ON POLYNOMIAL ALGEBRAS AND FREE ALGEBRAS

GEORGE GRÄTZER

1. Introduction. It is well known that given the polynomial algebra $\mathfrak{P}^{(\alpha)}(\tau)$ (for definitions, see §2), an algebra $\mathfrak{A}$ of type τ, and a sequence $\mathfrak{a}$ of elements of $\mathfrak{A}$, one can define a congruence relation $\theta_{\mathfrak{a}}$ of $\mathfrak{P}^{(\alpha)}(\tau)$ such that the factor algebra $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$ is isomorphic to the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{a}$, and the isomorphism is given in a very simple way.

It will be shown in this note that this result can be extended to the case when $\mathfrak{A}$ is a partial algebra. Theorem 1 gives the description of $\theta_{\mathfrak{a}}$. This is then used to describe the structure of $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$, which turns out to be the free algebra over $K(\tau)$ generated by $\mathfrak{A}$ in the sense of §6. Some elementary observations are made concerning the existence of algebras freely generated by partial algebras in §6.

It should be emphasized that the main results of the paper are the description of $\theta_{\mathfrak{a}}$ and of $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$. The results in §6 are not applications but only illustrations of these.

2. Preliminaries. A *partial algebra* $\mathfrak{A} = \langle A; F\rangle$ is a non-empty set A and a set F of finitary partial operations on A. Well-ordering $F = \langle f_0, \ldots, f_\gamma, \ldots\rangle_{\gamma<\beta}$ and associating with it the sequence $\tau = \langle n_0, \ldots, n_\gamma, \ldots\rangle_{\gamma<\beta}$ (where f_γ is an n_γ-ary partial operation) yields the type τ of $\mathfrak{A}$. β will be denoted by $o(\tau)$. $K(\tau)$ is the class of all algebras of type τ. The set of *α-ary polynomial symbols* $\mathbf{P}^{(\alpha)}(\tau)$ is defined by the following rules:

(i) $\mathbf{x}_\gamma \in \mathbf{P}^{(\alpha)}(\tau)$ for $\gamma < \alpha$;

(ii) if $\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma-1}$ are in $\mathbf{P}^{(\alpha)}(\tau)$, then $\mathbf{f}_\gamma(\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma-1})$ is in $\mathbf{P}^{(\alpha)}(\tau)$ for $\gamma < o(\tau)$;

(iii) α-ary polynomial symbols are those and only those which can be obtained from (i) and (ii) in a finite number of steps.

The algebra $\mathfrak{P}^{(\alpha)}(\tau) = \langle \mathbf{P}^{(\alpha)}(\tau); F\rangle$ is obtained in the natural manner, using (ii) to define the operations on $\mathbf{P}^{(\alpha)}(\tau)$. In the case of a partial algebra $\mathfrak{A}$, an α-ary polynomial symbol is not always associated with a mapping of A^α into A. Thus, we have to specify its interpretation:

Let $\mathfrak{A}$ be a partial algebra of type τ, $a_0, \ldots, a_\gamma, \ldots \in A$, $\gamma < \alpha$, $\mathbf{p} \in \mathbf{P}^{(\alpha)}(\tau)$. Then $p(a_0, \ldots, a_\gamma, \ldots)$ is defined and equals $a \in A$ if and only if it follows from the following rules:

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(i) if $\mathbf{p} = \mathbf{x}_\gamma$, $\gamma < \alpha$, then $p(a_0, \ldots, a_\gamma, \ldots) = a_\gamma$.

(ii) if $p_i(a_0, \ldots, a_\gamma, \ldots)$ is defined and equals b_i for $0 \leqslant i < n_\gamma$,
$f_\gamma(b_0, \ldots, b_{n_\gamma - 1})$ is defined and equals b,
$\mathbf{p} = \mathbf{f}_\gamma(\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma - 1})$,

then $p(a_0, \ldots, a_\gamma, \ldots)$ is defined and equals b.

We adopt the convention that an equation $p(a_0, \ldots, a_\gamma, \ldots) = q(a_0, \ldots, a_\gamma, \ldots)$ includes the assertion that the terms considered are defined.

