INEQUALITIES FOR GENERALIZED SYMMETRIC FUNCTIONS

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(received March 3, 1968)

The generating series for the elementary symmetric function E_r, the complete symmetric function H_r, are defined by

$$\prod_{i=1}^{m} (1 + \alpha_1 x) = 1 + \sum_{r=1}^{m} E_r x^r ,$$

$$\prod_{i=1}^{m} (1 - \alpha_i x)^{-1} = 1 + \sum_{r=1}^{\infty} H_r x^r$$

respectively. In [1] it is proved that if $\alpha_1, \alpha_2, \ldots, \alpha_m$ are non-negative reals then,

$$E_{a-\lambda} \cdot E_{b+\lambda} \geq E_{a-\lambda-1} \cdot E_{b+\lambda+1}, \quad 0 \leq \lambda < a \quad , \quad b \geq a .$$

In [3], the author has proved a similar relation for H_r,

(1) $$H_{a-\lambda} \; H_{b+\lambda} > H_{a-\lambda-1} H_{b+\lambda+1}, \quad 0 \leq \lambda < a \quad , \quad b \geq a .$$

In this paper we consider some generalizations of these inequalities. In [2] D.E. Littlewood has defined certain symmetric functions by means of the generating series

(2) $$\prod_{i=1}^{m} \left(\frac{1+\alpha_i tx}{1-\alpha_i x}\right) = 1 + \sum_{r=1}^{\infty} q_r^{(m)}(t) x^r .$$

Here (2) is a generating function for the symmetric functions $q_r^{(m)}(t)$. When $t = 0$ the complete symmetric funtion H_r is obtained, which can be used in the definition of S-functions; and when $t = 1$ the symmetric functions $q_r^{(m)}(1)$ can be used in defining Q-functions, which are of interest in the study of fractional linear representations of the symmetric group. (2) was first defined by D.E. Littlewood who showed that the resulting symmetric functions (subsequently called Hall functions) were similar to certain symmetric functions used by P. Hall in enumerating subroups of finite Abelian p-groups.

In this paper we consider a generalization of (2) defined by

$$\prod_{i=1}^{m} \left(\frac{1+\alpha_i tx}{1-\alpha_i x}\right)^k = 1 + \sum_{r=1}^{\infty} q_r^{(m)}(k,t)x^r ,$$

where k is a positive number. In [4] and [5] Whitely has given some properties of $q_r^{(m)}(k,t)$ for the case $t = 0$.

THEOREM 1. *If*

$$\prod_{i=1}^{m} \left(\frac{1+\alpha_i tx}{1-\alpha_i x}\right)^k = 1 + \sum_{r=1}^{\infty} q_r^{(m)}(k,t)x^r , \quad t \geq 0$$

where k *is a positive integer*, *then*

$$q_{a-\lambda}^{(m)}(k,t) \cdot q_{b+\lambda}^{(m)}(k,t) \geq q_{a-\lambda-1}^{(m)}(k,t) \cdot q_{b+\lambda+1}^{(m)}(k,t) ,$$

$0 \leq \lambda < a$, $b \geq a$, $\alpha_1, \alpha_2, \ldots, \alpha_m$ non-negative reals.

https://doi.org/10.4153/CMB-1969-079-6 Published online by Cambridge University Press

The inequality is strict unless all but one of the variables are zero and $k = 1$.

Proof. We prove this theorem by induction on k and m. Let $k = 1$; then

$$\prod_{i=1}^{m} \left(\frac{1+\alpha_i tx}{1-\alpha_i x}\right) = 1 + \sum_{r=1}^{\infty} q_r^{(m)}(1,t)x^r$$

and, when $m = 1$,

$$q_{a-\lambda}^{(1)}(1,t) \cdot q_{b+\lambda}^{(1)}(1,t) = q_{a-\lambda-1}^{(1)}(1,t) \cdot q_{b+\lambda+1}^{(1)}(1,t) ,$$

$$0 \leq \lambda < a \quad , \quad b \geq a \; .$$

Let the theorem be true for $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ and $k = 1$; then

$$(3) \qquad q_{a-\lambda}^{(m-1)}(1,t) \cdot q_{b+\lambda}^{(m-1)}(1,t) \geq q_{a-\lambda-1}^{(m-1)}(1,t) \cdot q_{b+\lambda+1}^{(m-1)}(1,t)$$

$$0 \leq \lambda < a \quad , \quad b \geq a \; .$$

Now

$$1 + \sum_{r=1}^{\infty} q_r^{(m)}(1,t)x^r = \prod_{i=1}^{m-1} \left(\frac{1+\alpha_i tx}{1-\alpha_i x}\right)\left(\frac{1+\alpha_m tx}{1-\alpha_m x}\right)$$

