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AN UNBOUNDED OPERATOR WITH SPECTRUM IN A STRIP AND MATRIX DIFFERENTIAL OPERATORS

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Abstract

Let A and $\tilde{A}$ be unbounded linear operators on a Hilbert space. We consider the following problem. Let the spectrum of A lie in some horizontal strip. In which strip does the spectrum of $\tilde{A}$ lie, if A and $\tilde{A}$ are sufficiently 'close'? We derive a sharp bound for the strip containing the spectrum of $\tilde{A}$, assuming that $\tilde{A} - A$ is a bounded operator and A has a bounded Hermitian component. We also discuss applications of our results to regular matrix differential operators.

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1. Introduction and statement of the main result

Let $\mathcal{H}$ be a complex separable Hilbert space with a scalar product $(\cdot,\cdot)$, norm given by $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ and unit operator I. By $\mathcal{L}(\mathcal{H})$ we denote the set of all bounded operators in $\mathcal{H}$. For an operator A on $\mathcal{H}$, $D(A)$ is its domain, A^* and A^{-1} are the adjoint and inverse operators, respectively, $\sigma(A)$ is the spectrum, $R_z(A) = (A - zI)^{-1}$ $(z \notin \sigma(A))$ is the resolvent, and $\lambda_j(A)$ $(j = 1, 2, \ldots)$ denote the eigenvalues of A taken with their multiplicities. In addition, for $\omega > 0$, we denote by

$$H_\omega := \{z \in \mathbb{C} : |\operatorname{Im} z| < \omega\}$$

the horizontal strip of height 2ω which is symmetric with respect to the real axis. Following [10, Section 4.1], we will say that an operator A on $\mathcal{H}$ is *a strip-type operator of height* ω (in short, $A \in \operatorname{Strip}(\omega)$) if $\sigma(A) \subset H_\omega$ and $\sup_{|\operatorname{Im} z| \geq \omega'} \|R_z(A)\| < \infty$ for all $\omega' > \omega$. Finally,

$$\omega_{\rm st}(A) := \inf\{\omega \geq 0 : A \in \operatorname{Strip}(\omega)\}$$

is called *the spectral height of* A.

We consider the following problem. Let A and $\tilde{A}$ be strip-type operators on $\mathcal{H}$. In which strip does the spectrum of $\tilde{A}$ lie if $\omega_{\rm st}(A)$ is known and $\tilde{A}$ and A are sufficiently 'close'? We also discuss applications of our results to matrix differential operators.

The strip-type operators form a wide class of unbounded operators in a Banach space. The important example here is the logarithm of a sectorial operator, arising in various applications (see [10, 16]). The natural functional calculus for strip-type operators appears first in [2]. It is discussed in [11] in a general setting and used in [3]. The theory of strip-type operators is developed in [9, 16, 17] and the references given therein. For more details, see [10, Ch. 4]. To the best of our knowledge, the above-mentioned problem has not been considered in the literature, although it is important for the localisation of spectra and in various applications.

Furthermore, A is said to be a strong strip-type operator of height ω, if for any $\omega' > \omega$ there is an $L_{\omega'}$ such that

$$\|R_z(A)\| \le \frac{L_{\omega'}}{|\mathrm{Im}\, z| - \omega'} \quad \text{for } |\mathrm{Im}\, z| > \omega'.$$

From [10, Example 4.1.1.2, page 92], if iA generates a C_0-group e^{iAt} in a Hilbert space, then A is a strong strip-type operator of height $\theta(e^{iAt})$, where $\theta(e^{iAt})$ is the group type of e^{iAt}. In particular,

$$\omega_{\mathrm{st}}(A) = \theta(e^{iAt}). \tag{1.1}$$

Throughout the paper it is assumed that $D(A)$ is dense in $\mathcal{H}$, $A = A_R + iA_I$, where A_R and A_I are self-adjoint operators, and

$$A_I \in \mathcal{L}(\mathcal{H}). \tag{1.2}$$

According to the Stone theorem (see [10, Section 4.1]), the operator iA_R generates a C_0-group e^{itA_R} $(-\infty < t < \infty)$ of unitary operators. In particular, for $t \ge 0$ it is a semigroup. Moreover, by [5, Theorem II.4.6], iA_R generates a bounded analytic semigroup. Hence, by [5, Proposition III.1.12], iA generates a bounded analytic semigroup, since A_I is bounded. Thus, under condition (1.2), A is a strip-type operator and therefore (1.1) holds.