3. The congruence relation $\theta_\mathfrak{a}$. Let $\mathfrak{A}$ be a partial algebra of type τ, $\mathfrak{a} \in A^\alpha$, $\mathfrak{a} = \langle a_0, \ldots, a_\gamma, \ldots \rangle_{\gamma < \alpha}$, and define a binary relation $\theta_\mathfrak{a}$ on $\mathbf{P}^{(\alpha)}(\tau)$ as follows:

$\mathbf{p} \equiv \mathbf{q}(\theta_\mathfrak{a})$ if and only if there exist $k \geqslant 1$, $\mathbf{r} \in \mathbf{P}^{(k)}(\tau)$, and $\mathbf{p}_i, \mathbf{q}_i \in \mathbf{P}^{(\alpha)}(\tau)$ $(0 \leqslant i < k)$ such that $p_i(a_0, \ldots, a_\gamma, \ldots)$ and $q_i(a_0, \ldots, a_\gamma, \ldots)$ exist, $p_i(a_0, \ldots, a_\gamma, \ldots) = q_i(a_0, \ldots, a_\gamma, \ldots)$, and $\mathbf{p} = r(\mathbf{p}_0, \ldots, \mathbf{p}_{k-1})$, $\mathbf{q} = r(\mathbf{q}_0, \ldots, \mathbf{q}_{k-1})$.

THEOREM 1. *$\theta_\mathfrak{a}$ is a congruence relation of $\mathfrak{P}^{(\alpha)}(\tau)$.*

Remark. If $\mathfrak{A}$ is an algebra, $\theta_\mathfrak{a}$ is defined simply by the rule: $\mathbf{p} \equiv \mathbf{q}(\theta_\mathfrak{a})$ if and only if $p(a_0, \ldots, a_\gamma, \ldots) = q(a_0, \ldots, a_\gamma, \ldots)$.

Proof. (i) $\theta_\mathfrak{a}$ is obviously symmetric, and an easy computation shows that it is reflexive.

(ii) To prove the substitution property, let $\mathbf{p} = \mathbf{f}_\gamma(\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma - 1})$, $\mathbf{q} = \mathbf{f}_\gamma(\mathbf{q}_0, \ldots, \mathbf{q}_{n_\gamma - 1})$, and $\mathbf{p}_i \equiv \mathbf{q}_i(\theta_\mathfrak{a})$, $0 \leqslant i < n_\gamma$. Then

$$\boldsymbol{p}_i = r_i(\boldsymbol{p}_0^{\,i}, \ldots, \boldsymbol{p}_{n_i - 1}^{\,i}), \qquad \boldsymbol{q}_i = r_i(\boldsymbol{q}_0^{\,i}, \ldots, \boldsymbol{q}_{n_i - 1}^{\,i}),$$

and
$$p_j^{\,i}(a_0, \ldots, a_\gamma, \ldots) = q_j^{\,i}(a_0, \ldots, a_\gamma, \ldots).$$

If follows easily from the definition of α-ary polynomials that there are n-ary polynomial symbols $\mathbf{r}'_i$, $n = n_0 + n_1 + \ldots + n_{n_\gamma - 1}$, such that

$$r_i(b_0, \ldots, b_{n_i - 1}) = r'_i(c_0, \ldots, c_{n_0 - 1}, c_{n_0}, \ldots, c_{n_0 + \ldots + n_{i-1} - 1}, b_0, \ldots, b_{n_i - 1}, c_{n_0 + \ldots + n_i}, \ldots)$$

for all $i = 0, \ldots, n_\gamma - 1$, $b_j \in A$, $c_j \in A$. Thus we have that

$$\boldsymbol{p}_i = r'_i(\boldsymbol{p}_0^{\,0}, \ldots, \boldsymbol{p}_{n_0 - 1}^{0}, \ldots, \boldsymbol{p}_0^{n_\gamma - 1}, \ldots, \boldsymbol{p}_{n_{n_\gamma - 1} - 1}^{n_\gamma - 1})$$

for all $0 \leqslant i < n_\gamma$. Setting $\mathbf{r} = \mathbf{f}_\gamma(\mathbf{r}'_0, \ldots, \mathbf{r}'_{n_\gamma - 1})$, we get

$$r(\boldsymbol{p}_0^0, \ldots, \boldsymbol{p}_{n_0 - 1}^0, \ldots, \boldsymbol{p}_0^{n_\gamma - 1}, \ldots, \boldsymbol{p}_{n_{n_\gamma - 1} - 1}^{n_\gamma - 1}) = \boldsymbol{p},$$

$$r(\boldsymbol{q}_0^0, \ldots, \boldsymbol{q}_{n_0 - 1}^0, \ldots, \boldsymbol{q}_0^{n_\gamma - 1}, \ldots, \boldsymbol{q}_{n_{n_\gamma - 1} - 1}^{n_\gamma - 1}) = \boldsymbol{q}.$$

