Canad. J. Math. 2024, pp. 1–31
http://dx.doi.org/10.4153/S0008414X23000834

Solvability of Hessian quotient equations in exterior domains

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Abstract. In this paper, we study the Dirichlet problem of Hessian quotient equations of the form $S_k(D^2u)/S_l(D^2u) = g(x)$ in exterior domains. For $g \equiv$ const., we obtain the necessary and sufficient conditions on the existence of radially symmetric solutions. For g being a perturbation of a generalized symmetric function at infinity, we obtain the existence of viscosity solutions by Perron's method. The key technique we develop is the construction of sub- and supersolutions to deal with the non-constant right-hand side g.

1 Introduction

In this paper, we study the exterior Dirichlet problem of the Hessian quotient equation

$$\frac{S_k(D^2u)}{S_l(D^2u)} = g(x) \quad \text{in } \mathbb{R}^n\backslash\bar{\Omega}, \tag{1.1}$$

$$u = \phi \quad \text{on } \partial\Omega, \tag{1.2}$$

where $n \geq 2$, $0 \leq l < k \leq n$, Ω is a smooth, bounded, strictly convex open set in $\mathbb{R}^n$, $\phi \in C^2(\partial\Omega)$, $0 < g \in C^0(\mathbb{R}^n\backslash\Omega)$, $S_0(D^2u) := \sigma_0(\lambda(D^2u)) := 1$,

$$S_j(D^2u) := \sigma_j(\lambda(D^2u)) := \sum_{1\leq i_1<\cdots<i_j\leq n} \lambda_{i_1}\cdots\lambda_{i_j}, \quad j = 1, 2, \ldots, n$$

denotes the jth elementary symmetric function of $\lambda(D^2u) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, the eigenvalues of the Hessian matrix of u.

Equation (1.1) has received a lot of attentions since the classical work of Caffarelli, Nirenberg, and Spruck [7] and Trudinger [22]. For $l = 0$, it is the k-Hessian equation. In particular, if $k = 1$, it is the Poisson equation, while it is the Monge–Ampère equation if $k = n$. For $n = k = 3$ and $l = 1$, it is the special Lagrangian equation which is closely connected with geometric problems: If u satisfies $\det D^2u = \Delta u$ in $\mathbb{R}^3$, then

Received by the editors April 11, 2023; accepted November 20, 2023.
Published online on Cambridge Core December 14, 2023.
Dai is supported by the Shandong Provincial Natural Science Foundation (Grant No. ZR2021MA054). Bao is supported by the Beijing Natural Science Foundation (Grant No. 1222017). Wang is supported by the National Natural Science Foundation of China (Grant Nos. 11971061 and 12271028), the Beijing Natural Science Foundation (Grant No. 1222017), and the Fundamental Research Funds for the Central Universities.
AMS subject classification: **35K55, 35D40, 35B40, 35B07.**
Keywords: Hessian quotient equations, exterior Dirichlet problem, radially symmetric solutions, asymptotic behavior, necessary and sufficient conditions.

the graph of Du in $\mathbb{C}^3$ is a special Lagrangian submanifold, that is, its mean curvature vanishes everywhere and the complex structure on $\mathbb{C}^3$ sends the tangent space of the graph to the normal space at every point.

A classical theorem of Jörgens ($n = 2$) [15], Calabi ($n \leq 5$) [8], and Pogorelov ($n \geq 2$) [21] states that any convex classical solution of $\det D^2 u = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. Caffarelli and Li [6] extended the Jörgens–Calabi–Pogorelov theorem and studied the existence of solutions of Monge–Ampère equations in exterior domains with prescribed asymptotic behavior. They proved that for $n \geq 3$, given any $b \in \mathbb{R}^n$ and any $n \times n$ real symmetric positive definite matrix A with $\det A = 1$, there exists some constant c^* depending on n, Ω, ϕ, b, and A such that for every $c > c^*$, the Monge–Ampère equation $\det D^2 u = 1$ in $\mathbb{R}^n \backslash \bar{\Omega}$ with the Dirichlet boundary condition (1.2) and the following prescribed asymptotic behavior at infinity

$$u(x) = \frac{1}{2} x^T A x + b \cdot x + c + O(|x|^{2-n}), \quad \text{as } |x| \to \infty, \tag{1.3}$$

admits a unique viscosity solution $u \in C^\infty(\mathbb{R}^n \backslash \bar{\Omega}) \cap C^0(\mathbb{R}^n \backslash \Omega)$. Based on this work, Li and Lu [19] completed the characterization of the existence and non-existence of solutions in terms of the above asymptotic behaviors. For more results concerning the exterior Dirichlet problem for Monge–Ampère equations, we refer to [1, 3–5, 13] and the references therein.

After the work of Caffarelli and Li [6], there have been extensive studies on the existence of fully nonlinear elliptic equations in exterior domains.

For the exterior Dirichlet problem of the Hessian equations, Dai and Bao [12] first obtained the existence of solutions satisfying the asymptotic behavior (1.3) with $A = (1/\binom{n}{k})^{\frac{1}{k}} I$. Later on, Bao, Li, and Li [2] proved that for $n \geq 3$, given any $b \in \mathbb{R}^n$ and any $n \times n$ real symmetric positive definite matrix A with $S_k(A) = 1$, there exists some constant c^* depending on n, Ω, ϕ, b, and A such that for every $c > c^*$, the Hessian equation $S_k(D^2 u) = 1$ in $\mathbb{R}^n \backslash \bar{\Omega}$ with the Dirichlet boundary condition (1.2) and the following prescribed asymptotic behavior at infinity

$$u(x) = \frac{1}{2} x^T A x + b \cdot x + c + O(|x|^{\theta(2-n)}), \quad \text{as } |x| \to \infty,$$

admits a unique viscosity solution $u \in C^\infty(\mathbb{R}^n \backslash \bar{\Omega}) \cap C^0(\mathbb{R}^n \backslash \Omega)$, where $\theta \in [\frac{k-2}{n-2}, 1]$ is a constant depending on n, k, and A. For more results concerning the exterior Dirichlet problem for Hessian equations, we refer to [9, 10] and the references therein.

For the exterior Dirichlet problem of the Hessian quotient equation (1.1) with $g \equiv 1$ and the Dirichlet boundary condition (1.2), Dai [11] first obtained the existence of solutions with asymptotic behavior

$$u(x) = \frac{\bar{c}}{2} |x|^2 + c + O(|x|^{2-k+l}), \quad \text{as } |x| \to \infty,$$

where $n \geq 3$, $\bar{c} = (\binom{n}{l} / \binom{n}{k})^{\frac{1}{k-l}}$, and $k - l \geq 3$. Subsequently, Li and Dai [17] obtained the existence result for the case $k - l = 1$ and $k - l = 2$. Later on, Li and Li [16] proved that for $n \geq 3$ and, given any $b \in \mathbb{R}^n$ and any A in the set

$\mathcal{A}_{k,l} := \{A$ is an $n \times n$ real symmetric positive definite matrix with $S_k(A)/S_l(A) = 1\}$, with $\frac{k-l}{\bar{t}_k - \underline{t}_l} > 2$, where

$$\bar{t}_k := \max_{1 \le i \le n} \frac{\frac{\partial}{\partial \lambda_i}\sigma_k(\lambda(A))\lambda_i}{\sigma_k(\lambda(A))} = \max_{1 \le i \le n} \frac{\sigma_{k-1;i}(\lambda)\lambda_i}{\sigma_k(\lambda)}, \tag{1.4}$$

and

$$\underline{t}_k := \min_{1 \le i \le n} \frac{\frac{\partial}{\partial \lambda_i}\sigma_k(\lambda(A))\lambda_i}{\sigma_k(\lambda(A))} = \min_{1 \le i \le n} \frac{\sigma_{k-1;i}(\lambda)\lambda_i}{\sigma_k(\lambda)}, \tag{1.5}$$

there exists some constant c^* depending on n, k, l, Ω, ϕ, b, and A such that for every $c > c^*$, the Hessian equation $S_k(D^2u)/S_l(D^2u) = 1$ in $\mathbb{R}^n \backslash \bar{\Omega}$ with the Dirichlet boundary condition (1.2) and the following prescribed asymptotic behavior at infinity

$$u(x) = \frac{1}{2}x^T Ax + b^T x + c + O(|x|^{2-m}), \text{ as } |x| \to \infty,$$

admits a unique viscosity solution $u \in C^0(\mathbb{R}^n \backslash \Omega)$, where $m \in (2, n]$ is a constant depending on n, k, l, and A. Recently, we have just learned that Jiang, Li, and Li [14] generalized this result to $g = 1 + O(r^{-\beta})$ with $\beta > 2$. For more results concerning the exterior Dirichlet problem for Hessian equations, we refer to [18] and the references therein.

Our paper consists of two parts. In the first part, we obtain the necessary and sufficient conditions on the existence of radially symmetric solutions of the exterior Dirichlet problem of Hessian quotient equations.

Before stating our result of the first part, we first give the definition of k-convex functions. For $k = 1, 2, \ldots, n$, we say a C^2 function u defined in a domain is k-convex (uniformly k-convex), if $\lambda(D^2u) \in \bar{\Gamma}_k$ (Γ_k), where

$$\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, 2, \ldots, k\}.$$

In particular, a uniformly n-convex function is a convex function. Note that (1.1) is elliptic for uniformly k-convex functions.

For $n \ge 3$, consider the following problem:

$$\begin{cases} S_k(D^2u) = S_l(D^2u), & \text{in } \mathbb{R}^n \backslash \bar{B}_1, \\ u = b, & \text{on } \partial B_1, \\ u = \frac{\hat{a}}{2}|x|^2 + c + O(|x|^{2-n}), & \text{as } |x| \to \infty, \end{cases} \tag{1.6}$$

where B_1 denotes the unit ball in $\mathbb{R}^n$, $2 \le k \le n$, $0 \le l \le k-1$, $\hat{a} = (\binom{n}{l}/\binom{n}{k})^{\frac{1}{k-l}}$, b, $c \in \mathbb{R}$. Our first result can be stated as follows.

Theorem 1.1 *For $n \ge 3$, let $2 \le k \le m \le n, 1 \le l < k$. Then problem (1.6) admits a unique radially symmetric uniformly m-convex solution*

$$u(x) = b + \int_1^{|x|} s\xi(\mu^{-1}(c)s^{-n})ds, \quad \forall\ |x| \ge 1,$$

if and only if $c \in [\mu(\alpha_0), \infty)$ *for* $m = k$ *and* $c \in [\mu(\alpha_0), \mu(\alpha_m)]$ *for* $k+1 \le m \le n$, *where* $\xi(t)$ *denotes the inverse function of*

$$t = \binom{n}{k}\xi^k - \binom{n}{l}\xi^l$$

in the interval $[a_0, \infty)$ *with*

$$a_0 = \left(\left(l\binom{n}{l}\right)\Big/\left(k\binom{n}{k}\right)\right)^{\frac{1}{k-l}},\ \alpha_0 = a_0^l\binom{n}{l}\frac{l-k}{k},\ \alpha_m = \left(\frac{(m-l)\binom{n}{l}}{(m-k)\binom{n}{k}}\right)^{\frac{l}{k-l}}\binom{n}{l}\frac{k-l}{m-k}$$

with $k+1 \le m \le n$ *and*

$$\mu(\alpha) = b - \frac{\hat{a}}{2} + \int_1^\infty s[\xi(\alpha s^{-n}) - \hat{a}]ds, \quad \forall\ \alpha \in [\alpha_0, \infty). \tag{1.7}$$

Remark 1.2 For the case $l = 0$, the necessary and sufficient conditions on the existence of radially symmetric solutions of the exterior Dirichlet problem of Hessian equations were obtained by Wang and Bao [23].

For $n = 2$, consider the following problem:

$$\begin{cases} S_2(D^2u) = S_1(D^2u), & \text{in } \mathbb{R}^n\backslash\bar{B}_1, \\ u = b, & \text{on } \partial B_1, \\ u = |x|^2 + \dfrac{\rho}{2}\ln|x| + c + O(|x|^{-2}), & \text{as } |x| \to \infty, \end{cases} \tag{1.8}$$

where b, c, and $\rho \in \mathbb{R}$.

