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Some problems on idempotent measures on semigroups

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Essentially this paper does the following: In Section 2 it gives necessary and sufficient conditions in order that the support of an idempotent measure on a locally compact semigroup S, be completely simple. In Section 3 it proves that if I is an ideal of S of positive measure μ (= any probability measure), then $\mu^n(I)$ strictly increases to the limit 1. If in addition μ is idempotent, then $\mu(N^{-1}N)$ and $\mu(NN^{-1})$ are positive for any open set N. In Section 4 certain compactness conditions are proven equivalent to joint weak*-continuity of the convolution of bounded measures and a limit theorem concerning the convolution powers (Cesarò sums) of μ is proven.

1. Introduction

In what follows, S is a locally compact Hausdorff topological semigroup, $\mathcal{B}$ its Borel σ-algebra generated by all the open subsets of S and μ is a (Borel) regular probability measure on S with support

$$F \equiv \{s \in S \ ; \ \text{for every open } V \ni s \ , \ \mu(V) > 0\} \ .$$

The closed subsemigroup generated by F will be denoted by $D \equiv \overline{\bigcup_n F^n}$.

For $A, B \subset S$, $x \in S$,

$AB^{-1} \equiv \{s \in S \ ; \ \text{there is } b \in B \text{ such that } sb \in A\}$ and

$Ax^{-1} \equiv \{s \in S \ ; \ sx \in A\}$. Analogously one defines $A^{-1}B$ and $x^{-1}A$.

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We denote by $\mu_x \equiv \mu\left(\cdot\, x^{-1}\right)$ the measure $\mu_x(B) \equiv \mu\left(Bx^{-1}\right)$, for $B \in \mathcal{B}$. (Similarly for ${}_x\mu(B) \equiv \mu\left(x^{-1}B\right)$.) μ_x is also a regular probability measure and the function of x, $\mu\left(Bx^{-1}\right)$, for fixed $B \in \mathcal{B}$, is (Borel) measurable. In fact, for fixed open $V \subset S$,

$$V_\alpha \equiv \left\{s \in S \; ; \; \mu\left(Vs^{-1}\right) > \alpha\right\}$$

is open so that the function $\mu\left(Vx^{-1}\right)$ is lower semicontinuous [5, p. 179]. Let $C_c(S)$, $C_\infty(S)$ be the space of all real valued continuous functions on S which have compact supports and vanish at ∞ respectively. If μ, ν are regular probability measures their convolution $\mu*\nu$ is defined as the regular probability measure on S generated by the linear functional on $C_c(S)$

$$L(f) \int_S \left[\int_S f(xy)\mu(dx)\right]\nu(dy) = \int_S \left[\int_S f(xy)\nu(dy)\right]\mu(dx)$$

for $f \in C_c(S)$. [13, p. 19]. μ is called idempotent if $\mu*\mu = \mu$. It can be shown that for $B \in \mathcal{B}$,

$$(1) \qquad \mu*\nu(B) = \int_S \mu\left(Bx^{-1}\right)\nu(dx) = \int_S \mu(dx)\nu\left(x^{-1}B\right) .$$

(See [5, p. 179] for the proof of (1); their proof applies to the locally compact case.) If μ is idempotent, $\overline{FF} = F$ and hence F is a (closed) subsemigroup. [7, p. 686].

It has been conjectured [9] that if μ is idempotent, then F is a completely simple subsemigroup. If F is completely simple, then for any $e \in E(F) \equiv$ the set of all idempotents of F, $X \equiv E(Fe)$ and $Y \equiv E(eF)$ are (closed) left-zero and right-zero semigroups in F respectively; $G \equiv eFe$ is a closed subgroup and $YX \subset G$; the product space $X \times G \times Y$ (product topology) is a locally compact semigroup and is isomorphic (both algebraically and topologically) to F. Multiplication on $X \times G \times Y$ is defined by

$$(2) \qquad (x_1, g_1, y_1)(x_2, g_2, y_2) = \left(x_1, g_1(y_1x_2)g_2, y_2\right) .$$

The isomorphism $\eta : X \times G \times Y \to F$ is given by

$$\eta(x, g, y) = xgy$$

$$\eta^{-1}(s) = \left(s(ese)^{-1}, ese, (ese)^{-1}s\right) , \quad s \in F .$$

[2, p. 49, 61, 62].