Thus, $\mathbf{p} \equiv \mathbf{q}(\theta_{\mathfrak{a}})$, which was to be proved. In order to establish the transitivity of $\theta_{\mathfrak{a}}$ we need a lemma.

LEMMA 1.* $\mathbf{p} = f_\gamma(\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma-1}) \equiv f_\delta(\mathbf{q}_0, \ldots, \mathbf{q}_{n_\delta-1}) = \mathbf{q}(\theta_{\mathfrak{a}})$ *holds if and only if* (a) $p(\mathfrak{a}) = q(\mathfrak{a})$, or (b) $\mathbf{p}_0 \equiv \mathbf{q}_0(\theta_{\mathfrak{a}}), \ldots, \mathbf{p}_{n_\gamma-1} \equiv \mathbf{q}_{n_\gamma-1}(\theta_{\mathfrak{a}})$ *and* $\gamma = \delta$.

Proof. The "if" part is obvious, so we prove the "only if" part: By the definition of $\theta_{\mathfrak{a}}$ there is an $\mathbf{r} \in P^{(k)}(\tau)$ such that

$$\mathbf{p} = r(\hat{\mathbf{p}}_0, \ldots, \hat{\mathbf{p}}_{k-1}), \quad \mathbf{q} = r(\hat{\mathbf{q}}_0, \ldots, \hat{\mathbf{q}}_{k-1}), \quad \text{and} \quad \hat{\mathbf{p}}_i(\mathfrak{a}) = \hat{\mathbf{q}}_i(\mathfrak{a}), \qquad i = 0, \ldots, n_\gamma - 1.$$

Hence, either $\mathbf{r} = \mathbf{x}_i$ for some i, i.e., $\mathbf{p} = \hat{\mathbf{p}}_i$, $\mathbf{q} = \hat{\mathbf{q}}_i$, and $p(\mathfrak{a}) = q(\mathfrak{a})$ (i.e. (a)) or $r = f_\nu(\mathbf{r}_0, \ldots, \mathbf{r}_{n_\nu-1})$. In the latter case,

$$\mathbf{p} = f_\gamma(\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma-1}) = f_\nu(r_0(\hat{\mathbf{p}}_0, \ldots, \hat{\mathbf{p}}_{k-1}), \ldots, r_{n_\nu-1}(\hat{\mathbf{p}}_0, \ldots, \hat{\mathbf{p}}_{k-1}))$$

and

$$\mathbf{q} = f_\delta(\mathbf{q}_0, \ldots, \mathbf{q}_{n_\delta-1}) = f_\nu(r_0(\hat{\mathbf{q}}_0, \ldots, \hat{\mathbf{q}}_{k-1}), \ldots, r_{n_\nu-1}(\hat{\mathbf{q}}_0, \ldots, \hat{\mathbf{q}}_{k-1}));$$

thus $\gamma = \delta = \nu$ and $\mathbf{p}_i = r_i(\hat{\mathbf{p}}_0, \ldots, \hat{\mathbf{p}}_{k-1})$, $\mathbf{q}_i = r_i(\hat{\mathbf{q}}_0, \ldots, \hat{\mathbf{q}}_{k-1})$. Since $\hat{\mathbf{p}}_i \equiv \hat{\mathbf{q}}_i(\theta_{\mathfrak{a}})$ and the substitution property has already been proved, we conclude that $\mathbf{p}_i \equiv \mathbf{q}_i(\theta_{\mathfrak{a}})$, $0 \leqslant i < n_\gamma$. This completes the proof of the lemma.