Theorem 1.3 *For* $n = 2$, *problem* (1.8) *admits a unique radially symmetric convex solution*

$$\begin{aligned} u(x) = b - \frac{1}{2} + \frac{1}{2}|x|^2 + \frac{1}{2}[|x|\sqrt{|x|^2+\rho} + \rho\ln(|x| + \sqrt{|x|^2+\rho})] \\ - \frac{1}{2}[\sqrt{1+\rho} + \rho\ln(1+\sqrt{1+\rho})] \end{aligned} \tag{1.9}$$

if and only if $\rho \ge -1$, *and* $c = \nu(\rho)$, *where*

$$\nu(\rho) = b - \frac{1}{2} + \frac{\rho}{4} + \frac{\rho}{2}\ln 2 - \frac{1}{2}[\sqrt{1+\rho} + \rho\ln(1+\sqrt{1+\rho})].$$

Corollary 1.4 *For* $n = 2$, *problem* (1.8) *admits a unique radially symmetric convex solution* (1.9) *if and only if* $c \le b - 1$.

In the second part of this paper, we obtain the existence of viscosity solutions of the exterior Dirichlet problem of the Hessian quotient equation (1.1) and (1.2).

Before stating our main result, we will first give the definition of viscosity solutions of (1.1) and (1.2).

Definition 1.5 A function $u \in C^0(\mathbb{R}^n\backslash\Omega)$ is said to be a viscosity subsolution (supersolution) of (1.1) and (1.2), if $u \le \phi$ ($u \ge \phi$) on $\partial\Omega$ and for any $\bar{x} \in \mathbb{R}^n\backslash\bar{\Omega}$ and any uniformly k-convex function $v \in C^2(\mathbb{R}^n \backslash \bar{\Omega})$ satisfying

$$u(x) \leq (\geq)\, v(x),\ x \in \mathbb{R}^n \backslash \bar{\Omega} \quad \text{and} \quad u(\bar{x}) = v(\bar{x}),$$

we have

$$\frac{S_k(D^2 v(\bar{x}))}{S_l(D^2 v(\bar{x}))} \geq (\leq)\, g(\bar{x}).$$

If $u \in C^0(\mathbb{R}^n \backslash \Omega)$ is both a viscosity subsolution and a viscosity supersolution of (1.1) and (1.2), we say that u is a viscosity solution of (1.1) and (1.2).

Suppose that $g \in C^0(\mathbb{R}^n)$ satisfies

$$0 < \inf_{\mathbb{R}^n} g \leq \sup_{\mathbb{R}^n} g < +\infty, \tag{1.10}$$

and for some constant $\beta > 2$,

$$g(x) = g_0(r_A(x)) + O(r_A(x)^{-\beta}), \quad \text{as } |x| \to \infty, \tag{1.11}$$

where

$$r_A(x) := \sqrt{x^T A x} \text{ for some } A \in \mathcal{A}_{k,l} \tag{1.12}$$

and $g_0 \in C^0([0, +\infty))$ satisfies

$$0 < \inf_{[0,+\infty)} g_0 \leq \sup_{[0,+\infty)} g_0 < +\infty,$$

and

$$\underline{C}_0 := \lim_{r \to +\infty} g_0(r) > 0.$$

Our main result can be stated as follows.

Theorem 1.6 *For $n \geq 3$, let $0 \leq l < k \leq n$ and $\frac{k-l}{\bar{t}_k - \underline{t}_l} > 2$. Assume that g satisfies (1.10) and (1.11). Then, for any given $b \in \mathbb{R}^n$ and $A \in \mathcal{A}_{k,l}$, there exists some constant $\tilde{c}$, depending only on $n, b, A, \Omega, g, g_0, \|\phi\|_{C^2(\partial\Omega)}$, such that for every $c > \tilde{c}$, there exists a unique viscosity solution $u \in C^0(\mathbb{R}^n \backslash \Omega)$ of (1.1) and (1.2) with the following prescribed asymptotic behavior at infinity:*

$$u(x) = u_0(x) + b \cdot x + c + O(|x|^{2-\min\{\beta, \frac{k-l}{\bar{t}_k - \underline{t}_l}\}}), \ \text{as } |x| \to \infty, \ \text{if } \beta \neq \frac{k-l}{\bar{t}_k - \underline{t}_l}, \tag{1.13}$$

or

(1.14)

$$u(x) = u_0(x) + b \cdot x + c + O(|x|^{2-\min\{\beta, \frac{k-l}{\bar{t}_k - \underline{t}_l}\}} \ln|x|), \ \text{as } |x| \to \infty, \ \text{if } \beta = \frac{k-l}{\bar{t}_k - \underline{t}_l},$$

https://doi.org/10.4153/S0008414X23000834 Published online by Cambridge University Press

where $u_0(x) = \int_0^{r_A(x)} \theta h_0(\theta) d\theta$ *with* h_0 *satisfying*

$$
\begin{cases}
\dfrac{h_0(r)^k + \bar{t}_k r h_0(r)^{k-1} h_0'(r)}{h_0(r)^l + \underline{t}_l r h_0(r)^{l-1} h_0'(r)} = g_0(r), \ r > 0, \\
h_0(0) > \sup_{r \in [0,+\infty)} g_0^{\frac{1}{k-l}}(r), \\
h_0(r)^k + \bar{t}_k r h_0(r)^{k-1} h_0'(r) > 0.
\end{cases}
\tag{1.15}
$$

Remark 1.7 Under the assumption of g_0, by the classical existence, uniqueness and extension theorem for the solution of the initial value problem of the ODE, we know that (1.15) admits a bounded solution h_0 in $C^0[0,+\infty) \cap C^1(0,+\infty)$. In particular, if $g_0 \equiv 1$, then $h_0 \equiv 1$ and $u_0(x) = \frac{1}{2}x^T Ax$.

The rest of this paper is organized as follows. In Section 2, we will prove Theorems 1.1 and 1.2. In Section 3, we will construct a family of generalized symmetric smooth k-convex subsolutions and supersolutions of (1.1) by analyzing the corresponding ODE. In Section 4, we will finish the proof of Theorem 1.6 by Perron's method. In Appendix A, we will give the Appendix.

2 Proof of Theorems 1.1 and 1.3

Before proving Theorems 1.1 and 1.3, we will first make some preliminaries.

Consider the function

$$t = t(\xi) = \binom{n}{k}\xi^k - \binom{n}{l}\xi^l, \quad \forall\ \xi \in \mathbb{R}.$$

It is easy to see that t is smooth and strictly increasing on the interval $[a_0, \infty)$. Let $\xi = \xi(t)$ denote the inverse function of t on the interval $[a_0, \infty)$. Then ξ is smooth and strictly increasing on $[\alpha_0, \infty)$. Moreover,

$$\xi(t) \geq a_0, \quad \forall\ t \geq \alpha_0.$$

Let $\mu(\alpha)$ be defined as in (1.7). Then μ has the following properties.

Lemma 2.1 *μ is smooth, strictly increasing on $[\alpha_0, \infty)$ and $\lim\limits_{\alpha\to\infty} \mu(\alpha) = \infty$.*

Proof The smoothness of μ follows directly from the smoothness of ξ. Then, by a direct computation,

$$\mu'(\alpha) = \int_1^\infty s^{1-n} \xi'(\alpha s^{-n}) ds.$$

Therefore, Lemma 2.1 follows directly from the facts that $\xi(t)$ is strictly increasing on $[\alpha_0, \infty)$ and $\lim\limits_{t\to\infty} \xi(t) = \infty$. ∎

Lemma 2.2 *Let $2 \leq k \leq m \leq n$ and $0 \leq l \leq k-1$. Assume that $\lambda = (\beta, \gamma, \ldots, \gamma) \in \mathbb{R}^n$ and $\sigma_k(\lambda) = \sigma_l(\lambda)$. Then $\lambda \in \Gamma_m$ if and only if*

$$a_0 < \gamma < \gamma_m,$$

where

$$\gamma_m = \begin{cases} \left(\frac{(m-l)\binom{n}{l}}{(m-k)\binom{n}{k}} \right)^{\frac{1}{k-l}}, & k+1 \le m \le n, \\ \infty, & m = k. \end{cases}$$

Proof If $\lambda \in \Gamma_m$, we have that for $j = 1, \ldots, m$,

$$\binom{n-1}{j-1} \beta \gamma^{j-1} + \binom{n-1}{j} \gamma^j > 0,$$

which implies that

$$\gamma^{j-1}(j\beta + (n-j)\gamma) > 0.$$

By [23, Lemma 1], we have that $\gamma > 0$. It follows that

$$j\beta + (n-j)\gamma > 0, \tag{2.1}$$

for $j = 1, \ldots, m$.

The equation $\sigma_k(\lambda) = \sigma_l(\lambda)$ can be equivalently written as

$$\frac{1}{n}\binom{n}{k}\gamma^{k-1}[k\beta + (n-k)\gamma] = \frac{1}{n}\binom{n}{l}\gamma^{l-1}[l\beta + (n-l)\gamma].$$

Since $\gamma > 0$, dividing the above equality by γ^{l-1}, we have that

$$\left[k\binom{n}{k}\gamma^{k-l} - l\binom{n}{l} \right]\beta = \gamma\left[(n-l)\binom{n}{l} - (n-k)\binom{n}{k}\gamma^{k-l} \right]. \tag{2.2}$$

It is easy to see that $\gamma \ne a_0$. Indeed, if $\gamma = a_0$, then the left-hand side of (2.2) equals to 0. However, the right-hand side of (2.2) equals $a_0^l \binom{n}{l} \frac{n(k-l)}{k} \ne 0$, which is a contradiction.

It follows from (2.2) and $\gamma \ne a_0$ that

$$\beta = \frac{(n-l)\binom{n}{l} - (n-k)\binom{n}{k}\gamma^{k-l}}{k\binom{n}{k}\gamma^{k-l} - l\binom{n}{l}}\gamma. \tag{2.3}$$

Inserting the above equality into (2.1), we have that

$$\frac{\binom{n}{k}(k-j)\gamma^{k-l} + \binom{n}{l}(j-l)}{k\binom{n}{k}\gamma^{k-l} - l\binom{n}{l}} > 0, \tag{2.4}$$

for $j = 1, \ldots, m$.

Now we claim that $k\binom{n}{k}\gamma^{k-l} - l\binom{n}{l} > 0$ or equivalently, $\gamma > a_0$. Indeed, if $k\binom{n}{k}\gamma^{k-l} - l\binom{n}{l} < 0$, on one hand, we have that by (2.4)

$$\binom{n}{k}(k-j)\gamma^{k-l} + \binom{n}{l}(j-l) < 0, \quad \forall\ j = 1, \ldots, m.$$

On the other hand,

$$\binom{n}{k}(k-j)\gamma^{k-l} + \binom{n}{l}(j-l) \ge \binom{n}{l}(j-l) > 0$$

for any $l+1 \le j \le k$, which is a contradiction.

For $m = k$, we have already obtained that $a_0 < \gamma < \infty$. For $k+1 \le m \le n$, by $\gamma > a_0$ and (2.4), we have that

$$\gamma^{k-l} < \frac{\binom{n}{l}(j-l)}{\binom{n}{k}(j-k)}, \quad \forall\, j = k+1, \dots, m.$$

Then we can conclude that $a_0 < \gamma < \gamma_m$.

Conversely, if $a_0 < \gamma < \gamma_m$, it is easy to check that $\lambda \in \Gamma_m$. Indeed, by (2.3), we have that for $j = 1, \dots, m$,

$$j\beta + (n-j)\gamma = \frac{\binom{n}{k}(k-j)\gamma^{k-l} + \binom{n}{l}(j-l)}{k\binom{n}{k}\gamma^{k-l} - l\binom{n}{l}} > 0.$$

It follows from the above inequality and $\gamma > 0$ that

$$\sigma_j(\lambda) = \frac{1}{n}\binom{n}{j}\gamma^{j-1}[j\beta + (n-j)\gamma] > 0, \quad \forall\, j = 1, \dots, m.$$

Lemma 2.2 is proved. ■

Proof of Theorem 1.1 If $c \in [\mu(\alpha_0), \infty)$ for $m = k$ or $c \in [\mu(\alpha_0), \mu(\alpha_m)]$ for $k+1 \le m \le n$, by the intermediate value theorem and Lemma 2.1, there exists uniquely $\alpha_0 \le \alpha < \infty$ for $m = k$ or $\alpha_0 \le \alpha \le \alpha_m$ for $k+1 \le m \le n$ such that $c = \mu(\alpha)$. Consider the function

$$u(r) = b + \int_1^r s\xi(\alpha s^{-n})ds, \quad \forall\, r \ge 1. \tag{2.5}$$

Here and throughout of this section, we use r to denote r_I defined as in (1.12).