Let

$$D(\tilde{A}) = D(A) \quad \text{and} \quad q := \|A - \tilde{A}\| < \infty. \tag{1.3}$$

Then $\|\tilde{A}_I\| \le q + \|A_I\|$ and therefore $\tilde{A}$ is also a strip-type operator.

We introduce the notation $x(t) = e^{itA}x_0$ $(x_0 \in D(A))$, $\alpha(A_I) = \sup\, \sigma(A_I)$ and $\beta(A_I) = \inf\, \sigma(A_I)$. Then

$$\frac{d}{dt}(x(t), x(t)) = 2\mathrm{Re}\,(iAx(t), x(t)) = -2(A_I x, x) \le -2\beta(A_I) \le 2\|A_I\|\,\|x(t)\|^2$$

and

$$\frac{d}{dt}(x(t), x(t)) = -2(A_I x, x) \ge -2\alpha(A_I)(A_I x, x).$$

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Consequently, $\|e^{iAt}x_0\| \le \|x_0\|e^{\|A_I\|t}$ for $t \ge 0$). Thus, from (1.1), $\omega_{\rm st}(A) \le \|A_I\|$. Similarly,

$$\omega_{\rm st}(\tilde{A}) \le \|\tilde{A}_I\|. \tag{1.4}$$

This inequality is rather rough. Below, we present a considerably sharper estimate.

To this end, note that according to (1.1), $\|e^{\pm iAt}\| \le \text{const.}\, e^{\omega_{\rm st}t}$ $(t \ge 0)$, and thus the operators $-(cI \pm iA)$, for $c \in \mathbb{R}$, generate the exponentially stable semigroups $e^{-(cI\pm iA)t}$, provided $c > \omega_{\rm st}$. Hence, the integral

$$X_c := \int_0^\infty e^{-(iA+cI)^*t}e^{-(iA+cI)t}\,dt \quad (c > \omega_{\rm st}) \tag{1.5}$$

strongly converges and

$$\|X_c\| \le \int_0^\infty e^{-2ct}\|e^{-iAt}\|^2\,dt.$$

We are now in a position to formulate our main result, which we prove in Section 2.

THEOREM 1.1. *Let conditions (1.2) and (1.3) hold. Let X_c be defined by (1.5) for some $c > \omega_{\rm st}$. Then $\omega_{\rm st}(\tilde{A}) < c$, provided $q\|X_c\| < 1/2$.*

Now put

$$w_c(A) := \frac{1}{2\pi}\int_{-\infty}^\infty \|(iA + (is + c)I)^{-1}\|^2\,ds.$$

By the classical Parseval–Plancherel equality [1, Theorem 5.2.1], for any $x \in \mathcal{H}$,

$$\begin{aligned}(X_cx, x) = \Big(\int_0^\infty e^{-(Ic+iA)^*t}e^{-(Ic+iA)t}x\,dt, x\Big) &= \int_0^\infty \|e^{-(Ai+Ic)t}x\|^2\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \|(iA + (is + c)I)^{-1}x\|^2\,ds.\end{aligned}$$

Hence,

$$\|X_c\| \le w_c(A). \tag{1.6}$$

If A is normal, that is, $AA^* = A^*A$, then by the spectral representation (see, for instance, [12]), we easily see that $\|e^{iAt}\| = e^{-t\beta(A)}$, where $\beta(A) := \inf \operatorname{Im}\ \sigma(A)$ and $t \ge 0$. But $\beta(A) \ge -\omega_{\rm st}(A)$. Therefore,

$$\|X_c\| \le \int_0^\infty e^{-2(c+\beta(A))}\,dt = \frac{1}{2(c+\beta(A))} = \frac{1}{2(c-\omega_{\rm st}(A))} \quad (c > \omega_{\rm st}(A)).$$

Making use of Theorem 1.1, we obtain $\omega_{\rm st}(\tilde{A}) \le \omega_{\rm st}(A) + q + \epsilon$ for $\epsilon > 0$. Hence, letting $\epsilon \to 0$, we arrive at the following result.

