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MODULES WHOSE CLOSED SUBMODULES ARE FINITELY GENERATED

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A module M is called a CC-module if every closed submodule of M is cyclic. It is shown that a cyclic module M is a direct sum of indecomposable submodules if all quotients of cyclic submodules of M are CC-modules. This theorem generalizes a recent result of B. L. Osofsky and P. F. Smith on cyclic completely CS-modules. Some further applications are given for cyclic modules which are decomposed into projectives and injectives.

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In [**6, 7**] Osofsky proved that a ring R is semisimple Artinian if every cyclic right R-module is injective. Since that time, this important theorem has been extensively investigated and extended by several authors (see e.g. [**2, 3, 10, 11**]). More recently, in Huynh–Dung [**4**], an attempt was made to generalize Osofsky's theorem to cyclic injective modules. Using a result of Damiano [**3**], it was shown in [**4**] that a cyclic finitely presented module is semisimple if all quotients of cyclic submodules of M are injective. In the recent work [**8**], Osofsky and Smith have shown that the hypothesis "finitely presented" can be removed. More generally, they have proved that a cyclic module M has finite uniform dimension if all quotients of cyclic submodules of M are CS-modules. This general theorem covers all previously known results in the area.

The purpose of this note is to present some extensions of Osofsky–Smith's theorem in [**8**]. First we show that a cyclic module M is a direct sum of indecomposable submodules if all quotients of cyclic submodules of M have closed submodules cyclic. Our arguments use the idea of proof in [**8**]. A similar result holds also for finitely generated modules with closed submodules finitely generated. Further we prove that a finitely generated module M has finite uniform dimension if every quotient of a cyclic submodule of M is a direct sum of a projective module and a CS-module. As a consequence, we obtain a module-theoretic version of a result in [**8**] that right CDPI-rings are right Noetherian. Finally, an application is given for right linearly topologized rings.

1. Definitions and notation

Throughout this paper we consider associative rings with identity element and unitary right modules. A module M is said to have finite uniform dimension if M does not contain infinite direct sums of non-zero submodules. A submodule K of M is called essential in M if $K \cap L \neq 0$ for every non-zero submodule L of M. In this case, M is called an essential extension of K. A submodule C of M is closed in M iff C is the only essential extension of C in M.

Following [1], a module M is called CS provided every closed submodule of M is a direct summand of M, or equivalently, every submodule of M is essential in a direct summand of M. Now we introduce some new notions which generalize the concept of cyclic CS-modules.

Definition 1. We will call a module M a CC-module if every closed submodule of M is cyclic.

Definition 2. A module M is called a CF-module if every closed submodule of M is finitely generated.

It is clear that M is a CC-module (resp. CF-module) iff every submodule of M has a cyclic (resp. finitely generated) essential extension.

M is called completely CC (resp. completely CF) provided every quotient of M is also a CC-module (resp. CF-module).

For a module M, $\mathrm{Soc}(M)$ will denote the socle of M. M is semisimple iff $M = \mathrm{Soc}(M)$.

2. CC-modules and CF-modules

It is obvious that the class of cyclic CC-modules contains properly the class of cyclic CS-modules. Therefore our next result may be regarded as an extension of the theorem of Osofsky and Smith mentioned in the introduction.

Theorem 2.1. *Let M be a cyclic module such that all cyclic submodules of M are completely CC-modules. Then M is a direct sum of indecomposable submodules.*