Next, we shall prove that u is the unique uniformly m-convex solution to (1.6). The uniqueness of u follows directly from the comparison principle (see [20, Theorem 1.7]). It is obvious that u satisfies the second line in (1.6) by taking $r = 1$ in (2.5). Differentiating (2.5) with respect to $r > 1$, it follows from the range of α and the monotonicity of ξ that

$$a_0 < \frac{u'(r)}{r} = \xi(\alpha r^{-n}) < \gamma_m, \quad \forall\, r > 1. \tag{2.6}$$

By (2.6) and Lemma 2.2, we can conclude that u is uniformly m-convex. Since $\xi(t)$ is the inverse function of $t = \binom{n}{k}\xi^k - \binom{n}{l}\xi^l$ on the interval $[a_0, \infty)$, then equation (2.6) can be equivalently written as

$$\begin{aligned} \alpha r^{-n} &= \binom{n}{k}\xi^k(\alpha r^{-n}) - \binom{n}{l}\xi^l(\alpha r^{-n}) \\ &= \binom{n}{k}\Big(\frac{u'}{r}\Big)^k - \binom{n}{l}\Big(\frac{u'}{r}\Big)^l, \quad \forall\, r > 1. \end{aligned} \tag{2.7}$$

By differentiating (2.7) with respect to r, we have that u satisfies the first line in (1.6). It only remains to prove that u satisfies the third line in (1.6). Since

$$\xi(t) = \xi(0) + \xi'(0)t + O(t^2)$$
$$= \hat{a} + \frac{t}{(k-l)\hat{a}^{l-1}\binom{n}{l}} + O(t^2),$$

as $t \to 0$, then we have that

$$u(r) = \frac{\hat{a}^2}{2}r^2 + b - \frac{\hat{a}}{2} + \int_1^r s[\xi(\alpha s^{-n}) - \hat{a}]ds$$
$$= \frac{\hat{a}^2}{2}r^2 + b - \frac{\hat{a}}{2} + \int_1^\infty s[\xi(\alpha s^{-n}) - \hat{a}]ds - \int_r^\infty s[\xi(\alpha s^{-n}) - \hat{a}]ds$$
$$= \frac{\hat{a}^2}{2}r^2 + \mu(\alpha) - \int_r^\infty s[\xi(\alpha s^{-n}) - \hat{a}]ds$$
$$= \frac{\hat{a}^2}{2}r^2 + \mu(\alpha) + O(r^{2-n})$$
$$= \frac{\hat{a}^2}{2}r^2 + c + O(r^{2-n}), \tag{2.8}$$

as $r \to \infty$.

Conversely, suppose that u is the unique radially symmetric uniformly m-convex solution to (1.6). By Lemma 2.2, we have that

$$a_0 < \frac{u'(r)}{r} < \gamma_m, \quad \forall\, r > 1. \tag{2.9}$$

The first line in (1.6) can be written as

$$\binom{n}{k}ku''(\frac{u'}{r})^{k-1} + \binom{n}{k}(n-k)(\frac{u'}{r})^k = \binom{n}{l}lu''(\frac{u'}{r})^{l-1} + \binom{n}{l}(n-l)(\frac{u'}{r})^l, \quad \forall\, r > 1.$$

By multiplying the above equation by r^{n-1}, we have that

$$\binom{n}{k}(r^{n-k}(u')^k)' = \binom{n}{l}(r^{n-l}(u')^l)', \quad \forall\, r > 1.$$

Then there exists $\alpha \in \mathbb{R}$ such that

$$\binom{n}{k}(\frac{u'}{r})^k - \binom{n}{l}(\frac{u'}{r})^l = \alpha r^{-n}, \quad \forall\, r > 1. \tag{2.10}$$

By (2.9), (2.10), and the monotonicity of $t = \binom{n}{k}\xi^k - \binom{n}{l}\xi^l$, we can conclude that $\alpha_0 \le \alpha < \infty$ for $m = k$ and $\alpha_0 \le \alpha \le \alpha_m$ for $k+1 \le m \le n$. By the definition of ξ, we can solve u'/r from (2.10), that is,

$$\frac{u'}{r} = \xi(\alpha r^{-n}), \quad \forall\, r > 1.$$

It follows that

$$u(r) = b + \int_1^r s\xi(\alpha s^{-n})ds, \quad \forall\, r > 1.$$

Then, by (2.8) and Lemma 2.2, $c = \mu(\alpha) \in [\mu(\alpha_0), \mu(\alpha_1)]$.

Theorem 1.1 is proved. ■

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Proof of Theorem 1.3 If $\rho \geq -1$ and $c = \nu(\rho)$, we will prove that u defined as in (1.9) is the unique convex solution to (1.8) as follows.

The uniqueness of u follows directly from the comparison principle (see [20, Theorem 1.7]). It is obvious that u satisfies the second line in (1.8) by taking $r = 1$ in (1.9). Differentiating (1.9) with respect to $r > 1$, we have that

$$\frac{u'(r)}{r} = 1 + \frac{\sqrt{r^2+\rho}}{r} > 1, \quad \forall\, r > 1. \tag{2.11}$$

By applying Lemma 2.2 with $n = m = k = 2$ and $l = 1$, we have that u is convex. By a direct computation,

$$\begin{aligned} &S_2(D^2u) - S_1(D^2u) \\ &= u''\frac{u'}{r} - (u'' + \frac{u'}{r}) \\ &= 0, \end{aligned}$$

which implies that u satisfies the first line in (1.8). It only remains to prove that u satisfies the third line in (1.8). Since

$$r\sqrt{r^2+\rho} = r^2 + \frac{\rho}{2} + O(r^{-2}),$$

and

$$\ln(r + \sqrt{r^2+\rho}) = \ln r + \ln 2 + O(r^{-2}),$$

as $r \to \infty$, we have that

$$u(r) = r^2 + \frac{\rho}{2}\ln r + \nu(\rho) + O(r^{-2}), \tag{2.12}$$

as $r \to \infty$, which implies that u satisfies the third line in (1.8).

Conversely, suppose that u is the unique radially symmetric convex solution to (1.8). The first line in (1.8) can be written as

$$u''\frac{u'}{r} = u'' + \frac{u'}{r}, \quad \forall\, r > 1.$$

By multiplying the above equation by $2r$, we have that

$$((u')^2)' = 2(ru')', \quad \forall\, r > 1.$$

Then there exists $\rho \in \mathbb{R}$ such that

$$(u')^2 - 2ru' = \rho, \quad \forall\, r > 1. \tag{2.13}$$

It follows that

$$\Delta(r) := 4(r^2 + \rho) \geq 0, \quad \forall\, r > 1,$$

which implies that $\rho \geq -1$. By Lemma 2.2 with $n = m = k = 2$ and $l = 1$, we have that

$$\frac{u'(r)}{r} > 1, \quad \forall\, r > 1. \tag{2.14}$$

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Combining (2.13) and (2.14), we can solve u' as

$$u'(r) = r + \sqrt{r^2 + \rho}, \quad \forall\, r > 1.$$

Integrating the above equality from 1 to r, we have that u must take the form as in (1.9). Moreover, by expanding u at infinity as in (2.12), we can conclude that $c = \nu(\rho)$.

Theorem 1.3 is proved. ■

Proof of Corollary 1.4 By the argument in [23], we have that $\nu(\rho)$ is increasing on $[-1, 0]$ and decreasing on $[0, \infty)$. Thus,

$$\nu(\rho) \le \nu(0) = b - 1, \quad \forall\, \rho \ge -1.$$

Then Corollary 1.4 follows from the above inequality and Theorem 1.3. ■

Remark 2.3 If $k = n = 2$, $l = 0$, we can refer to Theorem 2 in [23].

3 Generalized symmetric functions, subsolutions and supersolutions

In this section, we will construct a family of generalized symmetric smooth subsolutions and supersolutions of (1.1).

For any $A \in \mathcal{A}_{k,l}$, without loss of generality, we may assume that A is diagonal, that is,

$$A = \operatorname{diag}(a_1, \dots, a_n),$$

where $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $0 < a_1 \le a_2 \le \dots \le a_n$.

Definition 3.1 A function u is called a generalized symmetric function with respect to A if u is of the form

$$u(x) = u(r) = u(r_A(x)) = u\Big(\big(\sum_{i=1}^n a_i x_i^2\big)^{\frac{1}{2}}\Big),$$

where $r := r_A(x)$ is defined as in (1.12).

If u is both a generalized symmetric function with respect to A and a subsolution (supersolution, solution) of (1.1), then we say that u is a generalized symmetric subsolution (supersolution, solution) of (1.1).

By the assumptions on g, there exist functions $\overline{g}, \underline{g} \in C^0([0, +\infty))$ satisfying

$$0 < \inf_{r \in [0,+\infty)} \underline{g}(r) \le \underline{g}(r) \le g(x) \le \overline{g}(r) \le \sup_{r \in [0,+\infty)} \overline{g}(r) < +\infty, \forall\, x \in \mathbb{R}^n,$$

and

$$\underline{g}(r) \le g_0(r) \le \overline{g}(r), \forall\, x \in \mathbb{R}^n. \tag{3.1}$$

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Moreover, $\underline{g}(r)$ is strictly increasing in r and for some $C_1, \theta_0 > 0$, we have that

$$\overline{g}(r) = g_0(r) + C_1 r^{-\beta}, \ r > \theta_0 \tag{3.2}$$

and

$$\underline{g}(r) = g_0(r) - C_1 r^{-\beta}, \ r > \theta_0.$$

In order to construct the subsolutions of (1.1), we want to construct the generalized symmetric subsolutions or solutions of

$$\frac{S_k(D^2 v)}{S_l(D^2 v)} = \overline{g}(r). \tag{3.3}$$

However, Proposition A.1 tells us that it is impossible to construct the generalized symmetric solutions of the above equation for $1 \le k \le n-1$ directly unless $A = \hat{a} I$.

Thus, we will construct the generalized symmetric smooth subsolutions of (3.3). Indeed, we will construct the subsolutions of the form

$$w(r) := w_{\beta_1,\eta,\delta}(r) := \beta_1 + \int_\eta^r \theta h(\theta,\delta) d\theta, \ \forall\ r \ge \eta,$$

where $\beta_1 \in \mathbb{R}$, $\eta > 1$, $\delta > \sup_{r\in[1,+\infty)} \overline{g}^{\frac{1}{k-l}}(r)$, and $h = h(\theta,\delta)$ is obtained as follows.

Lemma 3.2 *For $n \ge 3$, $0 \le l < k \le n$, the following problem*

$$\begin{cases} \dfrac{h(r)^k + \overline{t}_k r h(r)^{k-1} h'(r)}{h(r)^l + \underline{t}_l r h(r)^{l-1} h'(r)} = \overline{g}(r), \ r > 1, \\ h(1) = \delta, \\ h(r)^k + \overline{t}_k r h(r)^{k-1} h'(r) > 0 \end{cases} \tag{3.4}$$

admits a smooth solution $h = h(r,\delta)$ on $[1,+\infty)$ satisfying:

(i) $\overline{g}^{\frac{1}{k-l}}(r) \le h(r,\delta) \le \delta$, *and* $\partial_r h(r,\delta) \le 0$.
(ii) $h(r,\delta)$ *is continuous and strictly increasing in* δ *and*

$$\lim_{\delta\to+\infty} h(r,\delta) = +\infty, \ \forall\ r \ge 1.$$

Proof For brevity, sometimes we write $h(r)$ or $h(r,\delta)$, $\overline{t}_k(a)$ or $\overline{t}_k$ and $\underline{t}_l(a)$ or $\underline{t}_l$, when there is no confusion. By (3.4), we have that

$$\begin{cases} \dfrac{\mathrm{d}h}{\mathrm{d}r} = -\dfrac{1}{r}\dfrac{h}{\overline{t}_k}\dfrac{h(r)^{k-l} - \overline{g}(r)}{h(r)^{k-l} - \overline{g}(r)\frac{\underline{t}_l}{\overline{t}_k}} =: V(h,r), \ r > 1, \\ h(1) = \delta. \end{cases} \tag{3.5}$$

Then $\partial V/\partial h$ is continuous and V satisfies the local Lipschitz condition in the domain $(\overline{g}^{\frac{1}{k-l}}(r), \delta) \times (1,+\infty)$. Since $\delta > \sup_{r\in[0,+\infty)} \overline{g}^{\frac{1}{k-l}}(r)$ and $\underline{t}_l/\overline{t}_k < 1$, so by the existence, uniqueness, and extension theorem for the solution of the initial value problem of the ODE, we obtain that problem (3.4) has a smooth solution $h(r,\delta)$ such that $\overline{g}^{\frac{1}{k-l}}(r) \le h(r,\delta) \le \delta$, and $\partial_r h(r,\delta) \le 0$. Then assertion (i) of this lemma is proved.