(iii) We prove the transitivity of $\theta_{\mathfrak{a}}$ by induction on the maximum rank of the polynomial symbols involved (the rank of a polynomial symbol $\mathbf{p}$, $\mathrm{rk}(\mathbf{p})$, is the number of symbols needed in building it up). Assume that $\mathbf{q} \equiv \mathbf{p}(\theta_{\mathfrak{a}})$ and $\mathbf{p} \equiv \mathbf{r}(\theta_{\mathfrak{a}})$ and $\max\{\mathrm{rk}(\mathbf{q}), \mathrm{rk}(\mathbf{p}), \mathrm{rk}(\mathbf{r})\} = 2$, i.e., all polynomial symbols are of the form $\mathbf{x}_i$. Then $\mathbf{q} \equiv \mathbf{r}(\theta_{\mathfrak{a}})$ is obvious. Assume that $\max\{\mathrm{rk}(\mathbf{q}), \mathrm{rk}(\mathbf{p}), \mathrm{rk}(\mathbf{r})\} = n$ and that transitivity has been proved for $k < n$. It follows from the definition of $\theta_{\mathfrak{a}}$ that either all of $p(\mathfrak{a})$, $q(\mathfrak{a})$, $r(\mathfrak{a})$ exist or none. In the first case $\mathbf{q} \equiv \mathbf{r}(\theta_{\mathfrak{a}})$ is clear; in the second case Lemma 1 shows that

$$\begin{aligned} \mathbf{p} &= f_\gamma(\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma-1}), \\ \mathbf{q} &= f_\gamma(\mathbf{q}_0, \ldots, \mathbf{q}_{n_\gamma-1}), \\ \mathbf{r} &= f_\gamma(\mathbf{r}_0, \ldots, \mathbf{r}_{n_\gamma-1}). \end{aligned}$$

$\mathbf{p} \equiv \mathbf{q}(\theta_{\mathfrak{a}})$ and Lemma 1 imply that $\mathbf{q}_i \equiv \mathbf{p}_i(\theta_{\mathfrak{a}})$; $\mathbf{p} \equiv \mathbf{r}(\theta_{\mathfrak{a}})$ and Lemma 1 imply that $\mathbf{p}_i \equiv \mathbf{r}_i(\theta_{\mathfrak{a}})$. Since $\max\{\mathrm{rk}(\mathbf{q}_i), \mathrm{rk}(\mathbf{p}_i), \mathrm{rk}(\mathbf{r}_i)\} < n$, we conclude that $\mathbf{q}_i \equiv \mathbf{r}_i(\theta_{\mathfrak{a}})$, $i = 0, \ldots, n_\gamma - 1$, and hence, by the substitution property, that $\mathbf{q} \equiv \mathbf{r}(\theta_{\mathfrak{a}})$. This settles the transitivity. Thus, $\theta_{\mathfrak{a}}$ has been shown to be a congruence relation of $\mathfrak{P}^{(\alpha)}(\tau)$, concluding the proof of Theorem 1.

4. An embedding theorem for partial algebras. Let $\mathfrak{A}$ be a partial algebra of type τ, $\mathfrak{a} = \langle a_0, \ldots, a_\gamma, \ldots \rangle_{\gamma<\alpha}$, and assume that each element of A

*This lemma and the resulting proof of the transitivity of $\theta_{\mathfrak{a}}$ are due to G. H. Wenzel.

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occurs once and only once in this sequence. Then the following embedding theorem shows that $\mathfrak{A}$ can be considered to be a relative subalgebra of $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$.

THEOREM 2. *Let A^* denote the set of elements of the form $[\mathbf{x}_\gamma]\theta_{\mathfrak{a}}$, i.e., the set of congruence classes of the $\mathbf{x}_\gamma$ in $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$. Then*

$$\phi\colon a_\gamma \longrightarrow [\mathbf{x}_\gamma]\theta_{\mathfrak{a}}$$

is an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}^* = \langle A^*; F\rangle$.

Proof. $[\mathbf{x}_\gamma]\theta_{\mathfrak{a}} = [\mathbf{x}_\delta]\theta_{\mathfrak{a}}$ can hold only if $\gamma = \delta$, since neither $\mathbf{x}_\gamma$ nor $\mathbf{x}_\delta$ have non-trivial representations $\mathbf{x}_\gamma = r(\mathbf{p}_0, \ldots, \mathbf{p}_{k-1})$ or $\mathbf{x}_\delta = r(\mathbf{q}_0, \ldots, \mathbf{q}_{k-1})$. Thus, ϕ is 1-1. Since ϕ is obviously onto, we just have to verify that $f_\gamma(a_{\delta_0}, \ldots, a_{\delta_{n_\gamma-1}}) = a_\delta$ holds if and only if

$$f_\gamma([\mathbf{x}_{\delta_0}]\theta_{\mathfrak{a}}, \ldots, [\mathbf{x}_{\delta_{n_\gamma-1}}]\theta_{\mathfrak{a}}) = [\mathbf{x}_\delta]\theta_{\mathfrak{a}}.$$

