Canad. Math. Bull. Vol. **60** (4), 2017 pp. 673–689
http://dx.doi.org/10.4153/CMB-2017-039-8

Character Amenability of the Intersection of Lipschitz Algebras

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Abstract. Let (X,d) be a metric space and let $J \subseteq [0,\infty)$ be nonempty. We study the structure of the arbitrary intersections of Lipschitz algebras and define a special Banach subalgebra of $\bigcap_{\gamma\in J} \operatorname{Lip}_\gamma X$, denoted by $\operatorname{ILip}_J X$. Mainly, we investigate the C-character amenability of $\operatorname{ILip}_J X$, in particular Lipschitz algebras. We address a gap in the proof of a recent result in this field. Then we remove this gap and obtain a necessary and sufficient condition for C-character amenability of $\operatorname{ILip}_J X$, specially Lipschitz algebras, under an additional assumption.

1 Introduction

Let (X,d) be a metric space and let $B(X)$ indicate the Banach space consisting of all bounded complex valued functions on X, endowed with the norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \quad (f \in B(X)).$$

Take $\alpha \in \mathbb{R}$ with $\alpha > 0$. Then $\operatorname{Lip}_\alpha X$ is the subspace of $B(X)$, consisting of all bounded complex-valued functions f on X such that

$$p_\alpha(f) := \sup\Big\{ \frac{|f(x)-f(y)|}{d(x,y)^\alpha} : x,y \in X, \quad x \neq y \Big\} < \infty. \tag{1.1}$$

It is known that $\operatorname{Lip}_\alpha X$, endowed with the norm $\|\cdot\|_\alpha$ given by

$$\|f\|_\alpha = p_\alpha(f) + \|f\|_\infty,$$

and pointwise product, is a unital commutative Banach algebra, called a *Lipschitz algebra*. These interesting Banach algebras were first considered by Sherbert [18]; see also Bishop [5]. Lipschitz spaces have also been considered recently. For example, in [14] the authors described a new method of using the theory of approximate resolutions to study Lipschitz functions. A topological property of function spaces in the Lipschitz category was studied in [7]. Another interesting work in this field is [11], which describes the general form of algebra, ring, and vector lattice homomorphisms between spaces of real-valued little Lipschitz functions on compact Holder metric spaces, for the case where $0 < \alpha < 1$. Character amenability of Lipschitz algebras was investigated by Hu, Monfared, and Traynor [10]. In particular, as an immediate consequence of

Received by the editors August 31, 2016; revised February 23, 2017.
Published electronically September 4, 2017.
This research was partially supported by the Banach algebra Center of Excellence for Mathematics, University of Isfahan.
AMS subject classification: **46H05**, 46J10, 11J83.
Keywords: amenability, character amenability, Lipschitz algebra, metric space.

[10, Theorem 3.8] regarding character amenability of the second dual of a Banach algebra, they showed that when X is an infinite compact metric space and $0 < \alpha < 1$, $\operatorname{Lip}_\alpha X$ is not character amenable. Furthermore in [6], C-character amenability of $\operatorname{Lip}_\alpha X$ was investigated for $\alpha > 0$.

This work is a detailed study of the construction of Lipschitz spaces. Recently, arbitrary intersections of some special Banach spaces, such as L^p−spaces, weighted L^p−spaces, and Lorentz spaces have been studied; see [1–3]. More precisely, in [1], for an arbitrary subset J of $[1,\infty)$ and Hausdorff locally compact group G, we introduced a special subset of $\bigcap_{p\in J} L^p(G)$, denoted by $IL_J(G)$, which was the set of all functions f in $\bigcap_{p\in J} L^p(G)$ such that

$$\|f\|_{IL_J} = \sup_{p\in J} \|f\|_p < \infty.$$

We proved that $IL_J(G)$, endowed with the norm $\|\cdot\|_{IL_J}$, is always a Banach space. Moreover, we provided some examples to show that $IL_J(G)$ can be a proper subset of $\bigcap_{p\in J} L^p(G)$. In fact, we showed that

$$\begin{aligned} IL_J(G) &= L^{m_J}(G) \cap L^{M_J}(G),\\ \|f\|_{IL_J} &= \max\{\|f\|_{m_J}, \|f\|_{M_J}\} \quad (f \in IL_J(G)), \end{aligned}$$

where

$$m_J = \inf_{\alpha\in J} \alpha \quad \text{and} \quad M_J = \sup_{\alpha\in J} \alpha.$$

Motivated by these works, we investigate arbitrary intersections of Lipschitz algebras, denoted by $\bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X$, where J is an arbitrary nonempty subset of $[0,\infty)$. By analogy with L^p-spaces, we introduce a special subset of $\bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X$, denoted by $\operatorname{ILip}_J X$, which is defined by the set of all functions f in $\bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X$ such that

$$\|f\|_J = \sup_{\alpha\in J} \|f\|_\alpha < \infty.$$

We prove that if $M_J < \infty$, then

$$\operatorname{ILip}_J X = \operatorname{Lip}_{M_J} X,$$

and for each $f \in \operatorname{ILip}_J X$

$$\frac{\|f\|_J}{3} \le \|f\|_{M_J} \le 3\|f\|_J.$$

In fact, $\|\cdot\|_J$ defines a norm on $\operatorname{ILip}_J X$, equivalent to the norm $\|\cdot\|_{M_J}$. Furthermore, $\operatorname{ILip}_J X$, endowed with $\|\cdot\|_J$ and pointwise product, is a Banach algebra.

We also study $\bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X$ for the case where $M_J = \infty$ and show that

$$\bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X = \bigcap_{\alpha\in[m_J,\infty)} \operatorname{Lip}_\alpha X = \bigcap_{\alpha\in[0,\infty)} \operatorname{Lip}_\alpha X.$$

We provide some examples to show that in this case, $\|\cdot\|_J$ can take value ∞ and so does not necessarily define a norm on $\bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X$. It leads us to introduce a special subset of $\bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X$, denoted by $\operatorname{Lip}_\infty X$, as

$$\operatorname{Lip}_\infty X = \Big\{ f \in \bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X : \|f\|_{\operatorname{Lip}_\infty} = \sup_{\alpha\ge 0} \|f\|_\alpha < \infty \Big\}.$$

https://doi.org/10.4153/CMB-2017-039-8 Published online by Cambridge University Press

Then we show that $\mathrm{Lip}_\infty X$, endowed with the norm $\|\cdot\|_{\mathrm{Lip}_\infty}$ and pointwise product, is a Banach algebra.

The last section contains the main results of this work. It is completely devoted to the C-character amenability of Lipschitz algebras. In [6, theorem 3.1], it has been stated that $\mathrm{Lip}_\alpha X$ is C-character amenable for some $C > 0$ and $\alpha > 0$ if and only if X is ε-uniformly discrete for some $\varepsilon > 0$. We found a gap in the proof of [6, Theorem 3.1], and give an example to show that [6, Theorem 3.1] is not true when $\alpha > 1$. We note that in the assumptions of [6, Theorem 3.1] an extra condition seems to be required. Indeed, we prove that the assertion of [6, Theorem 3.1] will be true for $\mathrm{Lip}_\alpha X$ with $\alpha > 1$ under the condition that $\mathrm{Lip}_\alpha X$ separates points of X, and under this condition the proof of [6, theorem 3.1] remains valid. Under this extra condition, we establish a necessary and sufficient condition for C-character amenability of $\mathrm{ILip}_J X$ in the case where $M_J < \infty$. This result also provides a necessary and sufficient condition for C-character amenability of Lipschitz algebras. In fact we show that $\mathrm{ILip}_J X$ (equivalently Lipschitz algebras) is C-character amenable for some $C > 0$ if and only if (X, d) is ε-uniformly discrete for some $\varepsilon > 0$. Finally, we investigate the C-character amenability of $\mathrm{Lip}_\infty X$. Indeed, under the assumption that $\mathrm{Lip}_\infty X$ separates points of X, we show that $\mathrm{Lip}_\infty X$ is C-character amenable for some $C > 0$ if and only if (X, d) is ε-uniformly discrete for some $\varepsilon \geq 1$.

