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A COMMUTATIVITY THEOREM FOR RINGS

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We prove the following theorem: Let R be a ring, l a positive integer, and n a non-negative integer. If for each $x, y \in R$, either $xy = yx$ or $xy = x^n f(y) x^l$ for some $f(X) \in X^2\mathbf{Z}[X]$, then R is commutative.

Throughout, R will represent a ring with centre $C = C(R)$, and $D = D(R)$ the commutator ideal of R. Let l, m be positive integers, and n a non-negative integer. We consider the following conditions:

$(*)_{(l,m,n)}$ For each $x, y \in R$, either $[x, y] = xy - yx = 0$ or $x^m y = x^n f(y) x^l$ for some $f(X) \in X^2\mathbf{Z}[X]$.

$(**)_{(l,m,n)}$ For each $x, y \in R$, either $[x, y] = 0$ or $x^m y - x^n f(y) x^l \in C$ for some $f(X) \in X^2\mathbf{Z}[X]$.

$(\dagger)_{(l,m,n)}$ For each $x, y \in R$, there exists $f(X) \in X^2\mathbf{Z}[X]$ such that $[x, x^m y - x^n f(y) x^l] = 0$.

(S) For each $x, y \in R$, there exists $f(X, Y) \in \mathbf{Z}\langle X, Y\rangle[X, Y]\mathbf{Z}\langle X, Y\rangle$ each of whose monomial terms is of length $\geqslant 3$ such that $[x, y] = f(x, y)$.

As is easily seen, $(*)_{(l,m,n)}$ implies $(**)_{(l,m,n)}$, and $(**)_{(l,m,n)}$ implies $(\dagger)_{(l,m,n)}$. Recently, Bell [1] announced that every ring R satisfying $(*)_{(1,1,0)}$ is commutative. The next result has been proved in [3, Theorem 1].

PROPOSITION 1. *Let R be a ring with 1. If R satisfies $(\dagger)_{(l,m,n)}$ then R is commutative.*

Our present objective is to prove the following theorem, by making use of Proposition 1.

THEOREM 1. *If a ring R satisfies $(*)_{(l,1,n)}$, then R is commutative.*

We start our preparation for proving Theorem 1 with the following proposition.

PROPOSITION 2. *Let R be a ring generated by two elements such that D is the heart of R and $RD = DR = 0$. Then R is nilpotent.*

PROOF: Obviously, D is $\mathbf{Z}$-isomorphic to $\mathbf{Z}/p\mathbf{Z}$ for some prime p. Noting that R/D is a homomorphic image of the subring $\langle X, Y\rangle$ of $\mathbf{Z}[X, Y]$ and every ideal of

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$\langle X, Y\rangle$ is an ideal of $\mathbf{Z}[X, Y]$, we see that R/D is Noetherian. Accordingly, R is right Noetherian.

Now, let x be an arbitrary element of R, and k a positive integer such that $r(x^k) = r(x^{k+1})$, where $r(*)$ denotes the right annihilator of $*$ in R. Since $Rx^k \subseteq (x^kR + D)R \subseteq x^kR$, x^kR is an ideal of R. Further, if $x^ka \in x^kR \cap D$ then $x^{k+1}a = x(x^ka) = 0$, and so $x^ka = 0$. Hence $x^kR \cap D = 0$, whence $x^kR = 0$ follows. We have thus seen that R is nil. Now, it is easy to see that R is nilpotent. □

Combining Proposition 2 with [**2**, Theorem S], we see that if R is not commutative then there exists a factorsubring of R which is of type $(\mathrm{a})_l$, $(\mathrm{a})_r$, (b), (c), (d) or (e):

$(\mathrm{a})_l$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

$(\mathrm{a})_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$ p a prime.

(b) $M_\sigma(K) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \middle| \alpha, \beta \in K \right\}$, where K is a finite field with a non-trivial automorphism σ.

(c) A non-commutative division ring.

(d) A simple radical ring with no non-zero divisors of zero.

(e) A finite nilpotent ring S such that $D(S)$ is the heart of S and $SD(S) = D(S)S = 0$.

In particular, if R is non-commutative and satisfies (S) then there exists a factorsubring of R which is of type $(\mathrm{a})_l$, $(\mathrm{a})_r$ (b), (c) or (d) (see [**2**, Corollary S.1]).

This result gives the following Meta-Theorem.

Meta-Theorem. *Let P be a ring property which is inherited by factorsubrings. If no rings of type $(\mathrm{a})_l$, $(\mathrm{a})_r$, (b), (c), (d) or (e) satisfy P, then every ring satisfying P is commutative. (If no rings of type $(\mathrm{a})_l$, $(\mathrm{a})_r$, (b), (c) or (d) satisfy P, then every ring satisfying (S) and P is commutative.)*

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1: In view of the Meta-Theorem, it suffices to show that R cannot be of type $(\mathrm{a})_l$, $(\mathrm{a})_r$, (b), (c), (d) or (e).

Noting that $e_{12} = e_{11}e_{12} \neq e_{11}^n f(e_{12})e_{11}^l = 0$ and $e_{12} = e_{12}e_{22} \neq e_{12}^n f(e_{22})e_{12}^l = 0$ for any $f(X) \in X^2\mathbf{Z}[X]$, we see that R cannot be of type $(\mathrm{a})_l$ or $(\mathrm{a})_r$. Further, by Proposition 1, no rings of type (b) or (c) satisfy $(*)_{(l,1,n)}$.

Now, suppose that R is of type (d), and choose $x, y \in R$ with $[x, y] \neq 0$. Then there exists $p(X) \in X\mathbf{Z}[X]$ such that $xy = x^n p(y)yx^l$. If $[x, y^l] \neq 0$ and $[x^l, y] \neq 0$, there exist $f(X)$, $g(X) \in X^2\mathbf{Z}[X]$ such that $xy^l = x^n f(y^l)x^l$ and $yx^l = y^n g(x^l)y^l$. Putting $f(y^l) = f_0(y)y$ and $g(x^l) = g_0(x)x$ with some $f_0(X)$, $g_0(X) \in X\mathbf{Z}[X]$, we obtain $xy^l = x^n f_0(y) y^n g_0(x^l) xy^l$. Since R is a radical ring, this forces a contradiction $xy^l = 0$. Next, if $[x^l, y] = 0$ then $xy = x^n p(y) x^{l-1} xy$, which implies a contradiction $xy = 0$. Similarly, $[x, y^l] = 0$ forces a contradiction. We have thus seen that R cannot be of type (d).

Finally, suppose that R is of type (e). Then $R^2 \subseteq C$. Given $x, y \in R$ with $[x, y] \neq 0$, we can take $p(X) \in X\mathbf{Z}[X]$ such that $xy = x^n y p(y) x^l = xyp(y)x^{l+n-1}$, whence $xy = 0$ follows; similarly $yx = 0$. But this is impossible. ∎

Corollary 1. *If R satisfies (S) and $(**)_{(l,1,n)}$, then R is commutative.*

Proof: In view of Proposition 1 and the Meta-Theorem, it suffices to show that R cannot be of type $(a)_l$, $(a)_r$, or (d). It is easy to see that R is not of type $(a)_l$ or $(a)_r$. If R is of type (d), then $C = 0$ and R satisfies $(*)_{(l,1,n)}$. Thus R is commutative by Theorem 1. But this is impossible. ∎

Finally, we remark that a ring with $(*)_{(l,m,n)}$ for $m > 1$ need not be commutative. Actually, there exists a non-commutative ring R with $R^3 = 0$.

References

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