The above conjecture is important because if it is true, then μ decomposes on $X \times G \times Y$ as a product measure $\mu = \mu_X \times \mu_G \times \mu_Y$, where μ_X, μ_Y are regular probability measures on X and Y respectively and μ_G is the normed Haar measure on G which turns out to be a compact group. (See the proof of the factorization of μ in [5, p. 183]; the argument applies to the locally compact case; a minor correction of the proof is given in [9].) Consider the following two compactness conditions (in increasing order of strength):

(K) there is compact C_0, $\mu(C_0) > 0$ such that $\emptyset \neq C_0C_0^{-1}$ is compact;

(L) AB^{-1} is compact for every pair of compact $A, B \subset S$.

A condition weaker than (L) is

(M) for every $f \in C_c(F)$, $g(x) \equiv \int f(xy)\mu(dy)$ vanishes at ∞ .

It was proved in [9] that if (L) holds and μ is idempotent, then $F = X \times G \times Y$ is completely simple with the last two factors compact. In this paper we prove the same result under condition (M), which turns out to be also necessary. In fact, if μ is idempotent, then (L) and (M) are equivalent conditions on F . These results given, in Section 2, might prove useful for attacking the general conjecture. In Section 3 we give some results about the two random walks induced on D by μ . If μ is idempotent, then Harris' recurrence condition [4] holds iff F is a compact group. Also if I is an ideal of D , $0 < \mu(I) < 1$, then $\{\mu^n(I)\}$ is strictly increasing and $\mu^n(I) \to 1$. In Section 4 we give some results about the convergence of probability measures and convolution powers of μ .

2.

The following example shows that for general μ condition (M) is weaker than (L). Let S be the Reals under multiplication and let

$$\mu_n(B) = (1/2^{1-n})\mu_0(B \cap [n, n+1)) \quad \text{or} \quad (1/2^{n+2})\mu_0(B \cap [n, n+1))$$

as $n < 0$ or $n \geq 0$, where μ_0 = the Lebesgue measure; then for any B in $\mathcal{B}$ let $\mu(B) = \sum_n \mu_n(B \cap [n, n+1))$. It follows easily that μ is a regular probability measure on S. Let $f \in C_c(S)$ and suppose $f(x) = 0$ for $x \notin [-m, m] = K$; then for $x \notin [-nm, nm]$, $x^{-1}K \subset \left[-\frac{1}{n}, \frac{1}{n}\right]$ and

$$|g(x)| \leq \int_{x^{-1}K} |f(xy)|\mu(dy) \leq \|f\|\frac{2}{n} .$$

Hence $g(x)$ vanishes at infinity; on the other hand $S = OO^{-1} = O^{-1}O$ is not compact.

THEOREM 2.1. *Let μ be right semi-invariant, i.e., $\mu(Cx) \geq \mu(C)$ for every compact C and $x \in S$, and let S satisfy* (K). *Then S is compact and has exactly one minimal left ideal.*

Proof. The compactness of S follows easily by contradiction modifying slightly the argument of [6, p. 538]. Since F is a right ideal and also $\mu(F-Fx) = 0$, $K \equiv \bigcap_x Sx \supset F$ and K = the unique minimal left ideal of S.

THEOREM 2.2. *Let F (or S) satisfy*

(m) *there is an upper semicontinuous function f such that*
$$g(x) \equiv \int f(xy)\mu(dy) \not\equiv 0 \text{ and } g(x) \text{ vanishes at infinity.}$$

Let μ be idempotent on S. Then F has a closed completely simple kernel (= minimal two-sided ideal) $K \equiv X \times G \times Y$ with the last two factors compact. For every $x \in K$, ${}_x\mu$ and μ_x are also idempotent measures on F. [In fact, μ_x is r^*-invariant [1] on its support which *equals* $Fx = Kx$.] *If in addition* $\mu(K) > 0$, *then* $K = F$.