2 Intersection of Lipschitz Algebras

Let (X, d) be a metric space. If $\alpha = 0$ is assumed in inequality (1.1), then all the functions of $B(X)$ satisfy the inequality. Thus, we define $\mathrm{Lip}_0 X$ to be the set $B(X)$ endowed with norm

$$\|f\|_0 = \|f\|_\infty + p_0(f) \quad (f \in \mathrm{Lip}_0 X).$$

In fact, $\mathrm{Lip}_0 X$ and $B(X)$ are the same spaces, and

$$\|f\|_\infty \leq \|f\|_0 \leq 3\|f\|_\infty.$$

Suppose that $J \subseteq [0, \infty)$ is nonempty. Similar to L^p-spaces, let

$$\mathrm{ILip}_J X = \Big\{ f \in \bigcap_{\alpha \in J} \mathrm{Lip}_\alpha X : \|f\|_J = \sup_{\alpha \in J} \|f\|_\alpha < \infty \Big\}.$$

The aim of this section is to study the structure of $\mathrm{ILip}_J X$. This requires some preparation, which will be provided in the following results. We begin with an elementary proposition that will be used several times in further results. First, recall that (X, d) is called ε*-uniformly* discrete for some $\varepsilon > 0$ if

$$d(x, y) \geq \varepsilon \quad (x, y \in X, x \neq y).$$

Proposition 2.1 *Let (X, d) be a metric space. Then (X, d) is ε-uniformly discrete for some $\varepsilon > 0$ if and only if there exists $\alpha > 0$ such that $\mathrm{Lip}_\alpha X = B(X)$. In this case, for all $\alpha > 0$, $\mathrm{Lip}_\alpha X = B(X)$ and for each $f \in \mathrm{Lip}_\alpha X$*

$$\|f\|_\infty \leq \|f\|_\alpha \leq \Big(1 + \frac{2}{\varepsilon^\alpha}\Big)\|f\|_\infty.$$

Proof First, suppose that (X,d) is ε-uniformly discrete for some $\varepsilon > 0$. Thus for all $x, y \in X$ with $x \neq y$ we have $d(x,y) \geq \varepsilon$. For each $f \in B(X)$ and $\alpha > 0$, we have

$$p_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} \leq \frac{1}{\varepsilon^\alpha} \sup_{x \neq y} |f(x) - f(y)| \leq \frac{2}{\varepsilon^\alpha} \|f\|_\infty < \infty.$$

It follows that $f \in \operatorname{Lip}_\alpha X$, and so $B(X) = \operatorname{Lip}_\alpha X$. Moreover

$$\|f\|_\infty \leq \|f\|_\alpha \leq \Big(1 + \frac{2}{\varepsilon^\alpha}\Big) \|f\|_\infty \quad (f \in \operatorname{Lip}_\alpha X).$$

Conversely, suppose that $B(X) = \operatorname{Lip}_\alpha X$ for some $\alpha > 0$. First note that since $B(X) = \operatorname{Lip}_\alpha X$, all of the characteristic functions $\chi_{\{x\}}$ $(x \in X)$ belong to $\operatorname{Lip}_\alpha X$, and so they are continuous. It follows that all the singleton subsets of X are open, which implies the discreteness of X. Thus, for each $x_0 \in X$, there exists $M_{x_0} > 0$ such that for all $x \in X$ with $x \neq x_0$, $d(x_0, x) \geq M_{x_0}$. Now suppose to the contrary that (X,d) is not uniformly discrete. Thus, there exist distinct elements $x_1, x_2 \in X$ such that $d(x_1, x_2) < 1$. Moreover, there exists $K_1 \in \mathbb{N}$ with $K_1 \geq 2$ such that

$$\frac{1}{K_1} \leq d(x_1, x_2) < 1.$$

Similarly, there exist distinct elements $x_3, x_4 \in X$ such that

$$d(x_3, x_4) < \min\Big\{\frac{1}{K_1}, M_{x_1}, M_{x_2}\Big\} \leq \frac{1}{2}.$$

Thus, there exists integer number $K_2 \geq 3$ such that

$$\frac{1}{K_2} \leq d(x_3, x_4) < \frac{1}{K_1} \leq 1/2. \tag{2.1}$$

Note that (2.1) implies that x_1, x_2, x_3, x_4 are distinct elements of X. Inductively, for each $n \in \mathbb{N}$ there exist distinct elements $x_{2n-1}, x_{2n} \in X$ and $K_n \in \mathbb{N}$ such that $K_n \geq n+1$ and

$$\frac{1}{K_n} \leq d(x_{2n-1}, x_{2n}) < \min\Big\{\frac{1}{K_{n-1}}, M_{x_1}, M_{x_2}, M_{x_3}, \dots, M_{x_{2n-2}}\Big\} \leq \frac{1}{K_{n-1}}.$$

Moreover, for each $n \in \mathbb{N}$, the elements $x_1, x_2, \dots, x_{2n}$ are distinct and

$$\frac{1}{d(x_{2n-1}, x_{2n})} > K_{n-1} \geq n. \tag{2.2}$$

Now consider the sequence (x_n) in X, constructed as above, and define the function $f: X \to \mathbb{C}$ by

$$f(x) = \begin{cases} 1 & x \in \{x_{2n}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $f \in B(X)$ and so by the hypothesis $f \in \operatorname{Lip}_\alpha X$. Consequently, for each $n \in \mathbb{N}$, we obtain from (2.2) that

$$n^\alpha < \frac{1}{d(x_{2n}, x_{2n-1})^\alpha} = \frac{|f(x_{2n}) - f(x_{2n-1})|}{d(x_{2n}, x_{2n-1})^\alpha} \leq p_\alpha(f) < \infty,$$

which is impossible. This contradiction implies that (X,d) is ε-uniformly discrete, for some $\varepsilon > 0$. ∎

Corollary 2.2 *Let (X,d) be a metric space and let $J \subseteq [0,\infty)$ with $M_J < \infty$. Then $\mathrm{ILip}_J X = B(X)$ if and only if (X,d) is ε-uniformly discrete for some $\varepsilon > 0$. In this case, for each $f \in \mathrm{ILip}_J X$,*

$$\|f\|_\infty \le \|f\|_J \le \max\Big\{3, \Big(1+\frac{2}{\varepsilon^{M_J}}\Big)\Big\}\|f\|_\infty.$$

Proposition 2.3 *Let (X,d) be a metric space and let $0 \le \gamma \le \alpha \le \beta < \infty$. Then*

$$\mathrm{Lip}_\gamma X \cap \mathrm{Lip}_\beta X \subseteq \mathrm{Lip}_\alpha X.$$