Clearly, $f_\gamma(a_{\delta_0}, \ldots, a_{\delta_{n_\gamma-1}}) = a_\delta$ implies that

$$f_\gamma([\mathbf{x}_{\delta_0}]\theta_{\mathfrak{a}}, \ldots, [\mathbf{x}_{\delta_{n_\gamma-1}}]\theta_{\mathfrak{a}}) = [\mathbf{x}_\delta]\theta_{\mathfrak{a}}.$$

Observing that $\mathbf{x}_\delta$ admits only trivial representations (an argument which we used once already), we conclude the converse statement. This completes the proof of the theorem.

Theorem 2 yields the "least economical" embedding of the partial algebra $\mathfrak{A}$ into an algebra. More precisely, $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$ is the largest algebra into which $\mathfrak{A}$ can be embedded such that the image of $\mathfrak{A}$ is a generating system. (This was anticipated in **(2)**.) The next section is devoted to a description of the structure of the algebra $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$ as defined above.

5. The structure of $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$. Let $\mathfrak{A}$ be a partial algebra of type $\tau, \mathfrak{a} \in A^\alpha$, and assume that each element of A occurs once and only once in $\mathfrak{a}$. We define certain subsets $A_{(n,\gamma)}$ and $A'_{(n,\gamma)}$ $(0 \leqslant n < \omega,\ 0 \leqslant \gamma < o(\tau))$ of $\mathbf{P}^{(\alpha)}(\tau)$ as follows:

$$A'_{(0,0)} = A^*,$$

where A^* was defined in Theorem 2. Defining $(m, \gamma) < (n, \delta)$ by (i) $m = n$ and $\gamma < \delta$ or (ii) $m < n$ (lexicographic ordering), we define recursively

$$A'_{(n,\delta)} = \bigvee\ (A_{(m,\gamma)};\ (m, \gamma) < (n, \delta)) \qquad ((n, \delta) \neq (0, 0))$$

and

$$A_{(n,\delta)} = A'_{(n,\delta)} \vee \{f_\delta(b_0, \ldots, b_{n_\delta-1});\ b_0, \ldots, b_{n_\delta-1} \in A'_{(n,\delta)}\}.$$

LEMMA 2. *The following equality holds*:

$$\mathbf{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}} = \bigvee\ (A_{(n,\delta)};\ 0 \leqslant n < \omega,\ 0 \leqslant \delta < o(\tau)).$$

Proof. The inclusions

(i) $A_{(n,\gamma)} \subseteq A'_{(n,\delta)} \subseteq A_{(n,\delta)}$ if $\gamma < \delta$,

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(ii) $A_{(n,\gamma)} \subseteq A'_{(m,\delta)} \subseteq A_{(m,\delta)}$ if $n < m$,

follow immediately from the definitions. Take $\mathbf{p} \in \mathbf{P}^{(\alpha)}(\tau)$. We shall prove by induction on the rank of $\mathbf{p}$ that $[\mathbf{p}]\theta_{\mathfrak{a}} \in A_{(n,\delta)}$ for some $n < \omega$, $\delta < o(\tau)$. If $\mathbf{p} = \mathbf{x}_\gamma$, then $[\mathbf{p}]\theta_{\mathfrak{a}} \in A_{(0,0)}$ by definition. Let $\mathbf{p} = \mathbf{f}_\gamma(\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma-1})$ and assume that $[\mathbf{p}_i]\theta_{\mathfrak{a}} \in A_{(n_i,\delta_i)}$, $0 \leqslant i < n_\gamma$. Setting $n = \max\{n_0, \ldots, n_{n_\gamma-1}\}$, $\delta = \max\{\delta_0, \ldots, \delta_{n_\gamma-1}\}$, we get $A_{(n_i,\delta_i)} \subseteq A_{(n,\delta)} \subseteq A_{(n+1,0)}$ from (i) and (ii). Thus, $[\mathbf{p}]\theta_{\mathfrak{a}} = \mathbf{f}_\gamma([p_0]\theta_{\mathfrak{a}}, \ldots, [p_{n_\gamma-1}]\theta_{\mathfrak{a}}) \in A_{(n+1,0)}$, which was to be proved. One more definition is needed to describe the detailed structure of $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$.