Proof. Assume that M can not be decomposed as a direct sum of indecomposable submodules. Then there are non-zero submodules A_1, B_1 of M such that $M = A_1 \oplus B_1$, and B_1 is not a direct sum of indecomposable submodules. Again we have $B_1 = A_2 \oplus B_2$, where B_2 is not a direct sum of indecomposable submodules. Continuing in this manner, by finite induction, we get infinite sequences $\{A_i\}$ and $\{B_i\}$ of non-zero submodules of M such that $M = (\bigoplus_{i=1}^n A) \oplus B_n$ and $\bigoplus_{j=n+1}^\infty A_j \subseteq B_n$ for each $n \geqq 1$. Each A_i is cyclic, hence A_i contains a maximal submodule X_i. Consider the quotient module $K = M / \bigoplus_{i=1}^\infty X_i$. Then $S = \bigoplus_{i=1}^\infty A_i / \bigoplus_{i=1}^\infty X_i$ is a semisimple submodule of K. By hypothesis, K is a CC-module, thus S has a cyclic essential extension N in K. Now we aim to show that N/S is not a CC-module which would give us the desired contradiction. First we notice that by the construction, for each n, $S_n = \bigoplus_{i=1}^n (A_i/X_i)$ is a direct summand of $K = ((\bigoplus_{i=1}^n A_i) \oplus B_n) / \bigoplus_{i=1}^\infty X_i$, hence S_n is a direct summand of N. It follows that every finitely generated submodule of S, being a direct summand of some S_n, must be a direct summand of N.

Now we proceed similarly as in [8]. Since S is infinitely generated, we can write $S=\bigoplus_{i=1}^{\infty} F_i$, where each F_i is infinitely generated. Since N is a CC-module, each F_i has a cyclic essential extension D_i in N. Clearly $D_i \neq F_i$ for each i. Assume that $N'=N/S$ is a CC-module, then $(\bigoplus_{i=1}^{\infty} D_i+S)/S$ has a cyclic essential extension E' in N'. There exists a cyclic submodule E in N such that $(E+S)/S=E'$. Clearly $D_i \subseteq E+S$ for each i. Since S is semisimple, there is a submodule T of S such that $S=(E \cap S) \oplus T$, then $E+S=E \oplus T$. Suppose that $D_i \cap E=0$ for some i, then D_i is isomorphic to a submodule of T, thus D_i is semisimple, a contradiction. It implies that $D_i \cap E \neq 0$, hence $F_i \cap E \neq 0$. So for each i we can take a non-zero simple submodule $V_i \subseteq F_i \cap E$. Since E is a CC-module, $V=\bigoplus_{i=1}^{\infty} V_i$ has a cyclic essential extension L in E. Obviously $L \nsubseteq S$, hence $L'=(L+S)/S \neq 0$. We claim that $L \cap \bigoplus_{i=1}^{\infty} D_i \subseteq S$. In fact, for each n,

$$\left(L \cap \bigoplus_{i=1}^{n} D_i\right) \cap S=L \cap \bigoplus_{i=1}^{n} F_i=\bigoplus_{i=1}^{n} V_i.$$

But as we have remarked, $\bigoplus_{i=1}^{n} V_i$ is a direct summand of N, and since S is essential in N, it follows that $L \cap \bigoplus_{i=1}^{n} D_i \subseteq S$. This shows that $L \cap \bigoplus_{i=1}^{\infty} D_i \subseteq S$, contradicting the fact that E' is an essential extension of $(\bigoplus_{i=1}^{\infty} D_i+S)/S$. This completes the proof of the theorem.

Corollary 2.2 (Osofsky–Smith [8]). *Let M be a cyclic module such that all quotients of cyclic submodules of M are CS-modules. Then M is a direct sum of uniform submodules.*

Proof. By Theorem 2.1, M is a direct sum of indecomposable submodules. Now the result follows from the easily-proved fact that an indecomposable module is CS iff it is uniform.

Corollary 2.3. *Let R be a ring such that every cyclic right R-module is a CC-module. Then every cyclic right R-module is a direct sum of indecomposable submodules.*

Recall that a right R-module M is called singular if for each element x in M there exists an essential right ideal K of R such that $xK=0$. From Theorem 2.1 we immediately derive:

Corollary 2.4. *Let R be a ring such that every cyclic singular right R-module is a CC-module. Then every cyclic singular right R-module is a direct sum of indecomposable submodules.*

Next we will consider CF-modules. The following result can be obtained with a proof similar to that of Theorem 2.1.