COROLLARY 1.2. *Let conditions (1.2) and (1.3) hold and let A be normal. Then $\omega_{\rm st}(\tilde{A}) \le \omega_{\rm st}(A) + q$. In particular, if A is self-adjoint, then $\omega_{\rm st}(\tilde{A}) \le q$.*

Let us show that Theorem 1.1 is sharp. To this end, assume that $K \in \mathcal{L}(\mathcal{H})$ and A are self-adjoint commuting operators and $\tilde{A} = A + iK$. Suppose also that $\sigma(A)$ and $\sigma(K)$ are discrete. Then $\sigma(\tilde{A})$ consists of the eigenvalues

$$\lambda_{jk}(\tilde{A}) = \lambda_j(A) + i\lambda_k(K) \quad (j, k = 1, 2, \ldots).$$

Hence, $\omega_{\mathrm{st}}(\tilde{A}) = \sup_k |\lambda_k(K)| = q$, since $q = \|\tilde{A} - A\| = \|K\| = \sup_k |\lambda_k(K)|$. But due to Corollary 1.2, $\omega_{\mathrm{st}}(\tilde{A}) \leq q$, since $\omega_{\mathrm{st}}(A) = 0$. So the bound in Theorem 1.1 is attained in this case.

2. Proof of Theorem 1.1

We need the following well-known theorem (see [4, Theorem 5.1.3, page 217]).

THEOREM 2.1. *Suppose that B is the infinitesimal generator of the C_0-semigroup $T(t)$ on a Hilbert space $\mathcal{H}$. Then $T(t)$ is exponentially stable if and only if there exists a bounded positive definite operator P such that*

$$(Bz, Pz) + (Pz, Bz) = -(z, z) \quad (z \in D(B)). \tag{2.1}$$

Moreover, if B is the infinitesimal generator of an exponentially stable C_0-semigroup then from [4, Section 5.5.3a, Equation (5.62)], for any $Q \in \mathcal{L}(\mathcal{H})$ the equation

$$(Bz_1, Pz_2) + (Pz_1, Bz_2) = -(z_1, Qz_2) \tag{2.2}$$

has a solution $P \in \mathcal{L}(\mathcal{H})$ which, again by [4, Section to 5.5.3a], is representable as

$$P = \int_0^\infty e^{B^* t} Q e^{Bt}\, dt. \tag{2.3}$$

For a self-adjoint operator S we write $S > 0$ ($S < 0$), if S is positive (negative) definite. Let $D(B) = D(B^*)$ and $B^*P + PB = -C^2$ (with $C > 0$) on $D(B)$ for some positive definite $P \in \mathcal{L}(\mathcal{H})$. Then

$$C^{-1}B^*PC^{-1} + C^{-1}PBC^{-1} = C^{-1}B^*CC^{-1}PC^{-1} + C^{-1}PC^{-1}CBC^{-1} = -I.$$

That is, $M^*Y + YM = -I$, where $M = CBC^{-1}$ and $Y = C^{-1}PC^{-1}$.

According to Theorem 2.1, M generates an exponentially stable semigroup. Since M and B are similar, we arrive at the following result.

COROLLARY 2.2. *Let $D(B) = D(B^*)$ and $B^*P + PB < 0$ on $D(B)$ for some positive definite $P \in \mathcal{L}(\mathcal{H})$. Then* $\sup \operatorname{Re} \sigma(B) < 0$.

PROOF OF THEOREM 1.1. From (2.3),

$$(cI + iA)^* X_c + X_c(cI + iA) = I. \tag{2.4}$$

Put $E = \tilde{A} - A$. Then from (2.4),

$$\begin{aligned}(i\tilde{A} + cI)^* X_c + X_c(i\tilde{A} + cI) &= (iA + cI)^* X_c + X_c(iA + cI) - iE^* X_c + iX_c E \\ &= I - iE^* X_c + iX_c E.\end{aligned}$$

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If $2q\|X_c\| < 1$, then $(i\tilde{A} + cI)^* X_c + X_c(i\tilde{A} + cI) > 0$. By Corollary 2.2, it follows that $\sup \mathrm{Re}\ \sigma(-i\tilde{A} - cI) < 0$. So $-c - \mathrm{Re}\,(ix - y) = -c + y < 0$ for any $x + iy \in \sigma(\tilde{A})$. Thus $\sup \mathrm{Im}\ \sigma(\tilde{A}) < c$. Replacing $\tilde{A}$ by $-\tilde{A}$ and proceeding in the same way, we find $-c + \mathrm{Re}\,(ix - y) = -c - y < 0$. Thus $\inf \mathrm{Im}\ \sigma(\tilde{A}) > -c$. This proves the theorem. □