Moreover, for each $f \in \mathrm{Lip}_\gamma X \cap \mathrm{Lip}_\beta X$,

$$\|f\|_\alpha \le \max\{\|f\|_\gamma, \|f\|_\beta\}.$$

Proof Suppose that $f \in \mathrm{Lip}_\gamma X \cap \mathrm{Lip}_\beta X$. Thus, for all $x, y \in X$

$$|f(x) - f(y)| \le p_\gamma(f)d(x,y)^\gamma,$$
$$|f(x) - f(y)| \le p_\beta(f)d(x,y)^\beta.$$

If $d(x,y) \ge 1$, then

$$|f(x) - f(y)| \le p_\gamma(f)d(x,y)^\gamma \le \max\big\{p_\gamma(f), p_\beta(f)\big\}d(x,y)^\alpha,$$

and if $d(x,y) < 1$, then

$$|f(x) - f(y)| \le p_\beta(f)d(x,y)^\beta \le \max\{p_\gamma(f), p_\beta(f)\}d(x,y)^\alpha.$$

Consequently, for all $x, y \in X$, we obtain

$$|f(x) - f(y)| \le \max\big\{p_\gamma(f), p_\beta(f)\big\}d(x,y)^\alpha.$$

It follows that $p_\alpha(f) < \infty$, and so $f \in \mathrm{Lip}_\alpha X$. Moreover,

$$p_\alpha(f) \le \max\{p_\gamma(f), p_\beta(f)\},$$

and so

$$\|f\|_\alpha \le \max\{\|f\|_\gamma, \|f\|_\beta\},$$

as claimed. ∎

Corollary 2.4 *Let (X,d) be a metric space and let $0 \le \alpha \le \beta < \infty$. Then $\mathrm{Lip}_\beta X \subseteq \mathrm{Lip}_\alpha X$. Moreover for each $f \in \mathrm{Lip}_\beta X$, $\|f\|_\alpha \le 3\|f\|_\beta$.*

Proof The result is immediately obtained by choosing $\gamma = 0$ in the proof of Proposition 2.3. ∎

We now state the main result of this section.

Theorem 2.5 *Let (X,d) be a metric space and $J \subseteq [0,+\infty)$ with $M_J < \infty$. Then*

$$\mathrm{ILip}_J X = \mathrm{Lip}_{M_J} X.$$

Moreover, for each $f \in \mathrm{ILip}_J X$,

$$\frac{\|f\|_J}{3} \le \|f\|_{M_J} \le 3\|f\|_J. \tag{2.3}$$

Proof By Corollary 2.4, we have

$$\mathrm{Lip}_{M_J} X \subseteq \bigcap_{\alpha \in J} \mathrm{Lip}_\alpha X, \quad \text{and} \quad \|f\|_\alpha \le 3\|f\|_{M_J},$$

for all $f \in \mathrm{Lip}_{M_J} X$ and $\alpha \in J$. It follows that

$$\|f\|_J = \sup_{\alpha \in J} \|f\|_\alpha \le 3\|f\|_{M_J} < \infty \quad (f \in \mathrm{Lip}_{M_J} X). \tag{2.4}$$

Inequality (2.4) implies that $\mathrm{Lip}_{M_J} X \subseteq \mathrm{ILip}_J X$. Now we prove the reverse of inclusion. Suppose that $f \in \mathrm{ILip}_J X$. For every $\varepsilon > 0$, there exists $\alpha_\varepsilon \in J$ with $M_J - \varepsilon < \alpha_\varepsilon \le M_J$. Thus, for distinct elements $x, y \in X$, we have

$$\begin{aligned}\frac{|f(x) - f(y)|}{d(x,y)^{M_J - \varepsilon}} &\le \max\Big\{2\|f\|_\infty, \frac{|f(x) - f(y)|}{d(x,y)^{\alpha_\varepsilon}}\Big\} \le \max\big\{2\|f\|_\infty, p_{\alpha_\varepsilon}(f)\big\} \\ &\le \max\big\{2\|f\|_\infty, \|f\|_{\alpha_\varepsilon}\big\} \le \max\big\{2\|f\|_\infty, \|f\|_J\big\}.\end{aligned}$$

The above inequalities hold for each $\varepsilon > 0$. Consequently,

$$\frac{|f(x) - f(y)|}{d(x,y)^{M_J}} \le \max\big\{2\|f\|_\infty, \|f\|_J\big\}. \tag{2.5}$$

Inequality (2.5) is valid for all distinct elements $x, y \in X$. It follows that $f \in \mathrm{Lip}_{M_J} X$ and

$$p_{M_J}(f) \le \max\big\{2\|f\|_\infty, \|f\|_J\big\}.$$

Moreover, $\|f\|_{M_J} \le 3\|f\|_J$. Thus, the proof is complete. ∎

Inequality (2.3) in Theorem 2.5 indicates that $\mathrm{ILip}_J X$ is exactly the space $\mathrm{Lip}_{M_J} X$, and $\|\cdot\|_J$ defines a norm on $\mathrm{ILip}_J X$ that is equivalent to norm $\|\cdot\|_{M_J}$. In fact, we have the following corollary.

Corollary 2.6 *Let (X, d) be a metric space and let $J \subseteq [0, +\infty)$ with $M_J < \infty$. Then $\mathrm{ILip}_J X$, endowed with norm $\|\cdot\|_J$ and pointwise product, is a Banach algebra.*

Remark 2.7 In [1], we investigated the intersections of L^p-spaces, denoted by $\bigcap_{p \in J} L^p(G)$, where G is a locally compact group and $J \subseteq (0, \infty)$ with $0 < m_J \le M_J < \infty$. In [1, Proposition 2.3] and [1, Example 2.4], we showed that the spaces $\bigcap_{p \in (m_J, M_J)} L^p(G)$, $\bigcap_{p \in [m_J, M_J)} L^p(G)$ and $\bigcap_{p \in [m_J, M_J]} L^p(G)$ can be different spaces. Then we introduced a special subset $IL_J(G)$ of $\bigcap_{p \in J} L^p(G)$, defined by

$$IL_J(G) = \Big\{ f \in \bigcap_{p \in J} L^p(G) : \|f\|_{IL_J} = \sup_{p \in J} \|f\|_p < \infty \Big\}.$$

We provided examples to show that $IL_J(G)$ is not necessarily equal to $\bigcap_{p \in J} L^p(G)$. Moreover, we proved that $IL_J(G)$, endowed with the norm $\|f\|_{IL_J}$, is a Banach space. In fact, analogously to Theorem 2.5, we proved that

$$\begin{aligned} IL_J(G) &= IL_{(0,M_J)}(G) = IL_{[0,M_J)}(G) = IL_{(0,M_J]}(G) = IL_{[0,M_J]}(G) \\ &= L^{m_J}(G) \cap L^{M_J}(G), \\ \|f\|_{IL_J} &= \max\{\|f\|_{m_J}, \|f\|_{M_J}\} \quad (f \in IL_J(G)); \end{aligned}$$

see [1, Theorem 3.2]. In Theorem 2.5, we showed that $\mathrm{ILip}_J X = \mathrm{Lip}_{M_J} X$, which is a subset of $\bigcap_{\alpha\in J} \mathrm{Lip}_\alpha X$. For some special metric spaces, the following equalities hold:

$$\mathrm{Lip}_{M_J} X = \mathrm{ILip}_J X = \bigcap_{\alpha\in J} \mathrm{Lip}_\alpha X. \tag{2.6}$$