Theorem 2.5. *Let M be a finitely generated module such that every finitely generated submodule of M is completely CF. Then M is a direct sum of indecomposable submodules.*

As an immediate consequence of Theorem 2.5 we have:

Corollary 2.6. *Let R be a ring for which every finitely generated right module is a CF-module. Then every finitely generated right R-module is a direct sum of indecomposable submodules.*

Corollaries 2.3 and 2.6 suggest the following natural question.

Question. Let R be a ring with the property that every cyclic right R-module is a CF-module. Is R necessarily a direct sum of indecomposable right ideals?

By Theorem 2.1, it is easily seen that the answer is "yes" if every finitely generated right ideal of R is principal. In particular, if R is a von Neumann regular ring, then R is semisimple Artinian iff every cyclic right R-module is a CF-module.

3. Decomposing cyclic modules into projectives and injectives

A ring R is called right PCI if every cyclic right R-module is injective or isomorphic to R_R (see Cozzens–Faith [2]). Damiano [3] proved that right PCI-rings are right Noetherian right hereditary. As a generalization of right PCI-rings, Smith [10, 11] introduced and investigated right CDPI-rings as those rings for which each cyclic right module is a direct sum of a projective module and an injective module. It was established recently in Osofsky–Smith [8, Proposition 2] that right CDPI-rings are right Noetherian right hereditary. In this section we shall prove a module-theoretic version of this result. It will be an easy consequence of the following more general theorem which is of independent interest.

Theorem 3.1. *Let M be a finitely generated module such that every quotient of a cyclic submodule of M is a direct sum of a projective module and a CS-module. Then M has finite uniform dimension.*

Proof. First we consider the case when M is a cyclic module. Let K be an essential submodule of M, then by hypothesis M/K is a direct sum of a projective module and a CS-module. It is easy to see that the projective direct summand must be zero, so M/K is CS. Similarly, all quotients of cyclic submodules of M/K are also CS. By Corollary 2.2, M/K has finite uniform dimension. It follows that if A and B are submodules of M such that A is essential in B, then B/A has finite uniform dimension. Let $S=\mathrm{Soc}(M)$ and E be a submodule of M with $S\subseteq E$. We show that M/E has finite uniform dimension. By Zorn's lemma, there is a submodule L of M such that $E\cap L=0$ and $E\oplus L$ is essential in M. Since $M/(E+L)$ has finite uniform dimension, it is enough to show that L also has this property. Similarly as in [5, Lemma 2], we assume that L contains an infinite direct sum $\bigoplus_{i=1}^{\infty} X_i$ of non-zero submodules X_i. Since $X_i\cap \mathrm{Soc}(M)=0$, each X_i contains a proper essential submodule Y_i. Then $\bigoplus_{i=1}^{\infty} Y_i$ is essential in $\bigoplus_{i=1}^{\infty} X_i$, and $\bigoplus_{i=1}^{\infty} X_i/\bigoplus_{i=1}^{\infty} Y_i$ has infinite uniform dimension, a contradiction. Thus M/E has finite uniform dimension.

Now consider a quotient module N of M such that N is a CS-module. We claim that N has finite uniform dimension. To see this, we first note that $N/\mathrm{Soc}(N)=M/E$ for some submodule E containing S, thus $N/\mathrm{Soc}(N)$ has finite uniform dimension. If $\mathrm{Soc}(N)$ is infinitely generated, then we can write $\mathrm{Soc}(N)=\bigoplus_{i=1}^{\infty} T_i$, where each T_i is infinitely generated. Since N is CS, each T_i is essential in a direct summand D_i of N. Clearly D_i is cyclic, hence $D_i \neq T_i$. Then $N/\mathrm{Soc}(N)$ contains an infinite direct sum $\bigoplus_{i=1}^{\infty}(D_i/T_i)$ of non-zero submodules, a contradiction. This shows that $\mathrm{Soc}(N)$ is finitely generated, thus N has finite uniform dimension.