By (3.4), we have that

$$\frac{(r^{\frac{k}{\overline{t}_k}}h^k(r))'}{(r^{\frac{l}{\underline{t}_l}}h^l(r))'}=\frac{\frac{k}{\overline{t}_k}r^{\frac{k}{\overline{t}_k}-1}(h(r)^k+\overline{t}_k rh(r)^{k-1}h'(r))}{\frac{l}{\underline{t}_l}r^{\frac{l}{\underline{t}_l}-1}(h(r)^l+\underline{t}_l rh(r)^{l-1}h'(r))}=\frac{\frac{k}{\overline{t}_k}r^{\frac{k}{\overline{t}_k}-1}}{\frac{l}{\underline{t}_l}r^{\frac{l}{\underline{t}_l}-1}}\overline{g}(r),\ r>1,$$

that is,

$$(r^{\frac{k}{\overline{t}_k}}h^k(r))'=\frac{k\underline{t}_l}{l\overline{t}_k}r^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(r)(r^{\frac{l}{\underline{t}_l}}h^l(r))'.$$

Integrating the above equality from 1 to r, we have that

$$\int_1^r(s^{\frac{k}{\overline{t}_k}}h^k(s))'ds=\frac{k\underline{t}_l}{l\overline{t}_k}\int_1^r s^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(s)(s^{\frac{l}{\underline{t}_l}}h^l(s))'ds.$$

Then

$$r^{\frac{k}{\overline{t}_k}}h^k(r)-\delta^k=\frac{k\underline{t}_l}{l\overline{t}_k}\int_1^r s^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(s)(s^{\frac{l}{\underline{t}_l}}h^l(s))'ds. \tag{3.6}$$

Since $h(r)^k+\overline{t}_k rh(r)^{k-1}h'(r)>0$, then $h(r)^l+\underline{t}_l rh(r)^{l-1}h'(r)>0$, so also $(r^{\frac{l}{\underline{t}_l}}h^l(r))'>0$. According to the mean value theorem of integrals, we have that there exists $1\le\theta_1\le r$ such that

$$r^{\frac{k}{\overline{t}_k}}h^k(r)-\delta^k=\frac{k\underline{t}_l}{l\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)\int_1^r(s^{\frac{l}{\underline{t}_l}}h^l(s))'ds,$$

i.e.,

(3.7)

$$F(h,\delta,\theta_1,r):=r^{\frac{k}{\overline{t}_k}}h^k(r)-\delta^k-\frac{k\underline{t}_l}{l\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)r^{\frac{l}{\underline{t}_l}}h^l(r)+\frac{k\underline{t}_l}{l\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)\delta^l=0.$$

Then we claim that

$$\frac{\partial F}{\partial h}=k\left[r^{\frac{k}{\overline{t}_k}}h^{k-1}-\frac{\underline{t}_l}{\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)r^{\frac{l}{\underline{t}_l}}h^{l-1}\right]>0. \tag{3.8}$$

Indeed, by (3.7), we have that

$$r^{\frac{k}{\overline{t}_k}}h^k(r)-\frac{k\underline{t}_l}{l\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)r^{\frac{l}{\underline{t}_l}}h^l(r)=\delta^l\left[\delta^{k-l}-\frac{k\underline{t}_l}{l\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)\right]. \tag{3.9}$$

Since $\underline{t}_l\le\frac{l}{n}<\frac{k}{n}\le\overline{t}_k$ and $\delta^{k-l}>\sup_{r\in[0,+\infty)}\overline{g}(r)$, we have that

$$\delta^{k-l}-\frac{k\underline{t}_l}{l\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)>0. \tag{3.10}$$

Inserting the above inequality into (3.9), we have that

$$r^{\frac{k}{\overline{t}_k}}h^k(r)-\frac{k\underline{t}_l}{l\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)r^{\frac{l}{\underline{t}_l}}h^l(r)>0.$$

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Since $-\frac{\underline{t}_l}{\overline{t}_k} > -\frac{k\underline{t}_l}{l\overline{t}_k}$, we have that

$$0 < r^{\frac{k}{\overline{t}_k}} h^k(r) - \frac{k\underline{t}_l}{l\overline{t}_k} \theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}} \overline{g}(\theta_1) r^{\frac{l}{\underline{t}_l}} h^l(r) < r^{\frac{k}{\overline{t}_k}} h^k(r) - \frac{\underline{t}_l}{\overline{t}_k} \theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}} \overline{g}(\theta_1) r^{\frac{l}{\underline{t}_l}} h^l(r),$$

that is, (3.8).

By the implicit function theorem and the existence and uniqueness of solutions of (3.4), (3.7) admits a unique function $h(r) := h(r,\delta) := h(r,\delta,\theta_1)$. Moreover,

$$\frac{\partial h}{\partial \delta} = -\frac{\partial F}{\partial \delta} \Big/ \frac{\partial F}{\partial h}.$$

By (3.8) and (3.10),

$$\frac{\partial h}{\partial \delta} = \frac{k\delta^{l-1}\left[\delta^{k-l} - \frac{\underline{t}_l}{\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)\right]}{kr^{\frac{l}{\underline{t}_l}} h^{l-1}[r^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}} h^{k-l} - \frac{\underline{t}_l}{\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)]} > 0. \tag{3.11}$$

By (3.9), we have that

$$h^l(r)\left[r^{\frac{k}{\overline{t}_k}} h^{k-l}(r) - \frac{k\underline{t}_l}{l\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1) r^{\frac{l}{\underline{t}_l}}\right] = \delta^l\left[\delta^{k-l} - \frac{k\underline{t}_l}{l\overline{t}_k}\theta_1^{\frac{k}{\overline{t}_k}-\frac{l}{\underline{t}_l}}\overline{g}(\theta_1)\right].$$

As $\delta \to +\infty$, the right side of the above equality tends to $+\infty$. Since h is increasing in δ by (3.11), we can conclude that $h(r,\delta,\theta_1) \to +\infty$, as $\delta \to +\infty$. This lemma is proved. ∎

The asymptotic behavior of w can be given as follows.

Lemma 3.3 *As $r \to \infty$,*

$$w(r) = \int_0^r \theta h_0(\theta)d\theta + \mu_{\beta_1,\eta}(\delta) + \begin{cases} O(r^{2-\min\{\beta,\frac{k-l}{\overline{t}_k-\underline{t}_l}\}}), & \text{if } \beta \neq \frac{k-l}{\overline{t}_k-\underline{t}_l}, \\ O(r^{2-\frac{k-l}{\overline{t}_k-\underline{t}_l}}\ln r), & \text{if } \beta = \frac{k-l}{\overline{t}_k-\underline{t}_l}, \end{cases} \tag{3.12}$$

where

$$\mu_{\beta_1,\eta}(\delta) \to +\infty, \text{as } \delta \to +\infty. \tag{3.13}$$

Proof Let h_0 satisfy (1.15). From the equation in (1.15), we have that

$$\frac{\frac{k-l}{\overline{t}_k-\underline{t}_l} r^{\frac{k-l}{\overline{t}_k-\underline{t}_l}-1} h_0(r)^{\frac{k\underline{t}_l-l\overline{t}_k}{\overline{t}_k-\underline{t}_l}} [h_0(r)^k + \overline{t}_k r h_0(r)^{k-1} h_0'(r)]}{\frac{k-l}{\overline{t}_k-\underline{t}_l} r^{\frac{k-l}{\overline{t}_k-\underline{t}_l}-1} h_0(r)^{\frac{k\underline{t}_l-l\overline{t}_k}{\overline{t}_k-\underline{t}_l}} [h_0(r)^l + \underline{t}_l r h_0(r)^{l-1} h_0'(r)]} = g_0(r), \ r > 0.$$

It follows that

$$\frac{(r^{\frac{k-l}{\overline{t}_k-\underline{t}_l}} h_0^{\frac{(k-l)\overline{t}_k}{\overline{t}_k-\underline{t}_l}})'}{(r^{\frac{k-l}{\overline{t}_k-\underline{t}_l}} h_0^{\frac{(k-l)\underline{t}_l}{\overline{t}_k-\underline{t}_l}})'} = g_0(r), r > 0. \tag{3.14}$$

Integrating the above equality on both sides, we have that

$$h_0(r) = \left(r^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}} \int_0^r g_0(s)(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}} h_0(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})' ds\right)^{\frac{\bar{t}_k-\underline{t}_l}{(k-l)\bar{t}_k}}. \tag{3.15}$$

Rewrite $w(r)$ as

$$\begin{aligned}
w(r) &= \beta_1 + \int_\eta^{+\infty} \theta h(\theta) d\theta - \int_r^{+\infty} \theta h(\theta) d\theta \\
&= \beta_1 + \int_\eta^{+\infty} \theta h(\theta) d\theta + \int_0^r \theta h_0(\theta) d\theta - \int_0^r \theta h_0(\theta) d\theta - \int_r^{+\infty} \theta h(\theta) d\theta \\
&= \beta_1 + \int_\eta^{+\infty} \theta h(\theta) d\theta + \int_0^r \theta h_0(\theta) d\theta - \int_0^\eta \theta h_0(\theta) d\theta - \int_\eta^{+\infty} \theta h_0(\theta) d\theta \\
&\quad + \int_r^{+\infty} \theta h_0(\theta) d\theta - \int_r^{+\infty} \theta h(\theta) d\theta \\
&= \int_0^r \theta h_0(\theta) d\theta + \beta_1 - \int_0^\eta \theta h_0(\theta) d\theta + \int_\eta^{+\infty} \theta[h(\theta) - h_0(\theta)] d\theta \\
&\quad - \int_r^{+\infty} \theta[h(\theta) - h_0(\theta)] d\theta \\
&= \int_0^r \theta h_0(\theta) d\theta + \mu_{\beta_1,\eta}(\delta) - \int_r^{+\infty} \theta[h(\theta) - h_0(\theta)] d\theta,
\end{aligned} \tag{3.16}$$

where

$$\mu_{\beta_1,\eta}(\delta) := \beta_1 - \int_0^\eta \theta h_0(\theta) d\theta + \int_\eta^{+\infty} \theta[h(\theta) - h_0(\theta)] d\theta.$$

By (3.4),

$$\frac{\frac{k-l}{\bar{t}_k-\underline{t}_l} r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}-1} h(r)^{\frac{k\underline{t}_l-l\bar{t}_k}{\bar{t}_k-\underline{t}_l}} [h(r)^k + \bar{t}_k r h(r)^{k-1} h'(r)]}{\frac{k-l}{\bar{t}_k-\underline{t}_l} r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}-1} h(r)^{\frac{k\underline{t}_l-l\bar{t}_k}{\bar{t}_k-\underline{t}_l}} [h(r)^l + \underline{t}_l r h(r)^{l-1} h'(r)]} = \overline{g}(r),\ r > 1.$$

It follows that

$$\frac{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}} h^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}})'}{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}} h^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'} = \overline{g}(r), r > 1, \tag{3.17}$$

that is,

$$(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}} h^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}})' = \overline{g}(r)(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}} h^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})', r > 1.$$

Integrating the above equality from 1 to r, we obtain that

$$r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}} h^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}} - \delta^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}} = \int_1^r \overline{g}(s)(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}} h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})' ds.$$

https://doi.org/10.4153/S0008414X23000834 Published online by Cambridge University Press

It follows that

$$h(r)=\left(\delta^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}r^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}+r^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_1^r\bar{g}(s)\big(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\big)'ds\right)^{\frac{\bar{t}_k-\underline{t}_l}{(k-l)\bar{t}_k}}\tag{3.18}$$

and

$$\begin{aligned}w(r)=\beta_1+\int_\eta^r\theta\Bigg(&\delta^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}+\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\\&\times\int_1^\theta\bar{g}(s)\big(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\big)'ds\Bigg)^{\frac{\bar{t}_k-\underline{t}_l}{(k-l)\bar{t}_k}}d\theta,\ \forall\, r\geq\eta>1.\end{aligned}$$