Proof. Let $a \in F$ such that $g(a) = \sup g(x)$; then $g(a) - g(ax) \geq 0$ for all $x \in F$ and by idempotence

$$\int_F [g(a) - g(ax)]\mu(dx) = 0 .$$

Now $\{x \in F; g(a) - g(ax) > 0\}$ is open in F and hence empty. (Note that $g(x)$ is also upper semicontinuous on F .) Hence

$$\overline{aF} \subset M \equiv \left\{x \in F; g(x) = \sup_{y \in F} g(y)\right\} .$$

Now $\overline{aF}$ as a compact right ideal of F contains a compact minimal right ideal R of itself and hence of the whole F . We may take $R \equiv eF$ where e is an idempotent in R . It follows that K (= the union of all minimal right ideals of F) is non-empty. Next, e is a primitive idempotent in eF (eF is compact and simple) and it turns out that e is primitive in K . For if $fe = ef = f$ for some idempotent $f \in K$, then $f = ef \in eF$ and $e = f$. Hence K is completely simple [2, p. 46, 49, 67]. It follows easily that $Ke = Fe$ is closed and $eK = eF$ and $eKe = eFe$ are compact. Also $K \cong E(Fe) \times eFe \times E(eF)$ both algebraically and topologically by [2, p. 62], and hence K is closed. (Note that $K = E(Fe) \times eFe \times E(eF)$ is locally compact.)

The support of ${}_x\mu$, $x \in K$, is $xK = xF$, which is closed. (Note that $x \in XK$.) As in [5, p. 181] for $B \subset xF$, $s \in xF$, $\mu\left(s^{-1}B\right)$ is an idempotent Markov transition measure on xF . By the method of [5], $\mu\left(s^{-1}B\right)$, for $s \in xF$, $B \in \mathcal{B}(xF)$, is constant independent of $s \in xF$. (Note that every invariant set of xF [5] is dense because xF is right simple.) Hence ${}_x\mu\left(s^{-1}B\right) = \mu\left((sx)^{-1}B\right) = {}_x\mu(B)$ and ${}_x\mu$ is idempotent on its support xF .

In the case $\mu(K) > 0$, by Theorem 3.1 in Section 3, $\mu(K) = 1$.

THEOREM 2.3. *Let* μ *be idempotent. Then a necessary and sufficient condition that* F *be completely simple is that* xF *is a closed minimal right ideal of* F *for every* $x \in F$.

Proof. By [2, p. 46] necessity is trivial. To prove sufficiency, we observe that $K = FF$ and since $\overline{FF} = F$, $\overline{K} = F$. In view of the proof of Theorem 2.2, it suffices to show that K has a primitive idempotent, i.e., that K is completely simple. Hence we only need to show that xF, for some $x \in F$, is a right group. By the second part of the Proof of Theorem 2.2, ${}_y\mu \equiv \nu$ for fixed $y \in xF$ has the property that

$\nu(B) = \nu(s^{-1}B)$ on its support $yF = xF$. By right simplicity of xF, $y = yz$ for some $z \in xF$. By [2, p. 96], $zs = s$ for every $s \in xF$ and in particular z is idempotent and $zF = xF$.

COROLLARY 2.4. *Let* μ *be idempotent on* S. *Then the following conditions are equivalent:*

(i) F *satisfies* (L);

(ii) F *satisfies* (M);

(iii) $F \equiv X \times G \times Y$ *is completely simple with the last two factors compact.*

Also the following statements are equivalent:

(1) F *satisfies* (L) *and its "dual" condition* (R) *(defined analogously using* $A^{-1}B$*);*

(2) F *satisfies* (M) *and its dual right condition (defined by using* $g(x) \equiv \int f(yx)\mu(dy)$*);*

(3) F *is compact completely simple;*

(4) F *satisfies* (L) *and* $x^{-1}x$ *is compact for some* $x \in F$.

Proof. *(ii)* $\to$ *(iii)*. In view of Theorem 2.2, it suffices to show $\overline{K} = K = F$. Suppose V is a compact neighborhood such that $V \cap K = \emptyset$. There is $f \in C_c(F)$ such that $f \equiv 1$ on V and $f \equiv 0$ on K, by complete regularity. Now $g(x) = \int f(xy)\mu(dy)$ vanishes at ∞ and since $\mu(V) > 0$, $\sup g(x) = g(e) \not\equiv 0$ where $e \in eF \subset K$. (See Proof of Theorem 2.2.) But this is a contradiction to $g(e) = \int f(ey)\mu(dy) \equiv 0$.

(iii) $\to$ *(ii)*. By using projections it suffices to prove (L) for

compact rectangles. Now

$$\begin{aligned}(A, B, C)(A', B', C')^{-1} &= \left\{(x, g, y);\ x \in A,\ g(yx') \in BG^{-1} \cap G\right\} \\ &= \left\{(x, g, y);\ x \in A,\ gy \in GA'^{-1} \cap eF\right\} \\ &= A \times \left\{(g, y);\ gy \in GA'^{-1} \cap eF\right\} \\ &= A \times \{\text{a compact subset of } G \times Y\} ,\end{aligned}$$

which is compact in $X \times G \times Y$. (Note that AB^{-1} is closed when A , B are compact and (L) holds on groups.) Now (L) implies (M). For if $f \in C_c(F)$ with support C and K is compact such that $\mu(K) > 1 - \varepsilon$, then for $x \in CK^{-1}$, $g(x) < \|f\|\varepsilon$.