To prove that $S=\mathrm{Soc}(M)$ is finitely generated, we apply the techniques used in the proof of [**12**, Lemma 2.6]. Assume the contrary that S is infinitely generated, then $S=S_1 \oplus S_2$, where each S_i is infinitely generated. By hypothesis, M/S_1 is a direct sum of a projective module and a CS-module. Then there is a direct summand A_1 of M such that $S_1 \subseteq A_1$ and A_1/S_1 is CS. From the argument above, we know that A_1/S_1 has finite uniform dimension. Let $M=A_1 \bigoplus A_2$, then $S=\mathrm{Soc}(A_1)\oplus\mathrm{Soc}(A_2)$. Since $\mathrm{Soc}(A_1)/S_1$ is finitely generated, it follows that $\mathrm{Soc}(A_2)$ is infinitely generated. Thus we have

$$M/S=(A_1/\mathrm{Soc}(A_1))\oplus(A_2/\mathrm{Soc}(A_2)),$$

where $A_i \neq \mathrm{Soc}(A_i)$, $i=1,2$. Since each A_i has the same properties as M does, we can apply the same argument to get a similar decomposition for A_i. Continuing in this manner, by induction, we conclude that M/S does not have finite uniform dimension, a contradiction. Therefore S is finitely generated which implies that M has finite uniform dimension.

If M is finitely generated, then $M=M_1+\cdots+M_n$, where each M_i is cyclic. As we have shown, all quotients of each M_i have finite uniform dimension. We have

$$(M_1+M_2)/M_1 \cong M_2/(M_1 \cap M_2).$$

From this it follows that M_1+M_2 has finite uniform dimension. By finite induction we get easily that M has finite uniform dimension. The proof is now complete.

As a consequence of Theorem 3.1 we obtain:

Proposition 3.2. *Let M be a finitely generated module such that each quotient of a cyclic submodule of M is a direct sum of a projective module and an injective module. Then M is Noetherian. In addition, if M is projective, then every submodule of M is projective.*

Proof. To prove that M is Noetherian, it is enough to consider the case when M is cyclic. Let K be an essential submodule of M. Similarly as in the proof of Theorem 3.1, we see that all quotients of cyclic submodules of M/K must be injective. Now from Corollary 2.2 it is clear that M/K must be semisimple. By [**5**, Lemma 2] we know that $M/\mathrm{Soc}(M)$ is Noetherian. On the other hand, by Theorem 3.1, M has finite uniform dimension, thus $\mathrm{Soc}(M)$ is finitely generated. This implies that M is Noetherian.

Now suppose that M is projective. Then every cyclic submodule of M is clearly projective. Let X be a cyclic submodule and P be a projective submodule of M. We will show that $X+P$ is projective. By hypothesis, $(X+P)/P=A\oplus B$, where A is injective and B is projective. Since the inverse image of B in $X+P$ is isomorphic to $P\oplus B$ which is projective, without loss of generality we may assume that $B=0$. Then

$$M/P=((X+P)/P)\oplus(Y/P)$$

for some submodule Y containing P. There is a homomorphism from $(X+P)\oplus Y$ onto M with a kernel isomorphic to P. Thus $X+P$ is isomorphic to a direct summand of $M\oplus P$, so $X+P$ is projective. Since every submodule of M is finitely generated, the result follows now by finite induction.

Finally, as an application of Proposition 3.2, we obtain an analogue to [**8**, Proposition 2] for linearly topologized rings. Recall that a topological ring R is called right linearly topologized if the open right ideals of R form a base of neighbourhoods of zero. In addition, if for each open right ideal A of R, R/A is a Noetherian right R-module, then R is called a right topologically Noetherian ring (see Sharpe–Vamos [**9**]). Now from Proposition 3.2 immediately follows:

Corollary 3.3. *Let R be a right linearly topologized ring. If each cyclic discrete right R-module is a direct sum of a projective module and an injective module, then R is right topologically Noetherian.*

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