}]}(\mathbb{Z}/p\mathbb{Z},\mu_p) \tag{8.3}$$

is zero. The first group in (8.3) can be identified with the set of deformations of $\mathbb{Q}_p/\mathbb{Z}_p\oplus\mu_{p^\infty}$ from k to R. The second group is isomorphic

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to $\mathrm{Ext}^1_K(\mathbb{Z}, \mu_p)$, which is isomorphic to the Galois cohomology group $H^1(\mathcal{G}_K, \mu_p) \cong K^*/(K^*)^p$. As in [La1, Lemma 7.2] it follows that the first arrow in (8.3) can be identified with the natural map $1 + \mathfrak{m}_R \to K^*/(K^*)^p$, whose image is uncountable since k is uncountable. Since for a finite extension K'/K the homomorphism $H^1(K, \mu_p) \to H^1(K', \mu_p)$ has finite kernel by the inflation-restriction exact sequence, the kernel of the second map in (8.3) is countable. Thus the composition (8.3) cannot be zero, and the proposition is proved for $\mathscr{P} = \mathscr{B}$.

Finally let $\mathscr{P}$ be arbitrary. Duality gives the following commutative diagram; see the end of Section 3.

$$\begin{array}{ccc} T(\mathscr{P}^{\mathrm{nr}}) \times T(\mathscr{P}^{t\,\mathrm{nr}}) & \longrightarrow & T(\mathscr{B}^{\mathrm{nr}}) \\ {\scriptstyle \tau(\mathscr{P}^{\mathrm{nr}})\times\tau(\mathscr{P}^{t\,\mathrm{nr}})}\downarrow & & \downarrow{\scriptstyle \tau(\mathscr{B}^{\mathrm{nr}})} \\ T(\hat{\mathscr{P}}_{\tilde{R}}) \times T(\hat{\mathscr{P}}^t_{\tilde{R}}) & \longrightarrow & T(\hat{\mathscr{D}}_{\tilde{R}}) \end{array} \tag{8.4}$$

The upper line of (8.4) is a bilinear form of free $\mathbb{Z}_p$-modules of rank h, whose scalar extension under $\mathbb{Z}_p \to \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ is perfect since (8.2) and (8.1) are bijective. Since this is a local homomorphism the upper line of (8.4) is perfect. Proposition 4.1 implies that the lower line of (8.4) is a bilinear form of free $\mathbb{Z}_p$-modules of rank h. We have seen that $\tau(\mathscr{B}^{\mathrm{nr}})$ is bijective. These properties imply that $\tau(\mathscr{P}^{\mathrm{nr}})$ is bijective. □

For a p-divisible group or commutative finite flat p-group scheme G over R let $(M(G), \phi)$ be the associated Breuil window or Breuil module. In the first case let $T(G)$ be the Tate module of G, and in the second case let $T(G) = G(\bar{K})$.

Corollary 8.6. *There is an isomorphism of $\mathcal{G}_{K_\infty}$-modules $T(G) \cong T^{\mathrm{nr}}(M(G), \phi)$.*

Proof. For p-divisible groups this is immediate from Propositions 4.1 and 8.5. The finite case follows from the p-divisible case as in the proof of [La3, Corollary 6.8]. More precisely, a finite G can be written as the kernel of an isogeny of p-divisible groups $G_0 \to G_1$, which gives exact sequences $0 \to T(G_0) \to T(G_1) \to T(G) \to 0$ and $0 \to M(G_0) \to M(G_1) \to M(G) \to 0$, and the latter gives an exact sequence $0 \to T^{\mathrm{nr}}(M(G_0)) \to T^{\mathrm{nr}}(M(G_1)) \to T^{\mathrm{nr}}(M(G)) \to 0$. The resulting isomorphism $T(G) \cong T^{\mathrm{nr}}(M(G))$ is independent of the resolution $G_0 \to G_1$ of G. □

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