Bull. Aust. Math. Soc. **90** (2014), 521–524
doi:10.1017/S0004972714000318

A NOTE ON SPACES WITH A RANK 3-DIAGONAL

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(Received 5 March 2014; accepted 4 April 2014; first published online 19 May 2014)

Abstract

We prove that if X is a space satisfying the discrete countable chain condition with a rank 3-diagonal then the cardinality of X is at most $\mathfrak{c}$.

2010 *Mathematics subject classification*: primary 54D20; secondary 54E35.
Keywords and phrases: cardinality, DCCC, rank 3-diagonal.

1. Introduction

Diagonal properties are useful in estimating the cardinality of a space. For example, Ginsburg and Woods [4] proved that the cardinality of a space with countable extent and a G_δ-diagonal is at most $\mathfrak{c}$. Therefore, if X is Lindelöf and has a G_δ-diagonal then $|X| \leq \mathfrak{c}$. However, the cardinality of a regular space with countable Souslin number and a G_δ-diagonal need not have an upper bound [6, 7]. Buzyakova [2] proved that if a space X with countable Souslin number has a regular G_δ-diagonal then the cardinality of X does not exceed $\mathfrak{c}$.

Rank 3-diagonal is one type of diagonal property. In this paper, we prove that if X is a DCCC space (defined below) with a rank 3-diagonal then the cardinality of X is at most $\mathfrak{c}$.

2. Notation and terminology

All spaces are assumed to be Hausdorff unless otherwise stated.

The cardinality of a set X is denoted by $|X|$, and $[X]^2$ will denote the set of two-element subsets of X. We write ω for the first infinite cardinal and $\mathfrak{c}$ for the cardinality of the continuum.

Definition 2.1 [8]. We say that a space X satisfies the discrete countable chain condition (or that X is DCCC) if every discrete family of nonempty open subsets of X is countable.

The authors are supported by NSFC project 11271178.

If A is a subset of X and $\mathcal{U}$ is a family of subsets of X, then $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We also put $\mathrm{St}^0(A,\mathcal{U}) = A$ and, for a nonnegative integer n, $\mathrm{St}^{n+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^n(A,\mathcal{U}),\mathcal{U})$. If $A = \{x\}$ for some $x \in X$, then we write $\mathrm{St}^n(x,\mathcal{U})$ instead of $\mathrm{St}^n(\{x\},\mathcal{U})$.

Definition 2.2 [1]. A diagonal sequence of rank k on a space X, where $k \in \omega$, is a countable family $\{\mathcal{U}_n : n \in \omega\}$ of open coverings of X such that $\{x\} = \bigcap\{\mathrm{St}^k(x,\mathcal{U}_n) : n \in \omega\}$ for each $x \in X$.

Definition 2.3 [1]. A space X has a rank k-diagonal, where $k \in \omega$, if there is a diagonal sequence $\{\mathcal{U}_n : n \in \omega\}$ on X of rank k.

Therefore, a space X has a rank 3-diagonal if there exists a diagonal sequence on X of rank three, that is, there is a countable family $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap\{\mathrm{St}^3(x,\mathcal{U}_n) : n \in \omega\}$.

All notation and terminology not explained here is given in [3].

3. Results

We will use the following countable version of a set-theoretic theorem due to Erdős and Radó.

Lemma 3.1 [5, Theorem 2.3]. *Let X be a set with $|X| > \mathfrak{c}$ and suppose that $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then there exist $n_0 < \omega$ and a subset S of X with $|S| > \omega$ such that $[S]^2 \subseteq P_{n_0}$.*

Lemma 3.2. *Let X be a space with a rank 3-diagonal. If $|X| > \mathfrak{c}$, then there exists an uncountable closed discrete subset of X which has a disjoint open expansion.*

Proof. Since X has a rank 3-diagonal, there exists a sequence $\{\mathcal{U}_m : m \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\mathrm{St}^3(x,\mathcal{U}_m) : m \in \omega\}$ for every $x \in X$. We may assume that $\mathrm{St}^3(x,\mathcal{U}_{m+1}) \subseteq \mathrm{St}^3(x,\mathcal{U}_m)$ for any $m \in \omega$. For $n \in \omega$, let

$$P_n = \{\{x,y\} \in [X]^2 : n = \min\{m \in \omega : x \notin \mathrm{St}^3(y,\mathcal{U}_m)\}\}.$$

Thus, $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then, by Lemma 3.1, there exists a subset S of X with $|S| > \omega$ and $[S]^2 \subseteq P_{n_0}$ for some $n_0 \in \omega$.

Now we show that S is closed and discrete and it has a disjoint open expansion.