For example, suppose that (X,d) is ε-uniformly discrete for some $\varepsilon > 0$ and $J \subseteq [0,\infty)$ with $M_J < \infty$. Thus, by Corollary 2.2, $\mathrm{ILip}_J X = B(X)$, and so

$$\mathrm{Lip}_{M_J} X = \mathrm{ILip}_J X = \bigcap_{\alpha\in J} \mathrm{Lip}_\alpha X = B(X).$$

But ε-uniform discreteness of X is not a necessary condition for equality (2.6). For instance, consider $\mathbb{R}$ endowed with the usual Euclidean metric and take $J = (1,2)$. It is not hard to see that for each $\alpha \in J$, $\mathrm{Lip}_\alpha \mathbb{R} = \mathrm{Cons}(\mathbb{R})$, where $\mathrm{Cons}(\mathbb{R})$ is the space consisting of all constant functions on $\mathbb{R}$. Thus

$$\mathrm{Cons}(\mathbb{R}) = \mathrm{Lip}_2 \mathbb{R} = \mathrm{Lip}_{M_J} \mathbb{R} = \mathrm{ILip}_J \mathbb{R} = \bigcap_{\alpha\in J} \mathrm{Lip}_\alpha \mathbb{R}.$$

At the present time, we are not able to prove equality (2.6) in general, or provide a special metric space (X,d) and a subset $J \subseteq [0,\infty)$ with $M_J < \infty$, such that

$$\mathrm{ILip}_J X \subsetneq \bigcap_{\alpha\in J} \mathrm{Lip}_\alpha X.$$

The following question arises naturally from the above discussion.

Question 2.8 For which metric space (X,d) and $J \subseteq [0,\infty)$ does equality (2.6) hold? Is there a metric space (X,d) and $J \subseteq [0,\infty)$ such that $\mathrm{ILip}_J X$ is a proper subset of $\bigcap_{\alpha\in J} \mathrm{Lip}_\alpha X$?

3 Introduction of $\mathrm{Lip}_\infty X$

Let (X,d) be a metric space and let $J \subseteq [0,\infty)$ be nonempty. In this section, we study $\bigcap_{\alpha\in J} \mathrm{Lip}_\alpha X$ for the case where $M_J = \infty$. The following result is required to justify our next definition.

Proposition 3.1 *Let (X,d) be a metric space and let J be an arbitrary subset of $[0,+\infty)$ such that $M_J = \infty$. Then*

$$\bigcap_{\alpha\in J} \mathrm{Lip}_\alpha X = \bigcap_{\alpha\in[0,\infty)} \mathrm{Lip}_\alpha X.$$

Proof It is obvious that

$$\bigcap_{\alpha\in[0,\infty)} \mathrm{Lip}_\alpha X \subseteq \bigcap_{\alpha\in J} \mathrm{Lip}_\alpha X.$$

Thus, it is sufficient to show that $\bigcap_{\alpha\in J} \mathrm{Lip}_\alpha X \subseteq \bigcap_{\alpha\in[0,\infty)} \mathrm{Lip}_\alpha X$. To that end, suppose that $f \in \bigcap_{\alpha\in J} \mathrm{Lip}_\alpha X$. Since $M_J = \infty$, for each $\alpha \geq 0$, there exists $\beta \in J$ such that $0 \leq \alpha < \beta$. Since $f \in \mathrm{Lip}_0 X = B(X)$, by Proposition 2.3 we obtain $f \in \mathrm{Lip}_\alpha X$. It follows that $f \in \bigcap_{\alpha\in[0,\infty)} \mathrm{Lip}_\alpha X$. Therefore, the desired equalities are satisfied. ∎

Example 3.2 Suppose that (X,d) is a metric space and $J \subseteq [0,\infty)$ such that $M_J = \infty$. Then the function $\|\cdot\|_J$, defined by

$$\|f\|_J = \sup_{\alpha\in J} \|f\|_\alpha \quad \Big(f \in \bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X\Big),$$

can take value ∞. For example, take the subset $X = \{0, \frac{1}{2}\}$ of $\mathbb{R}$, with the induced Euclidean metric, and let $J = [0,\infty)$. Then for all $f \in B(X)$ and $\alpha \geq 0$, we have

$$p_\alpha(f) = \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} = 2^\alpha |f(1) - f(0)| < \infty.$$

In particular, define the function f_0 on X by

$$f_0(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x = \frac{1}{2}. \end{cases}$$

Thus, $p_\alpha(f_0) = 2^\alpha$, and so $f_0 \in \operatorname{Lip}_\alpha X$ for each $\alpha \geq 0$. However,

$$\|f_0\|_J = \sup_{\alpha\geq 0} \|f_0\|_\alpha = \infty.$$

It follows that $\|\cdot\|_J$ is not a norm on $\bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X$, as claimed.

Example 3.2 leads us to introduce the linear subspace $\operatorname{Lip}_\infty X$ of $\bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X$ for the case where $M_J = \infty$. For $f \in B(X)$, let

$$p_\infty(f) = \sup_{\alpha\geq 0} p_\alpha(f) = \sup_{\alpha\geq 0} \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}. \tag{3.1}$$

Note that by Corollary 2.4 and Proposition 3.1, for each $f \in \bigcap_{\alpha\in J} \operatorname{Lip}_\alpha X$,

$$\sup_{\alpha\in J} \|f\|_\alpha \leq \sup_{\alpha\geq 0} \|f\|_\alpha \leq 3 \sup_{\alpha\in J} \|f\|_\alpha,$$

and thus for such f, we have

$$\sup_{\alpha\in J} \|f\|_\alpha < \infty \text{ if and only if } \sup_{\alpha\geq 0} \|f\|_\alpha < \infty.$$

As in (3.1), it let us introduce the function p_∞, independent of the choice of J. Now let

$$\operatorname{Lip}_\infty X = \{f \in B(X) : p_\infty(f) < \infty\}.$$

Moreover, for each $f \in \operatorname{Lip}_\infty X$ let

$$\|f\|_{\operatorname{Lip}_\infty} = \|f\|_\infty + p_\infty(f) = \sup_{\alpha\geq 0} \|f\|_\alpha.$$

Theorem 3.3 *Let (X,d) be a metric space. Then $\|\cdot\|_{\operatorname{Lip}_\infty}$ defines a norm on $\operatorname{Lip}_\infty X$, such that $\operatorname{Lip}_\infty X$, endowed with $\|\cdot\|_{\operatorname{Lip}_\infty}$ and pointwise product, is a Banach algebra.*

Proof It is easily verified that $\|\cdot\|_{\operatorname{Lip}_\infty}$ defines a norm on $\operatorname{Lip}_\infty X$. Moreover, for all $f, g \in \operatorname{Lip}_\infty X$ we have

$$\begin{aligned} \|fg\|_{\operatorname{Lip}_\infty} &= \|fg\|_\infty + p_\infty(fg) \leq \|f\|_\infty \|g\|_\infty + \|f\|_\infty p_\infty(g) + \|g\|_\infty p_\infty(f) \\ &\leq \big(\|f\|_\infty + p_\infty(f)\big)\big(\|g\|_\infty + p_\infty(g)\big) = \|f\|_{\operatorname{Lip}_\infty} \|g\|_{\operatorname{Lip}_\infty}. \end{aligned}$$