Let $\mathfrak{B}$ be a partial algebra, $X \subseteq B$ and $Y = X \vee \{f_\gamma(x_0, \ldots, x_{n_\gamma-1});\ x_i \in X\}$ for some $f_\gamma \in F$. We shall write $Y = X[f_\gamma]$ if

(i) $f_\gamma(x_0, \ldots, x_{n_\gamma-1}) = f_\delta(x'_0, \ldots, x'_{n_\delta-1}) \notin X$ implies that $\gamma = \delta$, $x_i = x'_i$, $0 \leqslant i < n_\gamma$;

(ii) $x_0, \ldots, x_{n_\delta-1} \in Y$ and $x_i \notin X$ for some $0 \leqslant i < n_\delta$ implies that $f_\delta(x_0, \ldots, x_{n_\delta-1})$ does not exist in $\mathfrak{B}$ or is not in Y, for any $\delta < o(\tau)$.

Using this terminology we get the following result concerning the structure of $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$.

THEOREM 3. *$\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$ contains an isomorphic copy $\mathfrak{A}^*$ of the partial algebra $\mathfrak{A}$. If we start with A^* and we perform two kinds of constructions,*

(i) *taking the set union of previously constructed sets,*

(ii) *constructing $X[f_\gamma]$ from X,*

then we get an increasing transfinite sequence of subsets of $\mathbf{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$ such that the union of all these subsets is the whole set.

In the light of Lemma 2 and the preceding definitions, it suffices to prove the following lemma.

LEMMA 3. $A_{(n,\gamma)} = A'_{(n,\gamma)}[f_\gamma]$.

Proof. Lemma 1 immediately yields part (i) in the definition of $A'_{(n,\gamma)}[f_\gamma]$. Moreover, the same lemma yields that $a_0, \ldots, a_{n_\delta-1} \in A_{(n,\gamma)}$ and, say, $a_i \notin A'_{(n,\gamma)}$, and $f_\delta(a_0, \ldots, a_{n_\delta-1}) \in A_{(n,\gamma)} - A'_{(n,\gamma)}$ is impossible. Thus, we assume that $a_0, \ldots, a_{n_\delta-1} \in A_{(n,\gamma)}$, $a_j \notin A'_{(n,\gamma)}$ and $f_\delta(a_0, \ldots, a_{n_\delta-1}) \in A'_{(n,\gamma)}$. Setting $a_i = [\mathbf{p}_i]\theta_{\mathfrak{a}}$, $f_\delta(a_0, \ldots, a_{n_\delta-1}) = [\mathbf{p}]\theta_{\mathfrak{a}}$, we get $f_\delta(\mathbf{p}_0, \ldots, \mathbf{p}_{n_\delta-1}) \equiv \mathbf{p}(\theta_{\mathfrak{a}})$ and hence, by Lemma 1, $\mathbf{p} = f_\delta(\mathbf{p}'_0, \ldots, \mathbf{p}'_{n_\delta-1})$ and

$$\mathbf{p}_i \equiv \mathbf{p}'_i(\theta_{\mathfrak{a}}),\ 0 \leqslant i < n_\delta - 1.$$

Since $[\mathbf{p}]\theta_{\mathfrak{a}} \in A'_{(n,\gamma)}$, there is a smallest $(m, \lambda) < (n, \gamma)$ such that $[\mathbf{p}]\theta_{\mathfrak{a}} \in A_{(m,\lambda)}$. Since $[\mathbf{p}]\theta_{\mathfrak{a}} \notin A'_{(0,0)}$ by Theorem 2, $[\mathbf{p}]\theta_{\mathfrak{a}} \in A_{(m,\lambda)} - A'_{(m,\lambda)}$ and so therefore $\mathbf{p} \equiv f_\lambda(\mathbf{q}_0, \ldots, \mathbf{q}_{n_\lambda-1})(\theta_{\mathfrak{a}})$, for some $[\mathbf{q}_i]\theta_{\mathfrak{a}} \in A'_{(m,\lambda)}$. Lemma 1 shows that $\lambda = \delta$ and $[\mathbf{q}_i]\theta_{\mathfrak{a}} = [\mathbf{p}_i]\theta_{\mathfrak{a}}$. Hence, $a_j = [\mathbf{p}_j]\theta_{\mathfrak{a}} \in A'_{(m,\delta)} \subseteq A'_{(n,\gamma)}$, a contradiction. This completes the proof of Lemma 3 and also of Theorem 3.