(4) → (3). Since F is completely simple, $x^{-1}x$ contains an idempotent e and $e^{-1}e \supset E(Fe)$. (Note that $x^{-1}x \neq \emptyset$.) Hence all the three factors of F are compact.

3.

The elementary right (respectively left) random walk induced by μ on D is defined as the sequence

$$Z_n = X_1X_2 \dots X_n , \quad \left(\text{resp. } Z_n = X_nX_{n-1} \dots X_1\right) , \quad n \geq 1$$

where $\{X_i\}$ is a sequence of independent random variables with values in D , identically distributed according to μ . [One uses the sequence space $\left(\Omega = \Pi D_i ,\ \times\mathcal{B}\left(D_i\right)\right)$ to construct such a sequence $\{X_i\}$ as coordinate projections, where $D_i = D$ for all i .] The proper right (respectively left) random walk on D induced by μ is defined as the Markov process $\{W_n\}$ with values in D corresponding to the transition probability functions

$$P^n(x, B) \equiv \mu^n\left(x^{-1}B\right) , \quad \left[\text{resp. } P^n(x, B) \equiv \mu^n\left(Bx^{-1}\right)\right] .$$

(See [3] and [4].)

It can be verified that the Z_n process has also the same

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transition probability functions.

THEOREM 3.1. *If* I *is an ideal of* D *such that* $0 < \mu(I) < 1$ *, then* $\{\mu^n(I)\}$ *is strictly increasing and* $\mu^n(I) \to 1$.

Proof. We have

$$\mu^2(I) = \int_I \mu\left(Ix^{-1}\right)\mu(dx) + \int_{D-I} \mu\left(Ix^{-1}\right)\mu(dx) .$$

Since $\int_I \mu\left(Ix^{-1}\right)\mu(dx) = \mu(I)$ we have $\mu^2(I) \geq (I)$ and by induction we have $\mu^{n+1}(I) \geq \mu^n(I)$ so that if $\mu(I) = 1$, $\mu^n(I) = 1$ for all n . By [11, p. 155] $\mu^{n+1}\left(I^c\right) \leq \mu^n\left(I^c\right)\mu\left(I^c\right)$. Hence if $0 < \mu(I) < 1$, we have $\mu^{n+1}\left(I^c\right) < \mu^n\left(I^c\right)$ and $\mu^{n+1}(I) > \mu^n(I)$.

Now $\sum P\left[X_n \in I\right] = \sum \mu(I) = \infty$. By the well known Borel-Cantelli Lemma, given $\varepsilon > 0$ there is N such that

$$P\left\{\bigcup_{m=1}^{N} \left[X_m \in I\right]\right\} \geq 1 - \varepsilon .$$

For $n \in N$, we have $\bigcup_{m=1}^{N} \left[X_m \in I\right] \subset \left[Z_n \in I\right]$. Hence

$$\mu^n(I) = P\left[Z_n \in I\right] > 1 - \varepsilon \quad \text{for} \quad n \geq N .$$

THEOREM 3.2. *Let* μ *be idempotent on* D $(= F)$. *Then for every open set* $N \subset F$, $N^{-1}N \neq \emptyset$ *and* $NN^{-1} \neq \emptyset$.

Proof. We prove that $N^{-1}N \neq \emptyset$ using the right elementary chain Z_n . Suppose $N^{-1}N = \emptyset$. For $n > k$,

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$$\begin{aligned} P[Z_n \in N/Z_k \in N] &= P[Z_k X_{k+1} \ldots X_n \in N/Z_k \in N] \\ &\le P[X_{k+1} \ldots X_n \in N^{-1}N/Z_k \in N] \\ &= P[X_{k+1}X_{k+2} \ldots X_n \in N^{-1}N] \\ &= P[X_1X_2 \ldots X_{n-k} \in N^{-1}N] \\ &= \mu^{(n-k)}(N^{-1}N) = 0 . \end{aligned}$$