Thus, $\operatorname{Lip}_\infty X$ is a normed algebra under the norm $\|\cdot\|_{\operatorname{Lip}_\infty}$ and pointwise product. Now suppose that $\{f_n\}$ is a cauchy sequence in $\operatorname{Lip}_\infty X$. Thus, $\{f_n\}$ is a Cauchy sequence in $B(X)$ and also in $\operatorname{Lip}_\alpha X$ for all $\alpha \geq 0$. Thus, there are $f \in B(X)$ and $f_\alpha \in \operatorname{Lip}_\alpha X$ such that $\|f_n - f\|_\infty \to 0$ and $\|f_n - f_\alpha\|_\alpha \to 0$. Consequently, $f = f_\alpha$ for all $\alpha \geq 0$, which implies that $f \in \bigcap_{\alpha\geq 0} \operatorname{Lip}_\alpha X$. On the other hand, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|f_n - f_m\|_{\operatorname{Lip}_\infty} < \varepsilon$. Thus, for all $\alpha \geq 0$ and $m, n \geq N$,

$$\|f_n - f_m\|_\alpha < \varepsilon. \tag{3.2}$$

Letting m tend to infinity in (3.2), we deduce that $\|f_n - f\|_\alpha \leq \varepsilon$ whenever $\alpha \geq 0$ and $n \geq N$. Consequently, $\|f_n - f\|_{\operatorname{Lip}_\infty} \leq \varepsilon$. It follows that

$$\|f_n - f\|_{\operatorname{Lip}_\infty} \longrightarrow 0.$$

Moreover, $f = (f - f_N) + f_N$, which implies that $f \in \operatorname{Lip}_\infty X$. Therefore, $\operatorname{Lip}_\infty X$ is a Banach algebra under the norm $\|\cdot\|_{\operatorname{Lip}_\infty}$ and pointwise product. ∎

Remark 3.4 (i) Consider the function f_0 defined in Example 3.2. As we mentioned, $p_\alpha(f_0) = 2^\alpha$ and so $f_0 \in \operatorname{Lip}_\alpha X$, for each $\alpha \geq 0$. However,

$$p_\infty(f_0) = \sup_{\alpha\geq 0} p_\alpha(f_0) = \infty.$$

It follows that $f_0 \in \bigcap_{\alpha\geq 0} \operatorname{Lip}_\alpha X$ but $f_0 \notin \operatorname{Lip}_\infty X$. Therefore, $\operatorname{Lip}_\infty X$ can be a proper subset of $\bigcap_{\alpha\geq 0} \operatorname{Lip}_\alpha X$.

(ii) By the explanations preceding Theorem 3.3, $f \in \operatorname{Lip}_\infty X$ if and only if

$$\sup_{n\in\mathbb{N}}\{p_n(f), p_{\frac{1}{n}}(f)\} < \infty.$$

Moreover, by the proof of Proposition 2.3, we have

$$\sup_{\alpha>0} p_\alpha(f) = \sup_{n\in\mathbb{N}}\{p_n(f), p_{\frac{1}{n}}(f)\}.$$

(iii) It is clear that for each $\alpha \geq 0$, $\operatorname{Cons}(X) \subseteq \operatorname{Lip}_\infty X \subseteq \operatorname{Lip}_\alpha X$, where $\operatorname{Cons}(X)$ is the set consisting of all constant function on X. Moreover, for each $f \in \operatorname{Lip}_\infty X$ and $\alpha \geq 0$, $\|f\|_\alpha \leq \|f\|_{\operatorname{Lip}_\infty}$. Consequently, $\operatorname{Lip}_\infty X$ is continuously embedded in $\operatorname{Lip}_\alpha X$. Moreover, Corollary 2.4 can be improved. In fact, $0 \leq \alpha \leq \beta \leq \infty$ implies that $\operatorname{Lip}_\beta X \subseteq \operatorname{Lip}_\alpha X$.

(iv) Suppose that $J \subseteq [0, \infty)$. Then by Theorem 2.5, we have

$$\operatorname{ILip}_J X = \begin{cases} \operatorname{Lip}_{M_J} X & \text{if } M_J < \infty, \\ \operatorname{Lip}_\infty X & \text{if } M_J = \infty. \end{cases}$$

In the next proposition, we give a more precise structure for $\operatorname{Lip}_\infty X$.

Proposition 3.5 *Let (X, d) be a metric space. Then*

$$\operatorname{Lip}_\infty X = \Big\{ f \in \bigcap_{\alpha\geq 0} \operatorname{Lip}_\alpha X : d(x,y) < 1 \Rightarrow f(x) = f(y) \Big\}.$$

https://doi.org/10.4153/CMB-2017-039-8 Published online by Cambridge University Press

Proof Suppose that $f \in \operatorname{Lip}_\infty X$. Then $f \in \bigcap_{\alpha \ge 0} \operatorname{Lip}_\alpha X$, and by the definition there is $M > 0$ such that

$$p_\infty(f) = \sup_{\alpha \ge 0} p_\alpha(f) = M < \infty.$$

Suppose that $x, y \in X$ with $0 < d(x,y) < 1$. Then for each $\alpha \ge 0$, we have

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \le p_\alpha(f) \le M,$$

and so $|f(x) - f(y)| \le M d(x,y)^\alpha$. Letting α tend to infinity, we obtain $f(x) = f(y)$. Consequently,

$$\operatorname{Lip}_\infty X \subseteq \Big\{ f \in \bigcap_{\alpha \ge 0} \operatorname{Lip}_\alpha X : d(x,y) < 1 \Rightarrow f(x) = f(y) \Big\}.$$

The reverse of the inclusion is obvious. ∎

In the further results, we verify the structure of $\operatorname{Lip}_\infty X$ for some special metric spaces.

Proposition 3.6 *Let $(X, \|\cdot\|)$ be a normed space. Then* $\operatorname{Lip}_\infty X = \operatorname{Cons}(X)$.

Proof Suppose that $f \in \operatorname{Lip}_\infty(X)$ and take $x, y \in X$ to be distinct and let $\|x - y\| = m$. We show that $f(x) = f(y)$. Let

$$\Delta = \big\{ tx + (1-t)y : t \in [0,1] \big\},$$

and define

$$\theta : [0,1] \longrightarrow \Delta, \quad \theta(t) = tx + (1-t)y.$$

Choose $0 = t_0, t_1, \dots, t_n = 1 \in [0,1]$, where n is taken sufficiently large such that $|t_i - t_{i-1}| < \frac{1}{m}$ for each $i = 1, \dots, n$. Take $x_0, x_1, \dots, x_n \in \Delta$ such that $\theta(t_i) = x_i$ $(i = 1, \dots, n)$. Consequently, $\|x_i - x_{i-1}\| < 1$, for all $i = 1, \dots, n$. It follows from Proposition 3.5 that $f(x_i) = f(x_{i-1})$, for all $i = 1, \dots, n$. Therefore, $f(x) = f(y)$, as claimed. ∎

Proposition 3.7 *Let (X, d) be a metric space.*

(i) *If X is σ-compact and $f \in \operatorname{Lip}_\infty X$, then f has countable range.*
(ii) *If X is compact and $f \in \operatorname{Lip}_\infty X$, then f has finite range.*
(iii) *If X is σ-compact and connected then* $\operatorname{Lip}_\infty X = \operatorname{Cons}(X)$.