Since by (3.2), $\bar{g}(r)=g_0(r)+C_1r^{-\beta}$, $r>\theta_0$, then the last term in (3.16) is

$$\begin{aligned}&-\int_r^{+\infty}\theta[h(\theta)-h_0(\theta)]d\theta\\
&=-\int_r^{+\infty}\theta\left\{\left(\delta^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}+\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_1^\theta\bar{g}(s)\big(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\big)'ds\right)^{\frac{\bar{t}_k-\underline{t}_l}{(k-l)\bar{t}_k}}-h_0(\theta)\right\}d\theta\\
&=-\int_r^{+\infty}\theta\left\{\left[\delta_0\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}+\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_{\theta_0}^\theta\left(g_0(s)+C_1s^{-\beta}\right)\left(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\right)'ds\right)^{\frac{\bar{t}_k-\underline{t}_l}{(k-l)\bar{t}_k}}-h_0(\theta)\right\}d\theta\\
&=-\int_r^{+\infty}\theta\left\{\left[\delta_0\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}+\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\left(\int_0^\theta g_0(s)\left(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\right)'ds\right.\right.\right.\\
&\qquad\qquad\left.-\int_0^{\theta_0}g_0(s)\left(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\right)'ds\right)\\
&\qquad\left.\left.+\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_{\theta_0}^\theta C_1s^{-\beta}\big(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\big)'ds\right]^{\frac{\bar{t}_k-\underline{t}_l}{(k-l)\bar{t}_k}}-h_0(\theta)\right\}d\theta,\\
&=-\int_r^{+\infty}\theta\left\{\left[\delta_1\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}+\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_0^\theta g_0(s)\big(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\big)'ds\right.\right.\\
&\qquad\left.\left.+\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_{\theta_0}^\theta C_1s^{-\beta}\big(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\big)'ds\right]^{\frac{\bar{t}_k-\underline{t}_l}{(k-l)\bar{t}_k}}-h_0(\theta)\right\}d\theta\\
&=-\int_r^{+\infty}\theta h_0(\theta)\left\{\left[\frac{\delta_1\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}}{h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)}+\frac{\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_0^\theta g_0(s)\big(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\big)'ds-h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)}{h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)}+1\right.\right.\\
&\qquad\left.\left.+\frac{\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_{\theta_0}^\theta C_1s^{-\beta}\big(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\big)'ds}{h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)}\right]^{\frac{\bar{t}_k-\underline{t}_l}{(k-l)\bar{t}_k}}-1\right\}d\theta,\end{aligned}\tag{3.19}$$

where $\delta_0=\delta^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}+\int_1^{\theta_0}\overline{g}(s)(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'ds$ and $\delta_1=\delta_0-\int_0^{\theta_0}g_0(s)(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'ds$. In (3.19), we let

$$Q(\theta):=\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_{\theta_0}^{\theta}C_1s^{-\beta}(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'ds.$$

Then, if $\beta\neq\frac{k-l}{\bar{t}_k-\underline{t}_l}$,

$$\begin{aligned}
Q(\theta)&=\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_{\theta_0}^{\theta}C_1s^{-\beta}(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'ds\\
&=\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\Bigg(C_1\theta^{\frac{k-l}{\bar{t}_k-\underline{t}_l}-\beta}h^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}(\theta)-C_1\theta_0^{\frac{k-l}{\bar{t}_k-\underline{t}_l}-\beta}h^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}(\theta_0)\\
&\qquad+C_1\beta\int_{\theta_0}^{\theta}s^{-\beta-1}s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}ds\Bigg)\\
&=C_2\theta^{-\beta}+C_3\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}+C_1\beta h(\kappa_0)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_{\theta_0}^{\theta}s^{-\beta-1}s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}ds \qquad (3.20)\\
&=C_2\theta^{-\beta}+C_3\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}+\frac{C_1\beta h(\kappa_0)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}}{\frac{k-l}{\bar{t}_k-\underline{t}_l}-\beta}\theta^{-\beta}-\frac{C_1\beta h(\kappa_0)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}}{\frac{k-l}{\bar{t}_k-\underline{t}_l}-\beta}\theta_0^{\frac{k-l}{\bar{t}_k-\underline{t}_l}-\beta}\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\\
&=C_4\theta^{-\beta}+C_5\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}, \qquad (3.21)
\end{aligned}$$

where $C_2:=C_2(\theta)=C_1h^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}(\theta)$ and $C_3=-C_1\theta_0^{\frac{k-l}{\bar{t}_k-\underline{t}_l}-\beta}h^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}(\theta_0)$. In (3.20), we employ the integration by parts and the mean value theorem of integrals and $\kappa_0\in[\theta_0,\theta]$, $C_4=C_2+\frac{C_1\beta h(\kappa_0)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}}{\frac{k-l}{\bar{t}_k-\underline{t}_l}-\beta}$, $C_5=C_3-\frac{C_1\beta h(\kappa_0)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}}{\frac{k-l}{\bar{t}_k-\underline{t}_l}-\beta}\theta_0^{\frac{k-l}{\bar{t}_k-\underline{t}_l}-\beta}$.

In (3.19), we set

$$\begin{aligned}
R(\theta)&:=\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_0^{\theta}g_0(s)(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'ds-h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)\\
&=\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_0^{\theta}g_0(s)(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'ds-\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_0^{\theta}g_0(s)(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'ds\\
&=\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_0^{\theta}g_0(s)\Bigg((s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'-(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'\Bigg)ds.
\end{aligned}$$

(3.22)

According to (3.14) and (3.17), we can have that

$$\lim_{r\to+\infty}\frac{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}})'}{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}})'}\frac{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'}{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'}=\lim_{r\to+\infty}\frac{\overline{g}(r)}{g_0(r)}=1.$$

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Consequently,

$$\lim_{r\to+\infty}\frac{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}})'}{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}})'}=\lim_{r\to+\infty}\frac{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'}{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'}. \tag{3.23}$$

On the other hand, in light of (3.14) and (3.17), we know that

$$(h_0(r))^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}=r^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_0^r g_0(s)(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'ds$$

and

$$(h(r))^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}=r^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_0^r \bar{g}(s)(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'ds.$$

As a result,

$$\lim_{r\to+\infty}\frac{(h(r))^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}}{(h_0(r))^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}}=\lim_{r\to+\infty}\frac{\bar{g}(r)(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'}{g_0(r)(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'}=\lim_{r\to+\infty}\frac{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'}{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'}. \tag{3.24}$$

Likewise, we also have that

$$\lim_{r\to+\infty}\frac{(h(r))^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}}{(h_0(r))^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}}=\lim_{r\to+\infty}\frac{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}})'}{(r^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}})'}. \tag{3.25}$$

From (3.23)–(3.25), we get that

$$\lim_{r\to+\infty}\frac{(h(r))^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}}{(h_0(r))^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}}=\lim_{r\to+\infty}\frac{(h(r))^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}}{(h_0(r))^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}}.$$

So

$$\lim_{r\to+\infty}\frac{h(r)}{h_0(r)}=1.$$

And, therefore, the term $\int_0^\theta g_0(s)\left((s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'-(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'\right)ds$ in (3.22) is bounded and thus

$$\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\int_0^\theta g_0(s)(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'ds-h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)=C_{10}\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}, \tag{3.26}$$

where $c_{10}=c_{10}(\theta)=\int_0^\theta g_0(s)\left((s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'-(s^{\frac{k-l}{\bar{t}_k-\underline{t}_l}}h_0(s)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}})'\right)ds$. Hence, by (3.21) and (3.26), we know that

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$$-\int_r^{+\infty}\theta[h(\theta)-h_0(\theta)]d\theta$$

$$=-\int_r^{+\infty}\theta h_0(\theta)\left\{\left[\frac{\delta_1\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}}{h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)}+\frac{C_{10}\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}}{h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)}+1+\frac{C_4\theta^{-\beta}+C_5\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}}{h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)}\right]^{\frac{\bar{t}_k-\underline{t}_l}{(k-l)\bar{t}_k}}-1\right\}d\theta. \tag{3.27}$$

Thus, due to the fact that h_0 is bounded, then (3.27) becomes

$$\begin{aligned}-\int_r^{+\infty}\theta[h(\theta)-h_0(\theta)]d\theta=&-\int_r^{+\infty}O(\theta^{1-\frac{k-l}{\bar{t}_k-\underline{t}_l}})+O(\theta^{1-\beta})d\theta\\ =&O(r^{2-\min\{\beta,\frac{k-l}{\bar{t}_k-\underline{t}_l}\}}),\quad\text{as}\quad r\to+\infty.\end{aligned}$$

If $\beta=\frac{k-l}{\bar{t}_k-\underline{t}_l}$, then by (3.20),

$$\begin{aligned}Q(\theta)&=C_2\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}+C_3\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}+C_1\frac{k-l}{\bar{t}_k-\underline{t}_l}h(\kappa_0)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}(\ln\theta-\ln\theta_0)\\ &=C_6\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\ln\theta+C_7\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}},\end{aligned}\tag{3.28}$$

where $C_6=C_1\frac{k-l}{\bar{t}_k-\underline{t}_l}h(\kappa_0)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}$, $C_7:=C_2+C_3-C_1\frac{k-l}{\bar{t}_k-\underline{t}_l}h(\kappa_0)^{\frac{(k-l)\underline{t}_l}{\bar{t}_k-\underline{t}_l}}\ln\theta_0$. Therefore, by (3.19), (3.22), and (3.28), we know that

$$-\int_r^{+\infty}\theta[h(\theta)-h_0(\theta)]d\theta$$

$$=-\int_r^{+\infty}\theta h_0(\theta)\left\{\left[\frac{\delta_1\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}}{h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)}+\frac{C_{10}\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}}{h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)}+1+\frac{C_6\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\ln\theta+C_7\theta^{-\frac{k-l}{\bar{t}_k-\underline{t}_l}}}{h_0^{\frac{(k-l)\bar{t}_k}{\bar{t}_k-\underline{t}_l}}(\theta)}\right]^{\frac{\bar{t}_k-\underline{t}_l}{(k-l)\bar{t}_k}}-1\right\}d\theta. \tag{3.29}$$

Hence, by the fact that h_0 is bounded, then (3.29) turns into

$$\begin{aligned}-\int_r^{+\infty}\theta[h(\theta)-h_0(\theta)]d\theta=&-\int_r^{+\infty}O(\theta^{1-\frac{k-l}{\bar{t}_k-\underline{t}_l}})+O(\theta^{1-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\ln\theta)d\theta\\ =&O(r^{2-\frac{k-l}{\bar{t}_k-\underline{t}_l}}\ln r),\quad\text{as}\quad r\to+\infty.\end{aligned}$$

To sum up, we can get (3.12). By Lemma 3.2(ii), we know that (3.13) holds. ∎

Let

$$W(x):=W_{\beta_1,\eta,\delta,A}(x):=w(r):=w_{\beta_1,\eta,\delta}(r),\ \forall\ x\in\mathbb{R}^n\backslash E_\eta,$$

where $E_\eta:=\{x\in\mathbb{R}^n:r<\eta\}$. Then we can conclude that such W is a generalized symmetric smooth subsolution of (1.1) as follows.

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Theorem 3.4 *W is a smooth k-convex subsolution of (1.1) in $\mathbb{R}^n \backslash \bar{E}_\eta$.*

Proof By the definition of w, we have that $w' = rh$ and $w'' = h + rh'$. It follows that for $i, j = 1, \ldots, n$,

$$\partial_{ij} W = ha_i \delta_{ij} + \frac{h'}{r}(a_i x_i)(a_j x_j).$$

By Lemma A.2, we have that for $j = 1, \ldots, k$,

$$\begin{aligned} S_j(D^2 W) &= \sigma_j(\lambda(D^2 W)) \\ &= \sigma_j(a)h^j + \frac{h'}{r} h^{j-1} \sum_{i=1}^n \sigma_{j-1;i}(a) a_i^2 x_i^2 \\ &\geq \sigma_j(a)h^j + \bar{t}_j(a)\sigma_j(a) r h^{j-1} h' \\ &= \sigma_j(a) h^{j-1}(h + \bar{t}_j(a) r h'), \end{aligned} \tag{3.30}$$

where we have used the facts that $h \geq \bar{g}^{\frac{1}{k-l}} > 0$ and $h' \leq 0$ for any $r \geq 1$ by Lemma 3.2(i).