6. Free algebras generated by partial algebras. The congruence relation $\theta_{\mathfrak{a}}$ for algebras is used, among other things, to describe the free algebra over a class of algebras $K \subseteq K(\tau)$. In this section we shall show that $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$ can

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be given a similar interpretation if we find a suitable generalization of the concept of free algebras.

Let $\mathfrak{K}$ be a class of algebras of type τ and let $\mathfrak{A} = \langle A; F\rangle$ be a partial algebra of type τ. The algebra $\mathfrak{F}_K(\mathfrak{A})$ is called the *algebra freely generated by the partial algebra* $\mathfrak{A}$ if the following conditions are satisfied:

(i) $\mathfrak{F}_K(\mathfrak{A}) \in K$.

((i) $\mathfrak{F}_K(\mathfrak{A})$ is generated by A' and $\chi: A' \to A$ is an isomorphism between $\mathfrak{A} = \langle A; F\rangle$ and $\mathfrak{A}' = \langle A'; F\rangle$ which is a relative subalgebra of $\mathfrak{F}_K(\mathfrak{A})$.

(iii) If ϕ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{C} \in K$, then there exists a homomorphism ψ of $\mathfrak{F}_K(\mathfrak{A})$ into $\mathfrak{C}$ such that ψ is an extension of ϕ.

Using this definition, the following theorem is clear.

THEOREM 4. (i) *If $\mathfrak{A}$ is an algebra in K, then $\mathfrak{F}_K(\mathfrak{A}) \cong \mathfrak{A}$.*

(ii) *$\mathfrak{F}_K(\mathfrak{A})$ is unique up to isomorphism.*

(iii) *If the domain of each $f_\gamma \in F$ is empty, then $\mathfrak{F}_K(\mathfrak{A}) \cong \mathfrak{F}_K(\mathfrak{m})$ if $\mathfrak{F}_K(\mathfrak{m})$ is the free algebra on $\mathfrak{m}$ generators and $\mathfrak{m} = |A|$.*

We shall conclude this paper by giving sufficient conditions on a class K for the existence of $\mathfrak{F}_K(\mathfrak{A})$. Theorem 5 is based on an idea of G. Birkhoff **(1)**.

THEOREM 5. *Let K be a class of algebras and let $\mathfrak{A}$ be a partial algebra. Assume that the following conditions hold:*

(i) *$\mathfrak{A}$ is isomorphic to a weak subalgebra of an algebra in K.*

(ii) *K is closed under the formation of subalgebras and direct products.*

Then $\mathfrak{F}_K(\mathfrak{A})$ exists.

Proof. By obvious changes in a proof of **(1)**.

Theorem 6 constructs $\mathfrak{F}_K(\mathfrak{A})$ from $\mathfrak{F}_K(\mathfrak{m})$.

THEOREM 6. *Let K be a class of algebras and let $\mathfrak{A}$ be a partial algebra. $\mathfrak{F}_K(\mathfrak{A})$ exists if the following conditions are satisfied:*

(i) *$\mathfrak{A}$ is isomorphic to a relative subalgebra of an algebra in K.*

(ii) *$\mathfrak{F}_K(\mathfrak{m})$ exists for some $\mathfrak{m} \geqslant |A|$.*

(iii) *K is closed under the formation of homomorphic images.*

Remark. This is analogous to a result of Sikorski **(3)** on free products of algebras.