3. Spectral strips of differential operators with matrix coefficients

Let $L^2 = L^2([0, 1], \mathbb{C}^n)$ be the space of functions defined on $[0, 1]$ with values in $\mathbb{C}^n$ and the scalar product

$$(f, h)_{L^2} = \int_0^1 (f(x), h(x))_n\, dx \quad (f, h \in L^2),$$

where $(\cdot, \cdot)_n$ means the scalar product in $\mathbb{C}^n$. On the domain

$$D(A) = \{u \in L^2 : u'' \in L^2 \text{ and } u(0) = u(1) = 0\},$$

consider the operator

$$\tilde{A} = -\frac{d^2}{dx^2} + C(x) \quad (x \in (0, 1)), \tag{3.1}$$

where $C(x)$ is an $n \times n$ matrix continuously dependent on x. We consider this operator as a perturbation of the operator

$$A = -\frac{d^2}{dx^2} + C_0 \quad (x \in (0, 1)) \tag{3.2}$$

with a constant $n \times n$ matrix C_0. By way of example, one can take $C_0 = C(0)$ or $C_0 = \int_0^1 C(x)\, dx$.

Clearly,

$$(A_I f)(x) = C_{0I} f(x) \quad (f \in L^2, x \in [0, 1], C_{0I} = (C_0 - C_0^*)/2i)$$

and

$$q = \|A - \tilde{A}\|_{L^2} \le \sup_x \|C(x) - C_0\|_n.$$

Here $\|A - \tilde{A}\|_{L^2}$ is the operator norm in L^2 of $A - \tilde{A}$ and $\|\cdot\|_n$ means the spectral matrix norm (the operator norm with respect to the Euclidean vector norm).

Take into account that the operator S defined on $D(A)$ by $S := -d^2/dx^2$ commutes with constant matrices. Since the eigenvalues of S are $\pi^2 k^2$ $(k = 1, 2, \ldots)$, by simple calculations we can show that $\sigma(A)$ consists of the eigenvalues $\lambda_{jk}(A) = \pi^2 k^2 + \lambda_j(C_0)$ $(k = 1, 2, \ldots, j = 1, \ldots, n)$, where $\lambda_j(C_0)$ are the eigenvalues of C_0 taken with their multiplicities. Thus,

$$\omega_{\mathrm{st}}(A) = \omega_{\mathrm{st}}(C_0) := \max_j |\mathrm{Im}\ \lambda_j(C_0)|.$$

Since S and C_0 commute, we have $e^{iAt} = e^{iC_0 t} e^{iSt}$. Hence, taking into account that $S = S^*$ and therefore $\|e^{iSt}\| = 1$, we can write $\|e^{iAt}\|_{L^2} \le \|e^{iC_0 t}\|_n$ and

$$\|X_c\|_{L^2} \le \int_0^\infty e^{-2ct} \|e^{-iC_0 t}\|_n^2 \, dt. \tag{3.3}$$

To estimate $\|e^{iC_0 t}\|_n$, for an $n \times n$ matrix M, introduce the quantity $g(M)$ which measures the departure from normality:

$$g(M) := \Big[N_2^2(M) - \sum_{k=1}^n |\lambda_k(M)|^2 \Big]^{1/2},$$

where $N_2(M) := (\text{trace } (M^* M))^{1/2}$ is the Hilbert–Schmidt (Frobenius) norm of M and $\lambda_k(M)$ $(k = 1, \dots, n)$ are the eigenvalues of M taken with their multiplicities.