Since $l < k$, $\underline{t}_l \leq \underline{t}_k \leq \frac{k}{n} \leq \bar{t}_k$, then $-\frac{\underline{t}_l}{\bar{t}_k} \geq -1$. It follows that

$$0 \leq \frac{h^{k-l} - \bar{g}(r)}{h^{k-l} - \bar{g}(r)\frac{\underline{t}_l}{\bar{t}_k}} \leq 1 \leq \frac{\bar{t}_k}{\bar{t}_j}, j \leq k,$$

and

$$h' = -\frac{1}{r}\frac{h}{\bar{t}_k}\frac{h(r)^{k-l} - \bar{g}(r)}{h(r)^{k-l} - \bar{g}(r)\frac{\underline{t}_l}{\bar{t}_k}} \geq -\frac{1}{r}\frac{h}{\bar{t}_k}\frac{\bar{t}_k}{\bar{t}_j} = -\frac{1}{r}\frac{h}{\bar{t}_j}.$$

Thus, $h + \bar{t}_j r h' \geq 0$. Hence, by (3.30), $S_j(D^2 w) \geq 0$ for $j = 1, \ldots, k$. Moreover, by (3.5) and the fact that $\sigma_k(a) = \sigma_l(a)$, we have that for any $x \in \mathbb{R}^n \backslash \bar{E}_\eta$,

$$\begin{aligned} \frac{S_k(D^2 W)}{S_l(D^2 W)} &= \frac{\sigma_k(\lambda(D^2 W))}{\sigma_l(\lambda(D^2 W))} \\ &= \frac{\sigma_k(a)h^k + \frac{h'}{r}h^{k-1}\sum_{i=1}^n \sigma_{k-1;i}(a)a_i^2 x_i^2}{\sigma_l(a)h^l + \frac{h'}{r}h^{l-1}\sum_{i=1}^n \sigma_{l-1;i}(a)a_i^2 x_i^2} \\ &\geq \frac{\sigma_k(a)h^k + \bar{t}_k(a)\sigma_k(a) r h^{k-1} h'}{\sigma_l(a)h^l + \underline{t}_l(a)\sigma_l(a) r h^{l-1} h'} \\ &= \frac{h^k + \bar{t}_k(a) r h^{k-1} h'}{h^l + \underline{t}_l(a) r h^{l-1} h'} \\ &= \bar{g}(r) \geq g(x). \end{aligned}$$

Then we complete the proof. ∎

Next we shall construct the generalized symmetric supersolution of (1.1) of the form

$$\Psi(x) := \Psi_{\beta_2,\eta,\tau,A}(x) := \psi(r) := \psi_{\beta_2,\eta,\tau}(r) := \beta_2 + \int_\eta^r \theta H(\theta,\tau)d\theta, \ \forall\, x \in \mathbb{R}^n \backslash E_\eta,$$

where $\beta_2 \in \mathbb{R}$ and H is obtained from the following lemma.

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Lemma 3.5 *For $n \geq 3$ and $0 \leq l < k \leq n$. Let $\frac{\underline{t}_l}{\overline{t}_k}\underline{g}(1) < \tau^{k-l} < \underline{g}(1)$. Then the problem*

$$\begin{cases} \dfrac{H(r)^k + \overline{t}_k r H(r)^{k-1} H'(r)}{H(r)^l + \underline{t}_l r H(r)^{l-1} H'(r)} = \underline{g}(r), \; r > 1, \\ H(1) = \tau \end{cases} \tag{3.31}$$

admits a smooth solution $H(r) = H(r,\tau)$ on $[1, +\infty)$ satisfying:

(i) $\frac{\underline{t}_l}{\overline{t}_k}\underline{g}(r) < H^{k-l}(r,\tau) < \underline{g}(r), \partial_r H(r,\tau) \geq 0$ *for $r \geq 1$.*
(ii) *$H(r,\tau)$ is continuous and strictly increasing with respect to τ.*

Proof For brevity, we sometimes write $H(r)$ or $H(r,\tau)$ when there is no confusion. From (3.31), we have

$$\begin{cases} \dfrac{\mathrm{d}H}{\mathrm{d}r} = -\dfrac{1}{r}\dfrac{H}{\overline{t}_k}\dfrac{H(r)^{k-l} - \underline{g}(r)}{H(r)^{k-l} - \underline{g}(r)\frac{\underline{t}_l}{\overline{t}_k}}, \; r > 1, \\ H(1) = \tau. \end{cases} \tag{3.32}$$

Since $\frac{\underline{t}_l}{\overline{t}_k}\underline{g}(1) < \tau^{k-l} < \underline{g}(1)$ and $\underline{g}(r)$ is strictly increasing, by the existence, uniqueness, and extension theorem for the solution of the initial value problem of the ODE, we can get that the problem has a smooth solution $H(r,\delta)$ satisfying $\frac{\underline{t}_l}{\overline{t}_k}\underline{g}(r) < H^{k-l}(r,\tau) < \underline{g}(r)$, and $\partial_r H(r,\tau) \geq 0$, that is, (i) of this lemma.

Let

$$p(H) := p(H(r,\tau)) := \frac{H(r,\tau)}{\overline{t}_k}\frac{H(r,\tau)^{k-l} - \underline{g}(r)}{H(r,\tau)^{k-l} - \underline{g}(r)\frac{\underline{t}_l}{\overline{t}_k}}.$$

Then (3.32) becomes

$$\begin{cases} \dfrac{\partial H}{\partial r} = -\dfrac{1}{r}p(H(r,\tau)), \; r > 1, \\ H(1,\tau) = \tau. \end{cases} \tag{3.33}$$

Differentiating (3.33) with τ, we have that

$$\begin{cases} \dfrac{\partial^2 H}{\partial r \partial \tau} = -\dfrac{1}{r}p'(H)\dfrac{\partial H}{\partial \tau}, \\ \dfrac{\partial H(1,\tau)}{\partial \tau} = 1. \end{cases}$$

It follows that

$$\frac{\partial H(r,\tau)}{\partial \tau} = q(r) = \exp\int_1^r (-\frac{1}{s})p'(H(s,\tau))ds > 0,$$

that is, (ii) of this lemma. ∎

Remark 3.6 If $g \equiv 1$, then we choose $\overline{g} \equiv \underline{g} \equiv 1, \delta = \tau = 1$. Then $h \equiv 1$ and $H \equiv 1$ satisfy (3.4) and (3.32), respectively.

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Analogous to (3.12),

$$\psi(r) = \int_0^r \theta h_0(\theta)d\theta + \nu_{\beta_2,\eta}(\tau) + \begin{cases} O(r^{2-\min\{\beta,\frac{k-l}{\bar{t}_k-\underline{t}_l}\}}), \text{ if } \beta \neq \frac{k-l}{\bar{t}_k-\underline{t}_l}, \\ O(r^{2-\frac{k-l}{\bar{t}_k-\underline{t}_l}} \ln r), \text{ if } \beta = \frac{k-l}{\bar{t}_k-\underline{t}_l}, \end{cases} \tag{3.34}$$

as $r \to +\infty$, where

$$\nu_{\beta_2,\eta}(\tau) := \beta_2 - \int_0^\eta \theta h_0(\theta)d\theta + \int_\eta^{+\infty} \theta[H(\theta) - h_0(\theta)]d\theta.$$

Theorem 3.7 *Ψ is a k-convex supersolution of (1.1) in $\mathbb{R}^n \backslash E_\eta$.*

Proof By Lemma A.2, we have that for $j = 1, \ldots, k$,

$$\begin{aligned} S_j(D^2\Psi) &= \sigma_j(\lambda(D^2\Psi)) \\ &= \sigma_j(a)H(r)^j + \frac{H'(r)}{r}H(r)^{j-1}\sum_{i=1}^n \sigma_{j-1;i}(a)a_i^2x_i^2 \geq 0, \end{aligned}$$

where we have used the fact that $H' \geq 0$ by Lemma 3.5(i). Moreover, by (3.31), we have that for any $x \in \mathbb{R}^n \backslash E_\eta$,

$$\begin{aligned} \frac{S_k(D^2\Psi)}{S_l(D^2\Psi)} &= \frac{\sigma_k(\lambda(D^2\Psi))}{\sigma_l(\lambda(D^2\Psi))} \\ &= \frac{\sigma_k(a)H(r)^k + \frac{H'(r)}{r}H(r)^{k-1}\sum_{i=1}^n \sigma_{k-1;i}(a)a_i^2x_i^2}{\sigma_l(a)H(r)^l + \frac{H'(r)}{r}H(r)^{l-1}\sum_{i=1}^n \sigma_{l-1;i}(a)a_i^2x_i^2} \\ &\leq \frac{H(r)^k + \bar{t}_k rH(r)^{k-1}H'}{H(r)^l + \bar{t}_l rH(r)^{l-1}H'} \\ &= \underline{g}(r) \leq g(x). \end{aligned} \tag{3.35}$$

■

4 Proof of Theorem 1.6

Before proving Theorem 1.6, we will first give some lemmas which will be used later.

Lemma 4.1 *Suppose that $\phi \in C^2(\partial\Omega)$. Then there exists some constant C, depending only on g, n, $\|\phi\|_{C^2(\partial\Omega)}$, the upper bound of A, the diameter and the convexity of Ω, and the C^2 norm of $\partial\Omega$, such that, for each $\varsigma \in \partial\Omega$, there exists $\bar{x}(\varsigma) \in \mathbb{R}^n$ such that $|\bar{x}(\varsigma)| \leq C$,*

$$\rho_\varsigma < \phi \quad on \quad \partial\Omega\backslash\{\varsigma\} \quad and \quad \rho_\varsigma(\varsigma) = \phi(\varsigma),$$

where

$$\rho_\varsigma(x) = \phi(\varsigma) + \frac{\Xi}{2}[(x - \bar{x}(\varsigma))^T A(x - \bar{x}(\varsigma)) - (\varsigma - \bar{x}(\varsigma))^T A(\varsigma - \bar{x}(\varsigma))], \quad x \in \mathbb{R}^n,$$

and $\frac{\binom{n}{k}\Xi^k}{\binom{n}{l}\Xi^l} > \sup_{E_{2R}} \bar{g}$ for some $R > 0$ such that $E_1 \subset\subset \Omega \subset\subset E_R$.

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Proof The proof of Lemma 4.1 is similar to the proof of Lemma 3.1 in [9]. We only substitute the constant $F^{\frac{1}{k}}$ in Lemma 3.1 in [9] with Ξ. Here, we omit its proof. ∎

Lemma 4.2 [11, Lemma 2.2] *Let B be a ball in $\mathbb{R}^n$, and let $f \in C^0(\overline{B})$ be nonnegative. Suppose that $\underline{u} \in C^0(\overline{B})$ satisfies $S_k(D^2\underline{u}) \geq f(x)$ in B in the viscosity sense. Then the Dirichlet problem*

$$\frac{S_k(D^2u)}{S_l(D^2u)} = f(x), \quad x \in B,$$
$$u = \underline{u}(x), \quad x \in \partial B$$

admits a unique k-convex viscosity solution $u \in C^0(\overline{B})$.

Lemma 4.3 [11, Lemma 2.3] *Let D be a domain in $\mathbb{R}^n$, and let $f \in C^0(\mathbb{R}^n)$ be nonnegative. Assume that $v \in C^0(\overline{D})$ and $u \in C^0(\mathbb{R}^n)$ are two k-convex functions satisfying in the viscosity sense*

$$\frac{S_k(D^2v)}{S_l(D^2v)} \geq f(x), x \in D,$$

and

$$\frac{S_k(D^2u)}{S_l(D^2u)} \geq f(x), x \in \mathbb{R}^n,$$

respectively, $u \leq v$ on $\overline{D}$ and $u = v$ on ∂D.

Let

$$w(x) := \begin{cases} v(x), & x \in D, \\ u(x), & x \in \mathbb{R}^n \backslash D. \end{cases}$$

Then $w \in C^0(\mathbb{R}^n)$ is a k-convex function satisfying

$$\frac{S_k(D^2w)}{S_l(D^2w)} \geq f(x) \text{ in } \mathbb{R}^n \quad \text{in the viscosity sense.}$$

Proof of Theorem 1.6 Without loss of generality, we may assume that $A = \operatorname{diag}(a_1, \dots, a_n) \in \mathcal{A}_{k,l}$, $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and $b = 0$.

For any $\delta > \sup_{r\in[1,+\infty)} g^{\frac{1}{k-l}}(r)$, let

$$W_\delta(x) := \kappa_1 + \int_R^{r_A(x)} \theta h(\theta, \delta) d\theta \qquad \forall x \in \mathbb{R}^n \backslash \{0\},$$

where $r_A(x)$ is defined as in (1.12), where $\rho_\varsigma(x)$ and $h(r, \delta)$ are obtained from Lemmas 4.1 and 3.2, respectively, and $\kappa_1 := \min_{\substack{x\in\overline{E_R}\backslash\Omega \\ \varsigma\in\partial\Omega}} \rho_\varsigma(x)$.