Proof (i) For each $x \in X$ and $\varepsilon > 0$, suppose that $B(x, \varepsilon)$ denotes the open ball centered at x with radius ε. By the hypothesis, there is a sequence (K_n) of compact subsets of X such that $X = \bigcup_{n=1}^\infty K_n$. By the compactness of K_n, for each $n \in \mathbb{N}$, there are $x_{1,n}, \dots x_{\ell_n,n} \in X$ such that

$$K_n \subseteq \bigcup_{i=1}^{\ell_n} B(x_{i,n}, 1/2).$$

This implies that

$$X = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\ell_n} B(x_{i,n}, 1/2).$$

Consider the countable subset $\{x_{i,n}\}$ of X and take $f \in \operatorname{Lip}_\infty X$. For each $x \in X$, there are $n \in \mathbb{N}$ and $1 \le i \le \ell_n$ such that $x \in B(x_{i,n}, 1/2)$. Since $d(x, x_{i,n}) < 1/2$, it follows from Proposition 3.5 that $f(x) = f(x_{i,n})$. Hence,

$$f(X) = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\ell_n} \{f(x_{i,n})\},$$

and so f has countable range.

(ii) The proof is similar to that of previous part.

(iii) It is known that all the connected subsets of $\mathbb{C}$ are either singletons or uncountable subsets of $\mathbb{C}$. It follows from part (i) that if $f \in \operatorname{Lip}_\infty(X)$, then f should be constant. Therefore $\operatorname{Lip}_\infty X = \operatorname{Cons}(X)$. ■

Proposition 3.8 *Let $X = \{x_n : n \in \mathbb{N}\}$ be a countable set and let d be a metric on X such that $d(x_n, x_{n+1}) < 1$ for all $n \in \mathbb{N}$. Then* $\operatorname{Lip}_\infty X = \operatorname{Cons}(X)$.

Proof Suppose that $f \in \operatorname{Lip}_\infty X$. Thus, Proposition 3.5 implies that $f(x_n) = f(x_{n+1})$ for all $n \in \mathbb{N}$. It follows that f is a constant function. This gives the proposition. ■

In Proposition 2.1, we give a necessary and sufficient condition for the equality $\operatorname{Lip}_\alpha X = B(X)$, where $\alpha > 0$. Moreover, in the previous results we obtained some sufficient conditions for the equality $\operatorname{Lip}_\alpha X = \operatorname{Cons}(X)$. The next results provide us with a more exact relation between these spaces. Before this, let us introduce a special subset of X that plays an important role in this verification. Let

$$B_X := \{x \in X : d(x, y) < 1 \text{ for some } y \neq x\}.$$

It is clear that if B_X is nonempty, then it contains at least two distinct elements of X. Moreover, B_X includes $D(X)$, the set consisting of all cluster points of X.

Proposition 3.9 *Let (X, d) be a metric space. Then the following assertions hold:*

(i) *If B_X is a nonempty subset of X, then* $\operatorname{Lip}_\infty X \subsetneqq B(X)$.

(ii) *If B_X is a nonempty and proper subset of X, then* $\operatorname{Cons}(X) \subsetneqq \operatorname{Lip}_\infty X$.

Proof (i) Suppose that x_0, x_1 are two distinct elements in B_X and define the complex function g on X by

$$g(x) = \begin{cases} 0 & \text{if } x = x_0, \\ 1 & \text{if } x \neq x_0. \end{cases}$$

Then for each $\alpha \ge 0$, we have

$$p_\alpha(g) = \sup_{x \neq x_0} \frac{|g(x) - g(x_0)|}{d(x, x_0)^\alpha} \ge \sup_{x \neq x_0, x \in B_X} \frac{1}{d(x, x_0)^\alpha} \ge \frac{1}{d(x_1, x_0)^\alpha}.$$

Consequently,

$$\sup_{\alpha \ge 0} p_\alpha(g) \ge \sup_{\alpha \ge 0} \frac{1}{(d(x_1, x_0))^\alpha} = \infty,$$

and so $g \notin \operatorname{Lip}_\infty(X)$. Therefore, $\operatorname{Lip}_\infty X \subsetneqq B(X)$.

(ii) Take the complex function f on X to be χ_{B_X}, defined by

$$f(x)=\begin{cases}1 & \text{if } x\in B_X,\\ 0 & \text{if } x\notin B_X.\end{cases}$$

Then for each $\alpha\ge 0$,

$$p_\alpha(f)=\sup_{x\in B_X, y\notin B_X}\frac{|f(x)-f(y)|}{d(x,y)^\alpha}=\sup_{x\in B_X, y\notin B_X}\frac{1}{d(x,y)^\alpha}\le 1,$$

and so $\sup_{\alpha\ge 0} p_\alpha(f)<\infty$. It follows that $f\in \operatorname{Lip}_\infty X$. Moreover, B_X is a proper subset of X, so f is a well-defined and non-constant function. This implies that $\operatorname{Cons}(X)\subsetneqq \operatorname{Lip}_\infty X$. ∎

Proposition 3.10 *Let (X,d) be a metric space. Then $\operatorname{Lip}_\infty X=B(X)$ if and only if X is ε-uniformly discrete for some $\varepsilon\ge 1$.*

Proof First, let $\operatorname{Lip}_\infty X=B(X)$. By Proposition 3.9(i), $B_X=\varnothing$. It follows that X is ε-uniformly discrete for some $\varepsilon\ge 1$. Conversely, suppose that X is ε-uniformly discrete for some $\varepsilon\ge 1$. Thus, for each $f\in B(X)$ we obtain

$$p_\infty(f)=\sup_{\alpha\ge 0}\sup_{x\ne y}\frac{|f(x)-f(y)|}{d(x,y)^\alpha}\le\sup_{\alpha\ge 0}\frac{2\|f\|_\infty}{\varepsilon^\alpha}\le 2\|f\|_\infty.$$

It follows that $f\in\operatorname{Lip}_\infty X$ and

$$\|f\|_\infty\le\|f\|_{\operatorname{Lip}_\infty}\le 3\|f\|_\infty.$$

This completes the proof. ∎

The following corollary is immediately obtained from Propositions 3.9 and 3.10.

Corollary 3.11 *Let (X,d) be a metric space. Then the following assertions hold:*

(i) $\operatorname{Lip}_\infty X=B(X)$ *if and only if* $B_X=\varnothing$.
(ii) *If* $\operatorname{Lip}_\infty X=\operatorname{Cons}(X)$, *then* $B_X=X$ *or* X *is a singleton.*
(iii) *If* $\varnothing\subsetneqq B_X\subsetneqq X$, *then* $\operatorname{Cons}(X)\subsetneqq\operatorname{Lip}_\infty X\subsetneqq B(X)$

Example 3.12 Consider the reverse of Corollary 3.11(ii). In the case where X is a singleton, $\operatorname{Cons}(X)=\operatorname{Lip}_\infty X=B(X)$. But the equality $B_X=X$ does not necessarily imply $\operatorname{Lip}_\infty X=\operatorname{Cons}(X)$. For example, suppose that $X=\{0,\frac{1}{2},2,\frac{5}{2}\}$, endowed with the induced Euclidean metric from $\mathbb{R}$. Define $f\colon X\to\mathbb{C}$ by

$$f(x)=\begin{cases}1 & \text{if } x\in\{0,1/2\},\\ 0 & \text{if } x\in\{2,5/2\}.\end{cases}$$

Then for each $\alpha\ge 0$ we have

$$p_\alpha(f)=\max\Big\{\Big(\frac{1}{2}\Big)^\alpha,\Big(\frac{2}{5}\Big)^\alpha,\Big(\frac{2}{3}\Big)^\alpha\Big\}=\Big(\frac{2}{3}\Big)^\alpha\le 1,$$

which implies that $f\in\operatorname{Lip}_\infty(X)$. It follows that $\operatorname{Cons}(X)\subsetneqq\operatorname{Lip}_\infty X$.