Hence $P[Z_n \in N/Z_k \in N] = 0$ for every $n > k$, for all k. Now $P[Z_1 \in N] = \mu(N) = \delta > 0$. Since $P[Z_2 \in N/Z_1 \in N] = 0$ we have $P[Z_1 \in N, Z_2 \in N] = 0$. Hence

$$P[Z_2 \in N, Z_1 \notin N] = P[Z_2 \in N] = \mu^2(N) = \delta .$$

Similarly

$$P[Z_3 \in N, Z_1 \notin N, Z_2 \notin N] = P[Z_3 \in N] = \mu^3(N) = \delta .$$

But

$$\Omega \supset [Z_1 \in N] \cup [Z_2 \in N, Z_1 \notin N] \cup [Z_3 \in N, Z_1 \notin N, Z_2 \notin N] \cup \ldots .$$

Hence $P(\Omega) = \infty$ which is a contradiction. Similarly using the elementary left random walk we prove $NN^{-1} \neq \emptyset$.

In view of the above Theorem one may conjecture that for every pair of open $U, V \subset F$, $U^{-1}V \neq \emptyset$ and $VU^{-1} \neq \emptyset$. The following Theorem indicates that in general this is not true.

THEOREM 3.3. *Let* μ *be idempotent. Then the following statements are equivalent:*

(i) *for every pair of open* $U, V \subset F$, $U^{-1}V \neq \emptyset$ *and* $VU^{-1} \neq \emptyset$;

(ii) $x^{-1}V \neq \emptyset$ *and* $Vx^{-1} \neq \emptyset$ *for every open* $V \subset F$ *and almost all* $x \in F$;

(iii) $x \in \overline{yF}$ *and* $x \in \overline{Fy}$ *for every pair* $x, y \in F$;

(iv) F *is a compact group.*

Proof. *(i)* $\to$ *(iv).* For any open $V \subset F$ let

$$V_\alpha = \left\{x \in F \; ; \; \mu\left(Vx^{-1}\right) > \alpha\right\} , \quad \alpha > 0 .$$

By [5, p. 183] there is a set $E \subset F$ of zero μ_s-measure for all $s \in F$ such that $\mu\left[\left(V_\alpha - E\right)s^{-1}\right] = \mu\left(V_\alpha s^{-1}\right) = 1$ for all $s \in V_\alpha - E$. We show first that if $V_\alpha \neq \emptyset$, then $\mu\left(\overline{V}x^{-1}\right) > \alpha$ for every $x \in F$. Since $\mu\left(\overline{V}x^{-1}\right)$ is upper semicontinuous, it suffices to prove that $\overline{V}_\alpha = F$. To see this let U be open; since $UV_\alpha^{-1} \neq \emptyset$ there exists $x_0 \in V_\alpha$, such that $Ux_0^{-1} \neq \emptyset$. Since $\mu\left(Ux^{-1}\right)$ is lower semicontinuous there exists an open subset D of V_α such that $\mu\left(Ux^{-1}\right) > 0$ for every $x \in D$. It follows that for some $y \in D$ we have $\mu\left(V_\alpha y^{-1}\right) = 1$ and hence $V_\alpha \cap U \neq \emptyset$. Next let $V_\alpha \neq \emptyset$, $\varepsilon > 0$. By local compactness and regularity of μ_x, $x \in V_\alpha$, we can find an open $U_\varepsilon \subset V$ such that $\overline{U}_\varepsilon$ is compact, $\overline{U}_\varepsilon \subset V$ and $\mu\left(V - U_\varepsilon x^{-1}\right) < \varepsilon$. Then $W_{\alpha,\varepsilon} = \left\{x \; ; \; \mu\left(U_\varepsilon x^{-1}\right) > \alpha - \varepsilon\right\} \neq \emptyset$ and $\mu\left(\overline{U}_\varepsilon x^{-1}\right) > \alpha - \varepsilon$ for all $x \in F$ by the first part of the proof. Hence $\mu\left(Vx^{-1}\right) > \alpha - \varepsilon$ for all x. Since ε is arbitrary $\mu\left(Vx^{-1}\right) \geq \alpha$ for all x. Since α is arbitrary $\mu\left(Vx^{-1}\right) = \text{constant} = \mu(V)$ (by idempotence) for every x. By regularity $\mu\left(Cx^{-1}\right) = \mu(C)$ for every compact $C \subset F$ and every $x \in F$. One shows similarly that $\mu\left(x^{-1}C\right) = \mu(C)$. By [12] F is a compact group.