$$= \left\{1 + \sum_{r=1}^{\infty} q_r^{(m-1)}(1,t)x^r\right\} \left\{1 + (1+t) \sum_{r=1}^{\infty} \alpha_m^r x^r\right\}$$

and so

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$$(4) \qquad q_r^{(m)}(1,t) = q_r^{(m-1)}(1,t) + (1+t) \sum_{j=1}^{r} \alpha_m^j \, q_{r-j}^{(m-1)}(1,t)$$

where $q_o(1,t)$ 1. From (4), we have

$$q_{a-\lambda}^{(m)}(1,t) \cdot q_{b+\lambda}^{(m)}(1,t) - q_{a-\lambda-1}^{(m)}(1,t) \cdot q_{b+\lambda+1}^{(m)}(1,t) =$$

$$q_{a-\lambda}^{(m)}(1,t) \left\{ q_{b+\lambda}^{(m)}(1,t) + (1+t) \sum_{j=1}^{b+\lambda} \alpha_m^j \cdot q_{b+\lambda-j}^{(m-1)}(1,t) \right\}$$

$$- \left\{ q_{a-\lambda-1}^{(m-1)}(1,t) + (1+t) \sum_{j=1}^{a-\lambda-1} \alpha_m^j \cdot q_{a-\lambda-1-j}^{(m-1)}(1,t) \right\} q_{b+\lambda+1}^{(m-1)}(1,t) \cdot$$

$$(5) \qquad = \left\{ q_{a-\lambda}^{(m-1)}(1,t) \cdot q_{b+\lambda}^{(m-1)}(1,t) - q_{a-\lambda-1}^{(m-1)}(1,t) \cdot q_{b+\lambda+1}^{(m-1)}(1,t) \right\} +$$

$$(1+t) \sum_{j=1}^{a-\lambda-1} \alpha_m^j \left\{ q_{a-\lambda}^{(m-1)}(1,t) \cdot q_{b+\lambda-j}^{(m-1)}(1,t) - q_{a-\lambda-1-j}^{(m-1)}(1,t) \cdot q_{b+\lambda+1}^{(m-1)}(1,t) \right\}$$

$$+ (1+t) \, q_{a-\lambda}^{(m-1)}(1,t) \left\{ \sum_{j=a-\lambda}^{b+\lambda} \alpha_m^j \cdot q_{b+\lambda-j}^{(m-1)}(1,t) \right\} .$$

Hence from (3) and (5) we have

$$q_{a-\lambda}^{(m)}(1,t) \, q_{b+\lambda}^{(m)}(1,t) \geqslant q_{a-\lambda-1}^{(m)}(1,t) \, q_{b+\lambda+1}^{(m)}(1,t), \qquad 0 \leqslant \lambda < a \quad , \quad b \geqslant a \ .$$

Let the theorem be true for (k-1); then

$$\prod_{i=1}^{m} \left(\frac{1+\alpha_i tx}{1-\alpha_1 x}\right)^{k-1} \quad 1 + \sum_{r=1}^{\infty} q_r^{(m)}(k-1,t)x^r ,$$

and

$$(6) \qquad q_{a-\lambda}^{(m)}(k-1,t) \cdot q_{b+\lambda}^{(m)}(k-1,t) \geqslant q_{a-\lambda-1}^{(m)}(k-1,t) \cdot q_{b+\lambda+1}^{(m)}(1-1,t), \quad 0 \leqslant \lambda < a \quad , \quad b \geqslant a.$$

Now

$$1 + \sum_{r=1}^{\infty} q_r^{(m)}(k,t)x^r = \prod_{i=1}^{m} \left(\frac{1+\alpha_i tx}{1-\alpha_i x}\right)^{k-1} \prod_{i=1}^{m} \left(\frac{1+\alpha_i tx}{1-\alpha_i x}\right)$$

$$= \left\{1 + \sum_{r=1}^{\infty} q_r^{(m)}(k-1,t)x^r\right\} \prod_{i=1}^{m} \left(\frac{1+\alpha_i tx}{1-\alpha_i x}\right)$$

$$= \left\{1 + \sum_{r=1}^{\infty} q_r^{(m)}(k-1,t)x^r\right\} \left(\frac{1+\alpha_1 tx}{1-\alpha_1 x}\right) \prod_{i=2}^{m} \left(\frac{1+\alpha_i tx}{1-\alpha_i x}\right)$$

Let

$$1 + \sum_{r=1}^{\infty} {}_r^{(m)}(k-1,t)x^r = \left\{1 + \sum_{r=1}^{\infty} q_r^{(m)}(k-1,t)x^r\right\} \left(\frac{1+\alpha_1 tx}{1-\alpha_1 x}\right)$$

$$= \left\{1 + \sum_{r=1}^{\infty} q_r^{(m)} k-1,t)\right\} \left\{\sum_{r=1}^{\infty} \alpha_1^r \, x^r (1+t) + 1\right\} .$$