Remark 3.13 We point out some remarkable features of B_X.

(i) The function $f = \chi_{B_X}$ in Proposition 3.9 belongs to $\mathrm{Lip}_\infty X$, and so it is continuous. It follows that B_X is a clopen subset of X, *i.e.*, both open and closed.

(ii) We show that if (X,d) is a connected metric space with at least two distinct elements, then $B_X = X$. To that end, first note that by part (i), $B_X = \varnothing$ or $B_X = X$. If $B_X = \varnothing$, then Corollary 3.11(i) implies that $\mathrm{Lip}_\infty X = B(X)$. Consequently by Proposition 3.10, X is ε-uniformly discrete for some $\varepsilon \geq 1$, and since X is connected, it follows that X is a singleton. This contradicts the hypothesis. Consequently, $B_X = X$.

(iii) Suppose that (X,d) is a metric space such that $X = D(X)$, where $D(X)$ is the set consisting of all cluster points of X. Then clearly $B_X = X$. But the converse of this statement is not necessarily true. For example, take X to be the subset $\{0, 1, \frac{1}{2}\}$ with the induced Euclidean metric from $\mathbb{R}$. It is obvious that $B_X = X$, whereas X does not have any cluster points.

Example 3.14 Let $X = \{0, \frac{1}{2}, 2\}$ with the induced usual Euclidean metric from $\mathbb{R}$. Then $B_X = \{0, \frac{1}{2}\}$ and so B_X is a nonempty and proper subset of X. Thus, by Corollary 3.11(iii),

$$\mathrm{Cons}(X) \subsetneqq \mathrm{Lip}_\infty X \subsetneqq B(X).$$

One can also obtain this result directly. Indeed, suppose that $f \in B(X)$. Thus

$$p_\alpha(f) = \max\Big\{ 2^\alpha |f(1/2) - f(0)|, \frac{|f(2) - f(0)|}{2^\alpha}, \Big(\frac{2}{3}\Big)^\alpha |f(2) - f(1/2)| \Big\}.$$

Now define

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1 & \text{if } x = \frac{1}{2}, \\ 0 & \text{if } x = 2. \end{cases}$$

For each $\alpha \geq 0$,

$$p_\alpha(f) = \max\Big\{ \frac{1}{2^\alpha}, \Big(\frac{2}{3}\Big)^\alpha \Big\} \leq 1,$$

which implies that $f \in \mathrm{Lip}_\infty X$. Thus, f is a non-constant function belonging to $\mathrm{Lip}_\infty X$. Consequently, $\mathrm{Cons}(X) \subsetneqq \mathrm{Lip}_\infty X$. Now define the function g on X by

$$g(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x = \frac{1}{2}, \\ 1 & \text{if } x = 2. \end{cases}$$

Then for each $\alpha > 0$,

$$p_\alpha(g) = \max\Big\{ 2^\alpha, \Big(\frac{2}{3}\Big)^\alpha \Big\} = 2^\alpha,$$

and so $g \notin \mathrm{Lip}_\infty X$. Therefore, $\mathrm{Lip}_\infty X \subsetneqq B(X)$.

We end this section with a brief discussion concerning different behaviors of $\mathrm{ILip}_J X$ and $\mathrm{Lip}_\infty X$ with respect to different metrics.

Remark 3.15 Suppose that (X, d_1) and (X, d_2) are two metric spaces such that d_1 and d_2 are equivalent; *i.e.*, there are $C_1, C_2 > 0$ such that

$$C_1 d_1(x,y) \leq d_2(x,y) \leq C_2 d_1(x,y) \quad (x, y \in X).$$

It is easily verified that for $\alpha \geq 0$, all the Lipchitz algebras $\mathrm{Lip}_\alpha(X, d_1)$ and $\mathrm{Lip}_\alpha(X, d_2)$ are the same spaces. Moreover, if $J \subseteq [0, \infty)$ with $M_J < \infty$, then by Theorem 2.5 we have

$$\mathrm{ILip}_J(X, d_1) = \mathrm{ILip}_J(X, d_2).$$

But the spaces $\mathrm{Lip}_\infty(X, d_1)$ and $\mathrm{Lip}_\infty(X, d_2)$ are not necessarily the same. For example, let $X = \{0, \frac{1}{2}\}$, d_1 be the induced Euclidean metric on X and let d_2 be the usual discrete metric on X, defined by

$$d_2(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

It is clear that d_1 and d_2 are equivalent metrics on X. However, Proposition 3.5 implies that $\mathrm{Lip}_\infty(X, d_1) = \mathrm{Cons}(X)$, whereas by Proposition 3.10, $\mathrm{Lip}_\infty(X, d_2) = B(X)$. These observations show that different metrics produce different spaces $\mathrm{Lip}_\infty X$.

4 Results on Character Amenability

Let $\mathcal{A}$ be a Banach algebra and let $\Delta(\mathcal{A})$ denote the spectrum of $\mathcal{A}$, consisting of all nonzero multiplicative linear functionals on $\mathcal{A}$. In [12, 13], Kaniuth, Lau, and Pym introduced and studied the concept of φ-amenability for Banach algebras, where $\varphi \in \Delta(\mathcal{A})$. A Banach algebra $\mathcal{A}$ is called *φ-amenable* if there exists a bounded linear functional m on $\mathcal{A}^*$ satisfying

$$m(\varphi) = 1 \quad \text{and} \quad m(f \cdot a) = m(f)\varphi(a)$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, where $f \cdot a \in \mathcal{A}^*$ is defined by $(f \cdot a)(b) = f(ab)$, $(b \in \mathcal{A})$. Moreover, for some $C > 0$, $\mathcal{A}$ is called *$C - \varphi$-amenable* if m is bounded by C; see Hu, Monfared, and Traynor [10]. The notion of (right) character amenability was introduced and studied by Monfared [15]. Character amenability of $\mathcal{A}$ is equivalent to $\mathcal{A}$ being φ-amenable for all $\varphi \in \Delta(\mathcal{A})$ and $\mathcal{A}$ having a bounded right approximate identity. The concept of C-character amenability is defined similarly; see [10] for more details in this field.

There are valuable works related to some notions of amenability of Lipschitz algebras. Gourdeau [8] discussed amenability of Lipschitz algebras. Moreover, he proved that if a Banach algebra $\mathcal{A}$ is amenable, then $\Delta(\mathcal{A})$ is uniformly discrete with respect to norm topology induced by $\mathcal{A}^*$; see also Bade, Curtis, and Dales [4], Gourdeau [9], and Zhang [19].

Now suppose that (X, d) is a metric space. As an immediate consequence of [10, Theorem 3.8] on character amenability of the second dual of a Banach algebra, Hu, Monfared, and Traynor [10, p. 64], obtained that if X is an infinite compact metric space and $0 < \alpha < 1$, then $\mathrm{Lip}_\alpha X$ is not character amenable. Moreover, C-character amenability of Lipschitz algebras was recently studied by Dashti, Nasr Isfahani and Soltani for each $\alpha > 0$; see [6]. In fact, as a generalization of the above result from [10], they showed that for $0 < \alpha \leq 1$ and any locally compact metric space X, $\mathrm{Lip}_\alpha X$ is C-character amenable for some $C > 0$ if and only if X is ε-uniformly discrete for some $\varepsilon > 0$.