It is easy to see that *(iii)* → *(ii)* → *(i)* → *(iv)* → *(iii)*

REMARK. Conditions *(i)* and *(iii)* may be used to define communication of states in the elementary random walks. Then these conditions state that every two states in D communicate.

We consider the following recurrence condition introduced by Harris [4].

$Q(x, E) = P\left[W_n \in E \text{ infinitely often } / W_1 = x\right] = 1$ for every

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E such that $\mu(E) > 0$ and every $x \in F$.

THEOREM 3.4. *Let* μ *be idempotent. Then* Harris' *recurrence condition holds for both proper random walks on* F *iff* F *is a compact group.*

Proof. Let $L(x, E) = P[W_n \in E \text{ for some } n / W_1 = x]$. Suppose F is a compact group. By [3] for E such that $\mu(E) > 0$,

$$0 < Q(x, E) = 1 - \sum_n \int_E (1 - L(y, E))P^n(x, dy) .$$

But the integral above equals the constant $\int_E (1 - L(y, E))\mu(dy)$ (since μ is the Haar measure on F) and this constant must be zero. Hence $Q(x, E) = 1$. Conversely, since $\mu_x(B) = \int \mu(By^{-1})\mu_x(dy)$ and $\mu(B) = \int \mu(By^{-1})\mu(dy)$, both μ and μ_x are invariant σ-finite measures for $P(x, B) \equiv \mu(Bx^{-1})$. By Harris' uniqueness Theorem [4, pp. 120-121], μ_x and μ differ by a constant and the same is true for μ , ${}_x\mu$, ${}_y\mu$. Hence Theorem 3.3 *(ii)* applies. (Note that Harris' uniqueness proof goes through even when $\mathcal{B}$ is not separable.)

4.

We shall denote by $B(S)$ the space of all regular measures on S bounded in norm by 1 (the unit ball). The subspace of all regular probability measures on S will be denoted by $P(S)$. Both spaces are semigroups under convolution. By Alaoglou's Theorem $B(S)$ (as the dual of $C_\infty(S)$) is compact in the weak* topology. The net $\{\mu_\beta\} \in P(S)$ converges to μ in the weak* topology $(\mu_\beta \overset{*}{\to} \mu)$ if $\int f(x)\mu_\beta(dx) \to \int f(x)\mu(dx)$ for every $f \in C_c(S)$. The net μ_β converges to μ in the weak topology $(\mu_\beta \overset{w}{\to} \mu)$ if $\int f(x)\mu_\beta(dx) \to \int f(x)\mu(dx)$ for every bounded continuous function f on S . Since S is locally compact if $\mu_\beta \overset{*}{\to} \mu$ then we have also $\int f(x)\mu_\beta(dx) \to \int f(x)\mu(dx)$ for

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every $f \in C_\infty(S)$. Convolution may fail to be even separately continuous in $B(S)$ with the weak* topology. (See [13, p. 20] for such an example.) However, as we prove below, convolution is jointly (weak*) continuous if S satisfies condition (L) and

(r) $x^{-1}C$ is compact for every compact $C \subset S$ and every $x \in S$.

(Similarly one defines the weaker version (l) of condition (L) using Cx^{-1}.)

THEOREM 4.1. *(a) Joint weak* continuity of the convolution* $\mu * \nu$ *implies conditions* (L) *and* (R).

(b) The following statements are equivalent:

(i) the operation $\mu * \nu$ *is jointly weak* continuous in* $P(S)$*;*

(ii) conditions (L) *and* (r) *hold;*

(iii) conditions (R) *and* (l) *hold;*

(iv) conditions (L) *and* (R) *hold. (See Corollary* 2.4 *for definition.)*

Proof. *(a)* Suppose that $\emptyset \neq C = AB^{-1}$ is not compact for some compact sets A, B. Then there is a net $x_\alpha \in C$ with no convergent subnet. There is $b_\alpha \in B$ such that $x_\alpha b_\alpha = a_\alpha \in A$. Let μ_α be the point-mass at x_α and let $f \in C_c(S)$ with support K. If the x_α were frequently in K, then x_α would have a convergent subnet. Hence x_α is eventually in K^c so that $\int f(x)\mu_\alpha(dx) \to 0$ for every such f and $\mu_\alpha \overset{*}{\to} 0$. Since A, B are compact we can find subnets $a_{\alpha_\beta}, b_{\alpha_\beta}$ converging to some $a \in A$ and $b \in B$ respectively. Let ν_{α_β} be the point mass at b_{α_β}. Then $\nu_{\alpha_\beta} \to \mu_b$ = the point-mass at b. By continuity we have $\mu_{\alpha_\beta} * \nu_{\alpha_\beta} \to 0 * \mu_b = 0$. Since $x_{\alpha_\beta} b_{\alpha_\beta} = a_{\alpha_\beta}$, we obtain also $\mu_{\alpha_\beta} * \nu_{\alpha_\beta} \to \mu_a$ = the point mass at a, which is a

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contradiction.