Proof. (ii) and (iii) imply that $\mathfrak{F}_K(\mathfrak{m})$ exists for $\mathfrak{m} = |A|$. Let α be an ordinal with $\bar{\alpha} = \mathfrak{m}$ and let $A = \{a_\gamma;\ \gamma < \alpha\}$. We define a subset T of $(F_K(\alpha))^2 = (F_K(\mathfrak{m}))^2$ as follows:

$$\langle x, y\rangle \in T \text{ if and only if } x = p(x_{i_0}, \ldots, x_{i_{n_\gamma - 1}}),\ y = q(x_{j_0}, \ldots, x_{j_{n_\delta - 1}})$$

and $p(a_{i_0}, \ldots, a_{i_{n_\gamma-1}}) = q(a_{j_0}, \ldots, a_{j_{n_\delta-1}})$. We set $\theta = \bigcup (\theta_{xy};\ \langle x, y\rangle \in T)$ which, by definition, is the smallest congruence relation under which $\langle x, y\rangle \in T$ implies that $x \equiv y(\theta)$. (ii) and (iii) imply that $\mathfrak{F}_K(\alpha)/\theta \in K$, and we claim that $\mathfrak{F}_K(\alpha)/\theta \cong \mathfrak{F}_K(\mathfrak{A})$. Let $a'_\gamma = [x_\gamma]\theta$ and $A' = \{a'_\gamma;\ \gamma < \alpha\}$.

Take any homomorphism ϕ of $\mathfrak{A}$ into $\mathfrak{B} \in K$ with $a_\gamma\phi = c_\gamma$, $\gamma < \alpha$, and define T' in terms of the c_γ as T was defined in terms of the a_j. If Φ is the corresponding congruence relation of $\mathfrak{F}_K(\alpha)$, then $\theta \leqslant \Phi$ since ϕ is a homomorphism. Thus, by the second isomorphism theorem, ψ: $[x_\gamma]\theta \to c_\gamma$ induces a homomorphism ψ of $\mathfrak{F}_K(\alpha)/\theta$ into $\mathfrak{C}$ with $a'_\gamma\psi = c_\gamma$ since $\mathfrak{F}_K(\alpha)/\theta \cong \mathfrak{C}$. Now we embed $\mathfrak{A}$ into the algebra $\mathfrak{B} \in K$ (which can be done by (i)) and let i: $a_\gamma \to a_\gamma$ be a mapping of $\mathfrak{A}$ into $\mathfrak{B}$. Applying the above remark, we get a homomorphism χ: $\mathfrak{F}_K(\alpha)/\theta \to \mathfrak{B}$ with $a'_\gamma\chi = a_\gamma$. Thus, $\chi_{A'}$: $\mathfrak{A}' \to \mathfrak{A}$ is an onto homomorphism while it is trivial by the construction of θ that $\chi_A{}^{-1}$ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$. Thus, χ is an isomorphism and $\chi\psi = \phi$. This completes the proof.

We conclude this section with two corollaries:

COROLLARY 1. *If K is an equational class, then* (i) *is necessary and sufficient for the existence of* $\mathfrak{F}_K(\mathfrak{A})$.

COROLLARY 2. *Let $K = K(\tau)$. Then $\mathfrak{F}_K(\mathfrak{A})$ always exists and*

$$\mathfrak{F}_K(\mathfrak{A}) \cong \mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}},$$

where $A = \{a_\gamma;\ \gamma < \alpha\}$ and $\mathfrak{a} = \langle a_0, \ldots, a_\gamma, \ldots\rangle_{\gamma<\alpha}$ contains each element of A exactly once.

The last corollary, the proof of which is obvious ($\mathfrak{F}_{K(\tau)}(\alpha) = \mathfrak{P}^{(\alpha)}(\tau)$ and θ as constructed in Theorem 6 equals $\theta_{\mathfrak{a}}$), yields the desired representation of $\mathfrak{F}_{K(\tau)}(\mathfrak{A})$ as the factor algebra $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\mathfrak{a}}$.

Remark (added May 15, 1967). In a forthcoming paper, P. Burmeister and J. Schmidt give a result related to Theorem 3 of the present paper. Namely, they prove the existence of an algebra satisfying a set of axioms which can be easily shown to be equivalent to the conditions of Theorem 3. It should be pointed out, however, that the mere existence of an algebra answering the description of Theorem 3 has already been proved; see for instance **(2)**. The purpose of this paper is the construction of the "kernel" $\theta_{\mathfrak{a}}$.

Also, it should be mentioned that whenever the free algebra generated by a partial algebra exists, its existence can be proved using the adjoint functor theorem.

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The Pennsylvania State University,
University Park, Pennsylvania;
The University of Manitoba,
Winnipeg, Manitoba