Notification 4.1 We find a gap in the proof of [6, Theorem 3.1] and give an example to show that [6, Theorem 3.1] is not true when $\alpha > 1$. In the proof of [6, Theorem 3.1], the fact that $\mathrm{Lip}_\alpha X$ separates the points of X was applied. This fact is valid whenever $0 < \alpha \le 1$; see, for example, [17, Lemma 3.1] for the proof of the case $\alpha = 1$. In fact they defined the function f_s $(s \in X)$ as $f_s(x) = \min\{d(s,x),1\}$ $(x \in X)$ and showed that the family of functions $\{f_s : s \in X\}$ separates the points of X. Moreover, since the function d^α is also a metric on X whenever $0 < \alpha < 1$, it is obtained that the family of functions $\{f_s : s \in X\}$, where $f_s(x) = \min\{d^\alpha(s,x),1\}$ $(x \in X)$, separates the points of X as well. In fact, the proof of [17, Lemma 3.1] can be adapted for each $0 < \alpha \le 1$.

However, it should be noted that $\mathrm{Lip}_\alpha X$ does not necessarily separate the points of X whenever $\alpha > 1$. For example, consider $\mathbb{R}$, endowed with the usual Euclidean metric. Then $\mathrm{Lip}_\alpha \mathbb{R} = \mathrm{Cons}(\mathbb{R})$ for all $\alpha > 1$, and so in this situation, $\mathrm{Lip}_\alpha \mathbb{R}$ does not separate the points of $\mathbb{R}$. More importantly, this example shows that the equivalence given in [6, Theorem 3.1], is not preserved for $\alpha > 1$. Indeed, $\mathrm{Lip}_\alpha \mathbb{R} = \mathrm{Cons}(\mathbb{R})$ is amenable, and so by [10, Theorem 2.9] it is C-character amenable for some $C > 0$. However $\mathbb{R}$ is not uniformly discrete. Therefore [6, Theorem 3.1] is true for $0 < \alpha \le 1$, but not for $\alpha > 1$. These observations lead us to suggest an extra condition in the assumptions of [6, Theorem 3.1], which seems to be required. Indeed, we prove that the assertion of [6, Theorem 3.1] will be true for $\mathrm{Lip}_\alpha X$ with $\alpha > 1$ under the condition that $\mathrm{Lip}_\alpha X$ separates points of X, and under this condition the proof of [6, Theorem 3.1] remains valid.

Regarding Notification 4.1, one can modify [6, Theorem 3.1] by the following result. The proof of [6, Theorem 3.1] is still valid for $\mathrm{Lip}_\alpha X$ with $\alpha > 1$ under the added condition.

Theorem 4.2 *Let (X,d) be a metric space and $\alpha \in (0,+\infty)$. Suppose that $\mathrm{Lip}_\alpha X$ separates the points of X. Then the following assertions are equivalent:*

(i) $\mathrm{Lip}_\alpha X$ *is C-character amenable, for some $C > 0$;*
(ii) $\mathrm{Lip}_\alpha X$ *is amenable;*
(iii) (X,d) *is ε-uniformly discrete, for some $\varepsilon > 0$.*

The following result is obtained from Corollary 2.2, Theorems 2.5 and 4.2.

Corollary 4.3 *Let (X,d) be a metric space and $J \subseteq [0,+\infty)$ with $M_J < \infty$. Suppose that $\mathrm{ILip}_J X$ separates the points of X. Then the following assertions are equivalent:*

(i) $\mathrm{ILip}_J X$ *is C-character amenable, for some $C > 0$;*
(ii) $\mathrm{ILip}_J X$ *is amenable;*
(iii) $\mathrm{ILip}_J X = B(X)$*, as two Banach algebras;*
(iv) (X,d) *is ε-uniformly discrete, for some $\varepsilon > 0$.*

Now we consider $\mathrm{ILip}_J X$ for a subset J of $[0,\infty)$, where $M_J = \infty$, which is the space $\mathrm{Lip}_\infty X$ by part (iv) of Remark 3.4. We end this work with the investigation of C-character amenability of $\mathrm{Lip}_\infty X$. By [8], for each $0 < \alpha < \infty$, all of the linear maps $\phi_x\colon \mathrm{Lip}_\alpha X \to \mathbb{C}$ $(x \in X)$, defined by $\phi_x(f) = f(x)$ $(f \in \mathrm{Lip}_\alpha X)$, are bounded and multiplicative and so belong to $\Delta(\mathrm{Lip}_\alpha X)$.

Theorem 4.4 *Let (X,d) be a metric space such that* $\mathrm{Lip}_\infty X$ *separates the points of* X. *Then the following assertions are equivalent:*

(i) $\mathrm{Lip}_\infty X$ *is* C*-character amenable, for some* $C > 0$*;*
(ii) $\mathrm{Lip}_\infty X$ *is amenable;*
(iii) $\mathrm{Lip}_\infty X = B(X)$*;*
(iv) (X,d) *is* ε*-uniformly discrete, for some* $\varepsilon \geq 1$.

Proof (i)⇒(iv) The proof of [6, Theorem 3.1] is adapted here. Suppose that $\mathrm{Lip}_\infty X$ is C-character amenable for some $C > 0$. Thus, by [6, Corollary 2.2], $\Delta(\mathrm{Lip}_\infty X)$ is uniformly discrete, and so there is $\varepsilon > 0$ such that $\|\phi - \psi\| \geq \varepsilon$ for all distinct elements $\phi, \psi \in \Delta(\mathrm{Lip}_\infty X)$. Since $X \subseteq \Delta(\mathrm{Lip}_\infty X)$ and $\mathrm{Lip}_\infty X$ separates the points of X, for all $x, y \in X$ with $x \neq y$, we have $\phi_x \neq \phi_y$, and so

$$\sup_{\|f\|_{\mathrm{Lip}_\infty} \leq 1} |\phi_x(f) - \phi_y(f)| \geq \varepsilon.$$

It follows that for all $\alpha > 0$ and $x, y \in X$ with $x \neq y$,

$$\sup_{\|f\|_\alpha \leq 1} |f(x) - f(y)| \geq \sup_{\|f\|_{\mathrm{Lip}_\infty} \leq 1} |f(x) - f(y)| \geq \varepsilon.$$

Consequently,

$$\begin{aligned} \varepsilon &\leq \sup_{\|f\|_\alpha \leq 1} |f(x) - f(y)| = \sup_{\|f\|_\alpha \leq 1} \Big(\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \Big) d(x,y)^\alpha \\ &\leq \sup_{\|f\|_\alpha \leq 1} p_\alpha(f) d(x,y)^\alpha \leq d(x,y)^\alpha. \end{aligned}$$

So, $\sqrt[\alpha]{\varepsilon} \leq d(x,y)$ for each $\alpha > 0$, and consequently $d(x,y) \geq 1$. Thus, (iv) is obtained.

(iv)⇒(iii) It is immediately obtained from Proposition 3.10.

(iii)⇒(ii) Suppose that (iii) holds. By Proposition 3.10, X is ε-uniformly discrete for some $\varepsilon \geq 1$. Also in this case $B(X) = \ell^\infty(X)$, which is amenable by [16, Example 2.3.4]. Thus, (ii) is obtained

(ii)⇒(i). If $\mathrm{Lip}_\infty X$ is amenable, it has an approximate diagonal bounded by some $C > 0$; see [16]. Thus, $\mathrm{Lip}_\infty X$ is C-character amenable by [10, Theorem 2.9] and so (i) is obtained. ■

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