Fact 1. It is evident that $\{\mathrm{St}(x,\mathcal{U}_{n_0}) : x \in S\}$ is an uncountable pairwise-disjoint family of nonempty open sets of X.

Fact 2. S is closed and discrete. If not, let $x \in X$ and suppose that x is an accumulation point of S. Since X is T_1, each neighbourhood of x meets infinitely many members of S. Therefore, there exist distinct points y and z in $S \cap \mathrm{St}(x,\mathcal{U}_{n_0})$. Thus, $y,z \in \mathrm{St}(x,\mathcal{U}_{n_0})$; by symmetry, $x \in \mathrm{St}(y,\mathcal{U}_{n_0})$ and $x \in \mathrm{St}(z,\mathcal{U}_{n_0})$, which implies that $x \in \mathrm{St}^2(y,\mathcal{U}_{n_0}) \subseteq \mathrm{St}^3(y,\mathcal{U}_{n_0})$. This is a contradiction. Thus, S has no accumulation points in X; equivalently, S is a closed and discrete subset of X. This completes the proof. □

https://doi.org/10.1017/S0004972714000318 Published online by Cambridge University Press

Theorem 3.3. *If X is a DCCC space and has a rank 3-diagonal, then the cardinality of X does not exceed $\mathfrak{c}$.*

Proof. Assume the contrary. It follows from Lemma 3.2 that $\{\mathrm{St}(x, \mathcal{U}_{n_0}) : x \in S\}$ is an uncountable pairwise-disjoint family of nonempty open sets of X. It must have a cluster point $y \in X$, since X is DCCC. We show that $y \in \mathrm{St}^2(S, \mathcal{U}_{n_0})$ by proving the following statement.

Claim. $\overline{\mathrm{St}(S, \mathcal{U}_{n_0})} \subset \mathrm{St}^2(S, \mathcal{U}_{n_0})$.

To prove the claim, pick any $y \in \overline{\mathrm{St}(S, \mathcal{U}_{n_0})}$. Clearly, $\mathrm{St}(y, \mathcal{U}_{n_0}) \cap \mathrm{St}(S, \mathcal{U}_{n_0}) \neq \emptyset$. By symmetry, $y \in \mathrm{St}^2(S, \mathcal{U}_{n_0})$. This proves the claim.

Now we assume that $y \in \mathrm{St}^2(x_0, \mathcal{U}_{n_0})$ for some $x_0 \in S$. It is clear that $\mathrm{St}^2(x_0, \mathcal{U}_{n_0}) \cap \mathrm{St}(x, \mathcal{U}_{n_0}) \neq \emptyset$ for any $x \in S \setminus \{x_0\}$, since otherwise $x \in \mathrm{St}^3(x_0, \mathcal{U}_{n_0})$. This shows that y is not a cluster point of $\{\mathrm{St}(x, \mathcal{U}_{n_0}) : x \in S\}$. This is a contradiction and proves that $|X| \leq \mathfrak{c}$. □

If we drop the condition 'DCCC' in Theorem 3.3, the conclusion is no longer true, as can be seen in the following example.

Example 3.4. Let D be a discrete space with $|D| = 2^{\mathfrak{c}}$. Clearly, it has a rank 3-diagonal and it is not DCCC.

We say that a space X has the countable chain condition (CCC) if any disjoint family of open sets in X is countable; a space X is star countable if whenever $\mathcal{U}$ is an open cover of X, there is a countable subset A of X such that $\mathrm{St}(A, \mathcal{U}) = X$.

Proposition 3.5.

(1) *A CCC space X is DCCC.*

(2) *A star countable space X is DCCC.*

Proof. (1) is immediate from the definitions. To prove (2), assume that X has an uncountable discrete family $\{U_\alpha : \alpha < \Lambda\}$ of nonempty open sets of X. For each $\alpha < \Lambda$, pick a point $x_\alpha \in U_\alpha$. Let $F = \{x_\alpha : \alpha < \Lambda\}$. Clearly, F is an uncountable closed discrete subspace of X. Then $\mathcal{U} = \{U_\alpha : \alpha < \Lambda\} \cup \{X \setminus F\}$ is an open cover of X for which there is no countable subset $A \subseteq X$ such that $\mathrm{St}(A, \mathcal{U}) = X$, which is a contradiction. □

With the aid of the above observations, Theorem 3.3 would be compared to a recent result of [9]: if a space X with a rank 2-diagonal either has the countable chain condition or is star countable, then the cardinality of X is at most $\mathfrak{c}$. We finish this paper with the following question.

Question 3.6. *Is the cardinality of a DCCC space with a rank 2-diagonal at most $\mathfrak{c}$?*

https://doi.org/10.1017/S0004972714000318 Published online by Cambridge University Press

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