(b) Let us assume that (L) and (r) hold. Let $\mu_\alpha \overset{*}{\to} \mu$ and $\nu_\alpha \overset{*}{\to} \nu$, μ_α, ν_α being nets in $P(S)$. Let $f(x) \in C_c(S)$ with support A. By (r), $h(y) = f(xy)$ is also in $C_c(S)$. Hence $\int h(y)\nu_\alpha(dy) \to \int h(y)\nu(dy)$ so that

$$(1) \qquad \left|\iint f(xy)\nu_\alpha(dy)\mu_\alpha(dx) - \iint f(xy)\nu(dy)\mu_\alpha(dx)\right| < \varepsilon/2$$

if $\alpha \geq \alpha_0$ for some suitable α_0. Let $g(x) = \int f(xy)\nu(dy)$. It is easy to see that by (L), $g(x) \in C_\infty(S)$. Hence if $\alpha \geq \alpha_1 \geq \alpha_0$ (for suitable α_1)

$$(2) \qquad \left|\iint f(xy)\nu(dy)\mu_\alpha(dx) - \iint f(xy)\nu(dy)\mu(dx)\right| < \varepsilon/2 .$$

Now (1) and (2) imply that $\mu_\alpha * \nu_\alpha \overset{*}{\to} \mu * \nu$. [We note that we have used the fact that $f(xy)$ is measurable in the product space and hence Fubini's Theorem applies. Also we observe that (R) and (l) similarly imply joint w^*-continuity.]

REMARK. From the above Theorem a purely topological Theorem that conditions *(ii)*, *(iii)* and *(iv)* are equivalent, is obtained (which is not, by any means, obvious otherwise).

LEMMA 4.1 *Let* $\{\mu_\alpha\} \overset{*}{\to} \mu$ *where* $\mu_\alpha \in P(S)$ *and* $\mu \in P(S)$. *Then we have also* $\mu_\alpha \overset{w}{\to} \mu$ *(in the weak topology).*

Proof. Let $\varepsilon > 0$. There exists a compact set A such that $\mu(A) > 1 - \varepsilon/4$. We can find an open set U such that $A \subset U$ and $\bar{U}$ is compact. Also we can choose a compact set B such that $\mu(B) > 1 - \varepsilon/4$ and $C = U^c \cap B \neq \emptyset$. Let $C \subset V \subset \bar{V}$ where V is open, $\bar{V}$ is compact and $\mu(V-C) < \varepsilon/4$. Let $f(x) \in C_c(S)$, $0 \leq f(x) \leq 1$, such that $f(C) = 1$ and $f(V^c) = 0$. Now for $\alpha \geq \alpha_0$ (for suitable α_0) $\left|\int f(x)\mu_\alpha(dx) - \int f(x)\mu(dx)\right| < \varepsilon/4$ so that $\mu_\alpha(C) < 3\varepsilon/4$. Therefore for

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all $\alpha \geq \alpha_0$, $\mu_\alpha(U \cap B^c) > 1 - 3\varepsilon/4$ and hence $\mu_\alpha(\overline{U}) > 1 - \varepsilon$. Now let $g(x)$ be any bounded continuous function $(|g(x)| \leq M)$. Then there is $h(x) \in C_c(S)$ such that $h(x) = g(x)$ whenever $x \in \overline{U}$. Let α_1 $(\geq \alpha_0)$ be such that

$$\left| \int f(x)\mu_\alpha(dx) - \int h(x)\mu(dx) \right| < \varepsilon .$$

Then

$$\left| \int g(x)\mu_\alpha(dx) - \int g(x)\mu(dx) \right|$$
$$\leq \left| \int_{\overline{U}^c} g(x)\mu_\alpha(dx) - \int_{\overline{U}^c} g(x)\mu(dx) \right| + \left| \int_{\overline{U}} h(x)\mu_\alpha(dx) - \int_{\overline{U}} h(x)\mu(dx) \right|$$
$$< 2\varepsilon M + 2\varepsilon M + \varepsilon = (4M+1)\varepsilon .$$

This proves the Lemma.