Let

$$\varphi(x) := \max_{\varsigma\in\partial\Omega} \rho_\varsigma(x).$$

Since ρ_ς satisfies

$$\frac{S_k(D^2\rho_\varsigma)}{S_l(D^2\rho_\varsigma)} \geq g(x) \text{ in } E_{2R},$$

then φ satisfies

$$\frac{S_k(D^2\varphi)}{S_l(D^2\varphi)} \geq g(x) \text{ in } E_{2R} \quad \text{in the viscosity sense,} \tag{4.1}$$

and

$$\varphi = \phi \text{ on } \partial\Omega. \tag{4.2}$$

By Theorem 3.4, we have that W_δ is a smooth k-convex subsolution of (1.1), i.e.,

$$\frac{S_k(D^2 W_\delta)}{S_l(D^2 W_\delta)} \geq g(x) \text{ in } \mathbb{R}^n \backslash \bar{\Omega}. \tag{4.3}$$

Since $\Omega \subset\subset E_R$, we can conclude that

$$W_\delta \leq \kappa_1 \leq \rho_\varsigma \leq \varphi \text{ on } \bar{E}_R \backslash \Omega. \tag{4.4}$$

Moreover, by Lemma 3.2, W_δ is strictly increasing in δ and

$$\lim_{\delta \to +\infty} W_\delta(x) = +\infty, \quad \forall\, r_A(x) > R.$$

By (3.12), we have that

$$W_\delta(x) = \int_0^{r_A(x)} \theta h_0(\theta) d\theta + \mu(\delta) + \begin{cases} O(|x|^{2-\min\{\beta, \frac{k-l}{\bar{t}_k - \underline{t}_l}\}}), & \text{if } \beta \neq \frac{k-l}{\bar{t}_k - \underline{t}_l}, \\ O(|x|^{2-\frac{k-l}{\bar{t}_k - \underline{t}_l}} \ln|x|), & \text{if } \beta = \frac{k-l}{\bar{t}_k - \underline{t}_l}, \end{cases}$$

as $|x| \to +\infty$, where

$$\mu(\delta) := \kappa_1 - \int_0^R \theta h_0(\theta) d\theta + \int_R^{+\infty} \theta[h(\theta) - h_0(\theta)] d\theta.$$

Let

$$\overline{u}_{\kappa_2,\tau}(x) := \kappa_2 + \int_1^{r_A(x)} \theta H(\theta, \tau) d\theta, \ \forall\, x \in \mathbb{R}^n \backslash \Omega,$$

where κ_2 is any constant, $\frac{\underline{t}_l}{\bar{t}_k} \underline{g}(1) < \tau^{k-l} < \underline{g}(1)$ and H is obtained from Lemma 3.5. Then we have that, by (3.35),

$$\frac{S_k(D^2\overline{u}_{\kappa_2,\tau})}{S_l(D^2\overline{u}_{\kappa_2,\tau})} \leq g(x), \ \forall\, x \in \mathbb{R}^n \backslash \Omega, \tag{4.5}$$

and by (3.34), as $|x| \to +\infty$,

$$\overline{u}_{\kappa_2,\tau}(x) = \int_0^{r_A(x)} \theta h_0(\theta) d\theta + \nu_{\kappa_2}(\tau) + \begin{cases} O(|x|^{2-\min\{\beta, \frac{k-l}{\bar{t}_k - \underline{t}_l}\}}), & \text{if } \beta \neq \frac{k-l}{\bar{t}_k - \underline{t}_l}, \\ O(|x|^{2-\frac{k-l}{\bar{t}_k - \underline{t}_l}} \ln|x|), & \text{if } \beta = \frac{k-l}{\bar{t}_k - \underline{t}_l}, \end{cases}$$

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where

$$\nu_{\kappa_2}(\tau) := \kappa_2 - \int_0^1 \theta h_0(\theta)d\theta + \int_1^{+\infty} \theta[H(\theta,\tau) - h_0(\theta)]d\theta$$

is convergent.

Since W_δ is strictly increasing in δ, then there exists some $\hat{\delta} > \sup_{r\in[1,+\infty)} \overline{g}^{\frac{1}{k-l}}(r)$ such that $\min_{\partial E_{2R}} W_{\hat{\delta}} > \max_{\partial E_{2R}} \varphi$. It follows that

$$W_{\hat{\delta}} > \varphi \ \text{ on } \ \partial E_{2R}. \tag{4.6}$$

Clearly, $\mu(\delta)$ is strictly increasing in δ. By (3.13), we have that $\lim_{\delta\to+\infty} \mu(\delta) = +\infty$.

Let

$$\hat{c} := \hat{c}(\tau) := \sup_{E_{2R}\setminus\Omega} \varphi - \int_0^1 \theta h_0(\theta)d\theta + \int_1^{+\infty} \theta[H(\theta) - h_0(\theta)]d\theta$$

and

$$\tilde{c} := \max\{\hat{c}, \mu(\hat{\delta}), \max_{\substack{\varsigma\in\partial\Omega \\ x\in\overline{E_{2R}}\setminus\Omega}} \rho_\varsigma(x)\}.$$

Then, for any $c > \tilde{c}$, there is a unique $\delta(c)$ such that $\mu(\delta(c)) = c$. Consequently, we have that

$$W_{\delta(c)}(x) = \int_0^{r_A(x)} \theta h_0(\theta)d\theta + c + \begin{cases} O(|x|^{2-\min\{\beta, \frac{k-l}{\overline{t}_k - \underline{t}_l}\}}), \text{ if } \beta \neq \frac{k-l}{\overline{t}_k - \underline{t}_l}, \\ O(|x|^{2-\frac{k-l}{\overline{t}_k - \underline{t}_l}} \ln|x|), \text{ if } \beta = \frac{k-l}{\overline{t}_k - \underline{t}_l}, \end{cases} \tag{4.7}$$

as $|x| \to +\infty$, and

$$\delta(c) = \mu^{-1}(c) > \mu^{-1}(\mu(\hat{\delta})) = \hat{\delta}.$$

By the monotonicity of W_δ in δ and (4.6), we conclude that

$$W_{\delta(c)} \geq W_{\hat{\delta}} > \varphi \ \text{ on } \ \partial E_{2R}. \tag{4.8}$$

Taking κ_2 such that $\nu_{\kappa_2}(\tau) = c$. Then we have that

$$\kappa_2 = c + \int_0^1 \theta h_0(\theta)d\theta - \int_1^{+\infty} \theta[H(\theta,\tau) - h_0(\theta)]d\theta, \tag{4.9}$$

and as $|x| \to +\infty$,

$$\overline{u}_{\kappa_2,\tau}(x) = \int_0^{r_A(x)} \theta h_0(\theta)d\theta + c + \begin{cases} O(|x|^{2-\min\{\beta, \frac{k-l}{\overline{t}_k - \underline{t}_l}\}}), \text{ if } \beta \neq \frac{k-l}{\overline{t}_k - \underline{t}_l}, \\ O(|x|^{2-\frac{k-l}{\overline{t}_k - \underline{t}_l}} \ln|x|), \text{ if } \beta = \frac{k-l}{\overline{t}_k - \underline{t}_l}. \end{cases} \tag{4.10}$$

Define

$$\underline{u}(x) := \begin{cases} \max\{W_{\delta(c)}(x), \varphi(x)\}, & x \in E_{2R}\setminus\Omega, \\ W_{\delta(c)}(x), & x \in \mathbb{R}^n\setminus E_{2R}. \end{cases}$$

Then, by (4.8), we have that $\underline{u} \in C^0(\mathbb{R}^n\backslash\Omega)$. By (4.1), (4.3), and Lemma 4.3, $\underline{u}$ satisfies in the viscosity sense

$$\frac{S_k(D^2\underline{u})}{S_l(D^2\underline{u})} \geq g(x) \text{ in } \mathbb{R}^n\backslash\overline{\Omega}.$$

By (4.4) and (4.2), we obtain that $\underline{u} = \phi$ on $\partial\Omega$. Moreover, by (4.7), we have that $\underline{u}$ satisfies the asymptotic behavior (4.10) at infinity.

By the definitions of $\tilde{c}$, $\overline{u}_{\kappa_2}$, and φ, we have that

$$\overline{u}_{\kappa_2,\tau} \geq \kappa_2 \geq \varphi \quad \text{in} E_{2R}\backslash\Omega. \tag{4.11}$$

By (4.4), $W_{\delta(c)} \leq \varphi \leq \overline{u}_{\kappa_2,\tau}$ on $\partial\Omega$. By (4.3), (4.5), (4.7), (4.10), and the comparison principle, we have that

$$W_{\delta(c)} \leq \overline{u}_{\kappa_2,\tau} \quad \text{in } \mathbb{R}^n\backslash\Omega. \tag{4.12}$$

Let $\overline{u} := \overline{u}_{\kappa_2,\tau}$ in $\mathbb{R}^n\backslash\Omega$. By (4.11), (4.12), and the definition of $\underline{u}$, we have that $\underline{u} \leq \overline{u}$ in $\mathbb{R}^n\backslash\Omega$.

For any $c > \tilde{c}$, let $\mathcal{S}_c$ denote the set of $\rho \in C^0(\mathbb{R}^n\backslash\Omega)$ which are viscosity subsolutions of (1.1) and (1.2) satisfying $\rho = \phi$ on $\partial\Omega$ and $\rho \leq \overline{u}$ in $\mathbb{R}^n\backslash\Omega$. Apparently, $\underline{u} \in \mathcal{S}_c$, which implies that $\mathcal{S}_c \neq \varnothing$. Define

$$u(x) := \sup\{\rho(x) | \rho \in \mathcal{S}_c\}, \quad \forall\, x \in \mathbb{R}^n\backslash\Omega.$$

Then

$$\underline{u} \leq u \leq \overline{u} \text{ in } \mathbb{R}^n\backslash\Omega.$$

Hence, by the asymptotic behavior of $\underline{u}$ and $\overline{u}$ at infinity, we have that

$$u(x) = \int_0^{r_A(x)} \theta h_0(\theta)d\theta + c + \begin{cases} O(|x|^{2-\min\{\beta, \frac{k-l}{\bar{t}_k-\underline{t}_l}\}}), \text{ if } \beta \neq \frac{k-l}{\bar{t}_k-\underline{t}_l}, \\ O(|x|^{2-\frac{k-l}{\bar{t}_k-\underline{t}_l}} \ln|x|), \text{ if } \beta = \frac{k-l}{\bar{t}_k-\underline{t}_l}, \end{cases}$$

as $|x| \to +\infty$.

Next, we will show that $u = \phi$ on $\partial\Omega$. On one side, since $\underline{u} = \phi$ on $\partial\Omega$, we have that

$$\liminf_{x\to\varsigma} u(x) \geq \lim_{x\to\varsigma} \underline{u}(x) = \phi(\varsigma), \quad \varsigma \in \partial\Omega.$$

On the other side, we want to prove that

$$\limsup_{x\to\varsigma} u(x) \leq \phi(\varsigma), \quad \varsigma \in \partial\Omega.$$

Let $\vartheta \in C^2(\overline{E_{2R}\backslash\Omega})$ satisfy

$$\begin{cases} \Delta\vartheta = 0, & \text{in } E_{2R}\backslash\bar{\Omega}, \\ \vartheta = \phi, & \text{on } \partial\Omega, \\ \vartheta = \max_{\partial E_{2R}} \overline{u}, & \text{on } \partial E_{2R}. \end{cases}$$

By Newton's inequality, for any $\rho \in \mathcal{S}_c$, we have that $\Delta\rho \geq 0$ in the viscosity sense. Moreover, $\rho \leq \vartheta$ on $\partial(E_{2R}\backslash\Omega)$. Then, by the comparison principle, we have that $\rho \leq \vartheta$ in $E_{2R}\backslash\Omega$. It follows that $u \leq \vartheta$ in $E_{2R}\backslash\Omega$. Therefore,

$$\limsup_{x\to\varsigma} u(x) \leq \lim_{x\to\varsigma} \vartheta(x) = \phi(\varsigma) \text{ for } \varsigma \in \partial\Omega.$$

Finally, we will prove that $u \in C^0(\mathbb{R}^n\backslash\Omega)$ is a viscosity solution of (1.1). For any $x \in \mathbb{R}^n\backslash\overline{\Omega}$, choose some $\varepsilon > 0$ such that $B_\varepsilon = B_\varepsilon(x) \subset \mathbb{R}^n\backslash\overline{\Omega}$. By Lemma 4.2, the following Dirichlet problem

$$\begin{cases} \dfrac{S_k(D^2\tilde{u})}{S_l(D^2\tilde{u})} = g(y), & \text{in } B_\varepsilon, \\ \tilde{u} = u, & \text{on } \partial B_\varepsilon \end{cases} \tag{4.13}$$

admits a unique k-convex viscosity solution $\tilde{u} \in C^0(\overline{B_\varepsilon})$. By the comparison principle, $u \leq \tilde{u}$ in B_ε. Define

$$\tilde{w}(y) = \begin{cases} \tilde{u}(y), & \text{in } B_\varepsilon, \\ u(y), & \text{in } (\mathbb{R}^n\backslash\Omega)\backslash B_\varepsilon. \end{cases}$$

Then $\tilde{w} \in \mathcal{S}_c$. Indeed, by the comparison principle, $\tilde{u}(y) \leq \overline{u}(y)$ in $\bar{B}_\varepsilon$. It follows that $\tilde{w} \leq \overline{u}$ in $\mathbb{R}^n\backslash B_\varepsilon$. By Lemma 4.3, we have that $\frac{S_k(D^2\tilde{w})}{S_l(D^2\tilde{w})} \geq g(y)$ in $\mathbb{R}^n\backslash\overline{\Omega}$ in the viscosity sense. Therefore, $\tilde{w} \in \mathcal{S}_c$.