Various properties of $g(M)$ can be found in [8, Section 3.1]. In particular,

$$g^2(M) \le N_2^2(M) - |\text{trace } M^2|$$

and

$$g^2(M) \le 2N_2^2(M_I) \quad (\text{where } M_I = (M - M^*)/2i).$$

In addition, $g(zM) = |z| g(M)$ for $z \in \mathbb{C}$. If M is a normal matrix, that is, $MM^* = M^* M$, then $g(M) = 0$. By [8, Theorem 3.5], for any $n \times n$ matrix M,

$$\|e^{Mt}\| \le \exp[\alpha(M) t] \sum_{k=0}^{n-1} \frac{g^k(M) t^k}{(k!)^{3/2}} \quad (\alpha(M) = \max_k \operatorname{Re} \lambda_k(M), t \ge 0).$$

But $\alpha(iC_0) \le \omega_{\text{st}}(C_0)$ and $g(iC_0) = g(C_0)$. Thus,

$$\|e^{iC_0 t}\| \le \exp[\omega_{\text{st}}(C_0) t] \sum_{k=0}^{n-1} \frac{g^k(C_0) t^k}{(k!)^{3/2}} \quad (t \ge 0)$$

and from (3.3),

$$\begin{aligned} \|X_c\|_{L^2} &\le \int_0^\infty \exp[-2(c - \omega_{\text{st}}(C_0)) t] \Big(\sum_{k=0}^{n-1} \frac{g^k(C_0) t^k}{(k!)^{3/2}} \Big)^2 dt \\ &= \int_0^\infty \exp[-2(c - \omega_{\text{st}}(C_0)) t] \sum_{j,k=0}^{n-1} \frac{g^{j+k}(C_0) t^{k+j}}{(j!\, k!)^{3/2}} \, dt \quad (c > \omega_{\text{st}}(C_0)). \end{aligned}$$

Since

$$\int_0^\infty \exp[-st] t^k \, dt = \frac{k!}{s^{k+1}} \quad (s > 0),$$

we find $\|X_c\| \le \frac{1}{2} \zeta(c - \omega_{\text{st}}(C_0))$, where

$$\zeta(s) = \sum_{j,k=0}^{n-1} \frac{(j+k)!\, g^{j+k}(C_0)}{2^{j+k} s^{k+j+1} (j!\, k!)^{3/2}} \quad (s > 0).$$

Now Theorem 1.1 implies the following result.

COROLLARY 3.1. *Let $\tilde{A}$ be defined by (3.1) and, for some $c > \omega_{\rm st}(C_0)$, let the condition*

$$q\zeta(c - \omega_{\rm st}(C_0)) < 1$$

hold. Then $\omega_{\rm st}(\tilde{A}) < c$.

Let x_n be the unique nonnegative root of the equation

$$q\zeta(y) = q \sum_{j,k=0}^{n-1} \frac{(j+k)!\, g^{j+k}(C_0)}{2^{j+k} y^{k+j+1} (j!\, k!\,)^{3/2}} = 1 \quad (y > 0), \tag{3.4}$$

which is equivalent to the equation

$$y^{2n} = q \sum_{j,k=0}^{n-1} \frac{(j+k)!\, g^{j+k}(C_0)}{2^{j+k} (j!\, k!\,)^{3/2}} y^{2n-k-j-1} = 1. \tag{3.5}$$

If $y > x_n + \omega_{\rm st}(C_0)$, then $q\zeta(y) < q\zeta(x_n) = 1$. Now Corollary 3.1 implies $\omega_{\rm st}(\tilde{A}) < y$. Letting $y \to x_n + \omega_{st}(C_0)$, we obtain the following result.

COROLLARY 3.2. *Let $\tilde{A}$ be defined by (3.1). Then $\omega_{\rm st}(\tilde{A}) \le \omega_{\rm st}(C_0) + x_n$.*

If C_0 is normal, then $g(C_0) = 0$, and with $0^0 = 1$ we have $\zeta(s) = 1/s$ and thus $x_n = q$. The following lemma gives us an estimate for x_n in the case $g(C_0) \neq 0$.

LEMMA 3.3. *Let $q\zeta(1) \le 1$. Then*

$$x_n \le \sqrt[2n]{q\zeta(1)}.$$

PROOF. By (3.4), $q\zeta(x_n) = 1 \ge q\zeta(1)$. Since $\zeta(s)$ is monotonically decreasing, it follows that $x_n \le 1$. Now (3.5) proves the lemma. □

Corollary 3.2 and the Lemma 3.3 yield the following result.

COROLLARY 3.4. *Let $\tilde{A}$ be defined by (3.1) and $q\zeta(1) \le 1$. Then*

$$\omega_{\rm st}(\tilde{A}) \le \omega_{\rm st}(C_0) + \sqrt[2n]{q\zeta(1)}.$$

For recent results on the spectra of differential operators see, for instance, the works [6, 7, 13, 14, 15, 18, 19] and the references which are given therein.

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