THEOREM 4.4. *Let* $\mu*\nu$ *be separately continuous*[1] *(weak* topology) in* μ *and* ν *, in* $B(S)$ *. Then for any* $\mu \in P(S)$ *,* $\mu_n = \frac{1}{n}\sum_{k=1}^{n} \mu^k$ *converges in the weak* topology to* μ_0 *, where* $\mu_0 = 0$ *or* $\mu_0(S) = 1$ *. In the latter case, also* $\mu_n \overset{w}{\to} \mu_0$ *(in the weak topology).*

Proof. Since $B(S)$ is compact, $\{\mu_n\}$ has a cluster point μ_0 . Since $B(S)$ is Hausdorff there is an (infinite) net $\{\mu_\alpha\} \overset{*}{\to} \mu_0$. We shall prove that $\mu*\mu_0 = \mu_0*\mu = \mu_0$. We claim that $\mu*\mu_\alpha$ also converges to μ_0 . Let $f(x)$ be any continuous function with compact support and let $M = \max|f(x)|$. Let $\varepsilon > 0$. Let $f_0(x) = \sum_{i=1}^{m} c_i\chi_{A_i}(x)$ be a simple function such that $|f(x) - f_0(x)| < \varepsilon/3$ for every x . Let N be such that $n \geq N$ implies that $\|\mu*\mu_n-\mu_n\| < \varepsilon/3Mm$; then with no loss of generality, we can assume that $\|\mu*\mu_\alpha-\mu_\alpha\| < \varepsilon/3Mm$, where $\|\mu\|$ equals $\sup\{\mu(B);\ B \in \mathcal{B}\}$. It follows that

[1] It suffices that S satisfy (L) and (R) .

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$$\left|\int f(x)\mu*\mu_\alpha(dx) - \int f(x)\mu_\alpha(dx)\right| \le \sum_{i=1}^{m} |c_i|\left|\mu_\alpha*\mu(A_i)-\mu_\alpha(A_i)\right| + \frac{2\varepsilon}{3} < \varepsilon .$$

Hence $\mu*\mu_\alpha$ also converges to μ_0 so that by weak*-continuity $\mu*\mu_0 = \mu_0$. Similarly $\mu_0 = \mu_0*\mu$. It then follows that $\mu_n*\mu_0 = \mu_0*\mu_n = \mu_0$ for all n. We next prove that the cluster point μ_0 is unique. Suppose μ_1 is another cluster point of $\{\mu_n\}$. Let $\{\mu_\beta\}$ be a subnet converging to μ_1. Then we have by the above relations, $\mu_0*\mu_\beta = \mu_\beta*\mu_0 = \mu_0$. Hence $\mu_0*\mu_1 = \mu_1*\mu_0 = \mu_0$. If we interchange the roles of μ_0 and μ_1 in the above argument, we will obtain also $\mu_1*\mu_0 = \mu_0*\mu_1 = \mu_1$. Therefore $\mu_0 = \mu_1$. Since the sequence μ_n has a unique cluster point, the first part of the theorem follows. In case $\mu_0(S) = 1$, then Lemma 4.1 applies.

REMARK. The following example shows that $\{\mu_n\}$ can indeed converge to 0 even in the presence of (L) and (R). Let $S = (0, 1/2]$ with multiplication and the relative topology as subspace of the reals. This space satisfies (L) and (R) but does not have a compact subsemigroup. Since the support of μ_0 has to be a compact subsemigroup (Corollary 2.4), we must have for every μ, $\mu_n \to \mu_0 = 0$. [If A, B, are compact, and if k is the infimum of $A \cup B$, then $k > 0$ and the closed set $AB^{-1} \subset [k, 1/2]$ and hence (L) and (R) hold on S.]

REMARK. Using a similar argument as in the proof of Theorem 4.1 *(a)* one can show that the assumption that $\mu*\nu$ is weak*-continuous in the first factor μ and Ax^{-1} is not compact for some compact A, leads to a contradiction. Hence separate weak* continuity of the convolution implies conditions (l) and (r). We have not been able to show the converse. Nor were we able to find an example of a locally compact semigroup that satisfies conditions (l) and (r) but not (L) and (R).

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