By the definition of u, $u \geq \tilde{w}$ in $\mathbb{R}^n\backslash\Omega$. It follows that $u \geq \tilde{u}$ in B_ε. Hence, $u \equiv \tilde{u}$ in B_ε. Since $\tilde{u}$ satisfies (4.13), then we have that in the viscosity sense,

$$\frac{S_k(D^2u)}{S_l(D^2u)} = g(y), \ \forall\ y \in B_\varepsilon.$$

In particular, we have that in the viscosity sense,

$$\frac{S_k(D^2u)}{S_l(D^2u)} = g(x).$$

Since x is arbitrary, we can conclude that u is a viscosity solution of (1.1).

Theorem 1.6 is proved. ∎

A Appendix

In this appendix, we will show that it is impossible to construct the generalized symmetric solution of (3.3).

Proposition A.1 *If there exists a C^2 function defined on (r_1, r_2) such that $T(x) := G(r)$ is a generalized symmetric solution of (3.3), then*

$$k = n, \ \ or \ \ a_1 = \cdots = a_n = \hat{a} = \left(\binom{n}{l}\Big/\binom{n}{k}\right)^{\frac{1}{k-l}},$$

where $r = r_A$ is defined as in (1.12).

Before proving the above proposition, we will first give some elementary lemmas.

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Lemma A.2 *If $M = (p_i\delta_{ij} + sq_iq_j)_{n\times n}$ with $p, q \in \mathbb{R}^n$ and $s \in \mathbb{R}$, then*

$$\sigma_k(\lambda(M)) = \sigma_k(p) + s\sum_{i=1}^{n}\sigma_{k-1;i}(p)q_i^2,\ k = 1,\dots,n.$$

Proof See [2]. ∎

Lemma A.3 *Suppose that $\tilde{\phi} \in C^2[0,+\infty)$ and $\tilde{\Phi}(x) := \tilde{\phi}(r)$. Then $\tilde{\Phi}$ satisfies*

$$S_k(D^2\tilde{\Phi}) = \sigma_k(a)\tilde{h}(r)^k + \frac{\tilde{h}'(r)}{r}\tilde{h}(r)^{k-1}\sum_{i=1}^{n}\sigma_{k-1;i}(a)a_i^2x_i^2, \tag{A.1}$$

where $\tilde{h}(r) := \tilde{\phi}'(r)/r$.

Proof Since $r^2 = x^TAx = \sum_{i=1}^n a_ix_i^2$, we have that

$$2r\partial_{x_i}r = \partial_{x_i}(r^2) = 2a_ix_i \ \text{ and } \ \partial_{x_i}r = \frac{a_ix_i}{r}.$$

It follows that

$$\partial_{x_i}\tilde{\Phi}(x) = \tilde{\phi}'(r)\partial_{x_i}r = \frac{\tilde{\phi}'(r)}{r}a_ix_i,$$

$$\begin{aligned}\partial_{x_ix_j}\tilde{\Phi}(x) &= \frac{\tilde{\phi}'(r)}{r}a_i\delta_{ij} + \frac{\tilde{\phi}''(r) - \frac{\tilde{\phi}'(r)}{r}}{r^2}(a_ix_i)(a_jx_j)\\ &= \tilde{h}(r)a_i\delta_{ij} + \frac{\tilde{h}'(r)}{r}(a_ix_i)(a_jx_j).\end{aligned}$$

By Lemma A.2, we have that

$$\begin{aligned}S_k(D^2\tilde{\Phi}) &= \sigma_k(\lambda(D^2\tilde{\Phi}))\\ &= \sigma_k(a)\tilde{h}(r)^k + \frac{\tilde{h}'(r)}{r}\tilde{h}(r)^{k-1}\sum_{i=1}^{n}\sigma_{k-1;i}(a)a_i^2x_i^2.\end{aligned}$$ ∎

Lemma A.4 *Let $a = (a_1, a_2, \dots, a_n)$ satisfy $0 < a_1 \le a_2 \le \cdots \le a_n$. Then, for $1 \le k \le n$,*

$$0 < \underline{t}_k \le \frac{k}{n} \le \overline{t}_k \le 1, \tag{A.2}$$

$$0 = \overline{t}_0 < \frac{1}{n} \le \frac{a_n}{\sigma_1(a)} = \overline{t}_1 \le \overline{t}_2 \le \cdots \le \overline{t}_{n-1} < \overline{t}_n = 1,$$

and

$$0 = \underline{t}_0 < \frac{a_1}{\sigma_1(a)} = \underline{t}_1 \le \underline{t}_2 \le \cdots \le \underline{t}_{n-1} < \underline{t}_n = 1,$$

where $\underline{t}_k$ and $\overline{t}_k$ are defined as in (1.4) and (1.5). Moreover, for $1 \le k \le n-1$,

$$\underline{t}_k = \overline{t}_k = \frac{k}{n}$$

if and only if $a_1 = \cdots = a_n = \overline{C}$ for some $\overline{C} > 0$.

Remark A.5 From (A.2), we know that

$$0 < \frac{a_1}{\sigma_1(a)} \le \underline{t}_l \le \frac{l}{n} < \frac{k}{n} \le \overline{t}_k \le 1.$$

Then

$$0 < \overline{t}_k - \underline{t}_l < 1.$$

Now we give the proof of Proposition A.1.

Proof For the special case $l = 0, 1 \le k \le n$, the Hessian equation case, Proposition A.1 can be proved similarly with Proposition 2.2 in [9]. We only need to prove the case $1 \le l < k \le n$.

Let $J(r) = G'(r)/r$. By (A.1), we know that T satisfies

$$\frac{S_k(D^2T)}{S_l(D^2T)} = \frac{\sigma_k(a)J(r)^k + \frac{J'(r)}{r}J(r)^{k-1}\sum_{i=1}^{n}\sigma_{k-1;i}(a)a_i^2x_i^2}{\sigma_l(a)J(r)^l + \frac{J'(r)}{r}J(r)^{l-1}\sum_{i=1}^{n}\sigma_{l-1;i}(a)a_i^2x_i^2} = \overline{g}(r).$$

Set $x = (0, \dots, 0, \sqrt{r/a_i}, 0, \dots, 0)$. Then

$$\frac{S_k(D^2T)}{S_l(D^2T)} = \frac{\sigma_k(a)J(r)^k + J'(r)J(r)^{k-1}\sigma_{k-1;i}(a)a_i}{\sigma_l(a)J(r)^l + J'(r)J(r)^{l-1}\sigma_{l-1;i}(a)a_i} = \overline{g}(r).$$

So

$$\sigma_k(a)J(r)^k + J'(r)J(r)^{k-1}\sigma_{k-1;i}(a)a_i = \overline{g}(r)[\sigma_l(a)J(r)^l + J'(r)J(r)^{l-1}\sigma_{l-1;i}(a)a_i]. \tag{A.3}$$

Since $\sigma_k(a) = \sigma_l(a)$, then

$$\frac{J(r)^k - \overline{g}(r)J(r)^l}{J'(r)} = \frac{\overline{g}(r)J(r)^{l-1}\sigma_{l-1;i}(a)a_i - J(r)^{k-1}\sigma_{k-1;i}(a)a_i}{\sigma_k(a)}. \tag{A.4}$$

Noting that the left side of (A.4) is independent of i, so for any $i \neq j$, we have that

$$\overline{g}(r)J(r)^{l-1}\sigma_{l-1;i}(a)a_i - J(r)^{k-1}\sigma_{k-1;i}(a)a_i = \overline{g}(r)J(r)^{l-1}\sigma_{l-1;j}(a)a_j - J(r)^{k-1}\sigma_{k-1;j}(a)a_j.$$

As a result,

$$\overline{g}(r)J(r)^{l-1}[\sigma_{l-1;i}(a)a_i - \sigma_{l-1;j}(a)a_j] = J(r)^{k-1}[\sigma_{k-1;i}(a)a_i - \sigma_{k-1;j}(a)a_j]. \tag{A.5}$$

Applying the equality $\sigma_k(a) = \sigma_{k;i}(a) + a_i\sigma_{k-1;i}(a)$ for all i, we get that

$$\begin{aligned}
&\sigma_{l-1;i}(a)a_i - \sigma_{l-1;j}(a)a_j \\
&\quad = [\sigma_{l-1;ij}(a) + \sigma_{l-2;ij}(a)a_j]a_i - [\sigma_{l-1;ij}(a) + \sigma_{l-2;ij}(a)a_i]a_j \\
&\quad = \sigma_{l-1;ij}(a)(a_i - a_j).
\end{aligned}$$

Therefore, (A.5) becomes

$$\overline{g}(r)J(r)^{l-1}\sigma_{l-1;ij}(a)(a_i - a_j) = J(r)^{k-1}\sigma_{k-1;ij}(a)(a_i - a_j).$$

https://doi.org/10.4153/S0008414X23000834 Published online by Cambridge University Press

If $k = n$, then $\sigma_{k-1;ij}(a) = 0$ for any $i \neq j$. But $\sigma_{l-1;ij}(a) > 0$, so we get that

$$a_1 = \cdots = a_n = \hat{a}.$$

If $1 \leq l < k \leq n-1$, then $\sigma_{k-1;ij}(a) > 0$ and $\sigma_{l-1;ij}(a) > 0$. Suppose on the contrary that $a_i \neq a_j$, then

$$\overline{g}(r)J(r)^{l-1}\sigma_{l-1;ij}(a) = J(r)^{k-1}\sigma_{k-1;ij}(a).$$

Thus,

$$\text{(A.6)} \qquad \frac{\sigma_{k-1;ij}(a)}{\sigma_{l-1;ij}(a)} = \frac{\overline{g}(r)J(r)^{l-1}}{J(r)^{k-1}} = \overline{g}(r)J(r)^{l-k}.$$

Since the left side is independent of r, then $\overline{g}(r)J(r)^{l-k}$ is a constant $c_0 > 0$. So $\overline{g}(r) = c_0 J(r)^{k-l}$. Substituting into (A.4), we have that

$$\text{(A.7)} \qquad \frac{J(r)(1-c_0)}{c_0 J'(r)} = \frac{\sigma_{l-1;i}(a)a_i - \sigma_{k-1;i}(a)a_i}{\sigma_k(a)}.$$

Since the left side of the above equality is independent of i, then for any $i \neq j$,

$$\sigma_{l-1;i}(a)a_i - \sigma_{k-1;i}(a)a_i = \sigma_{l-1;j}(a)a_j - \sigma_{k-1;j}(a)a_j,$$

so

$$\sigma_{l-1;ij}(a)(a_i - a_j) = \sigma_{k-1;ij}(a)(a_i - a_j).$$

However, $a_i \neq a_j$; thus, $\sigma_{l-1;ij}(a) = \sigma_{k-1;ij}(a)$. Therefore, by (A.6), we can have $c_0 = 1$. Then, by (A.7), we can get that for all i,

$$\sigma_{l-1;i}(a)a_i = \sigma_{k-1;i}(a)a_i.$$

Recalling the equality

$$\sum_{i=1}^{n} a_i \sigma_{k-1;i}(a) = k\sigma_k(a),$$

we know that $k\sigma_k(a) = l\sigma_l(a)$. Since $A \in \mathcal{A}_{k,l}$, we can conclude that $\sigma_k(a) = \sigma_l(a)$ and $k = l$, which is a contradiction. ∎

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