Hence

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$$(7) \qquad {}^{(m)}_{r}(k-1),t) = q^{(m)}_{r}\,k-1,t) + (1+t)\sum_{j=1}^{r}\alpha_1^{j}\, q^{(m)}_{r-j}(k-1,t) .$$

Using (7) and (6) we have

$${}^{(m)}_{\alpha-\lambda}(k-1,t)\ {}^{(m)}_{b+\lambda}(k-1,t) \geq {}^{(m)}_{\alpha-\lambda-1}(k-1,t)\ {}^{(m)}_{b+\lambda+1}(k-1,t), \quad 0\leq\lambda<a \quad , \quad b\geq a.$$

Now consider $\left\{ 1 + \sum_{r=1}^{\infty} {}^{(m)}(k-1,t)x^{r} \right\} \left(\frac{1+\alpha_2 tx}{1-\alpha_2 x}\right)$ and similarly proceeding

for each variable $\alpha_3, \alpha_4, \ldots, \alpha_m$ we have

$$q^{(m)}_{a-\lambda}(k,t)\ q^{(m)}_{b+\lambda}(k,t) \geq q^{(m)}_{a-\lambda-1}(k,t)\ q^{(m)}_{b+\lambda+1}(k,t), \quad 0\leq\lambda<a \quad , \quad b\geq a.$$

When $k=1$, and $t=0$ we get (1).

THEOREM 2.

$$(8) \qquad \left\{ q^{(m)}_{i}(k,t) \right\}^{1/i} \geq \left\{ q^{(m)}_{i+1}(k,t) \right\}^{\frac{1}{i+1}} , \quad i=1,2,3,\ldots .$$

<u>The inequality is strict unless all but one variables are zeros and</u> $k=1$.

<u>Proof</u>. From (8), if $\lambda=0$ and $a=b$, we have

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$$\left[q_a^{(m)}(k,t)\right]^2 \geqslant q_{a-1}^{(m)}(k,t) \cdot q_{a+1}^{(m)}(k,t) . \tag{9}$$

Now (9) can be duduced from (8) as in [3].

THEOREM 3. *If* $1 + \sum_r E^{(m)}(k)x^r = \prod_{i=1}^{m} (1 + \alpha_i x)^k$, *where* k *is a positive integer and* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *are non-negative reals, then*

$$E_{a-\lambda}^{(m)}(k) \cdot E_{b+\lambda}^{(m)}(k) \geqslant E_{a-\lambda-1}^{(m)}(k) \quad E_{b+\lambda+1}^{(m)}(k) , \quad 0 \leqslant \lambda < a , \quad b \geqslant a.$$

The inequality is strict unless all but one of the variables are zero and $k=1$.

Proof. Same as theorem 1.

THEOREM 4.

$$\left\{E_i^{(m)}(k)\right\}^{1/i} \geqslant \left\{E_{i+1}^{(m)}(k)\right\}^{1/i+1} , \quad i=1,2,\ldots$$

The inequality is strict unless all but one of the variables are zero and $k=1$.

Proof. Same as theorem 2.

Note. The theorems proved here are true for a more general type of non-symmetric function whose generating series is defined by

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$$1 + \sum_r q_r^{(m)}(t,k_1,k_2,\ldots,k_m)x^r = \prod_{i=1}^{m} \left(\frac{1+\beta_i tx}{1-\alpha_i x}\right)^{k_i} ,$$

where $t \geq 0$, $k_1,k_2,\ldots,k_m$ are positive integers and $\alpha_1,\alpha_2,\ldots,\alpha_m$, $\beta_1,\beta_2,\ldots,\beta_m$ are non-negative reals. Using the same method in theorems 1 and 2, we can prove that

$$q_{a-\lambda}^{(m)}(t,k_1,k_2,\ldots,k_m) \cdot q_{b+\lambda}^{(m)}(t,k_1,k_2,\ldots,k_m) \geq$$

$$q_{a-\lambda-1}^{(m)}(t,k_1,k_2,\ldots,k_m) \cdot q_{b+\lambda+1}(t,k_1,k_2,\ldots,k_m), \quad 0 \leq \lambda < a \quad , \quad b \geq a$$

and

$$q_i^{(m)}(t,k_1,k_2,\ldots,k_m)^{1/i} \geq q_{i+1}(t,k_1,k_2,\ldots k_m)^{1/i+1} \quad , \quad i=1,2,3,\ldots$$

I wish to record my sincere thanks to the referee for suggestions which led to be a better presentation.

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