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Bending Flows for Sums of Rank One Matrices

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Abstract. We study certain symplectic quotients of n-fold products of complex projective m-space by the unitary group acting diagonally. After studying nonemptiness and smoothness of these quotients we construct the action-angle variables, defined on an open dense subset, of an integrable Hamiltonian system. The semiclassical quantization of this system reporduces formulas from the representation theory of the unitary group.

1 Introduction

The aim of this paper is to generalize results of Kapovich and Millson [KM96] and Klyachko [Kly92] on the moduli space $M_{\mathbf{r}}$ of (closed) polygons in $\mathbb{R}^3$ with prescribed sidelengths $\mathbf{r} = (r_1, \dots, r_n)$, polygons related by a Euclidean motion being identified. They showed that $M_{\mathbf{r}}$ is a (possibly singular) symplectic manifold and introduced a class of commuting Hamiltonian flows, the so-called *bending flows*. These flows bend the polygon about the diagonals emanating from one fixed vertex. The part of the polygon to one side of the diagonal does not move, while the other part rotates at constant speed. The lengths of the diagonals are action variables which generate the bending flows; the conjugate angle variables are the dihedral angles between the fixed and the moving parts.

We generalize this picture by replacing vectors in $\mathbb{R}^3$ by positive semidefinite rank-one Hermitean matrices. These have the form $e = rw \otimes w^*$, where $r > 0$ and w is a unit vector in $\mathbb{C}^{m+1}$. Explicitly, $e \colon v \mapsto r(v, w)w$ where $(\ ,\ )$ is the standard positive definite Hermitean form on $\mathbb{C}^{m+1}$. The edges of a polygon will be $e_i = r_i w_i \otimes w_i^*, i = 1, \dots, n$, $r_i > 0$ fixed, and *closed* will mean *closed up to a multiple of the identity*,

$$e_1 + \cdots + e_n = \Lambda \mathbb{I}.$$

Equality of traces forces $\Lambda = (r_1 + \cdots + r_n)/(m + 1)$. Polygons are identified if they are related by simultaneous rotation of the sides by an element of U(m+1). Since $w \otimes w^*$ is unchanged if w is multiplied by $\exp(\sqrt{-1}\,\theta)$, we may think of an edge $rw \otimes w^*$ as a weighted point in the projective space $\mathbb{CP}^m$ and of a polygon as a weighted configuration of points in $\mathbb{CP}^m$.

The paper has three parts.

(1) A study of the moduli space $M_{\mathbf{r}}$. Criteria for nonemptiness and nonsingularity of $M_{\mathbf{r}}$ are found.

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(2) A generalization of bending flows and their action-angle coordinates.
(3) A relation between the bending flows and representations of U(m+1); this is reminiscent of geometric quantization.

This is the logical progression of the material, but (1) stands alone and (2), about the bending flows, can be read independently of the rest. We give a brief outline of each part.

It follows from general results of [Kly98] (however we sketch a direct proof in what follows) that $M_{\mathbf{r}}$ is nonempty if and only if the side lengths $\mathbf{r}$ satisfy the *generalized triangle inequalities*,

$$mr_i \leq r_1 + \cdots + \hat{r}_i + \cdots + r_n, \quad 1 \leq i \leq n.$$

This defines a convex cone $C(n, m+1)$ in $\mathbb{R}^n$. When $n = 3$, the generalized triangle inequalities are the usual conditions for r_1, r_2, r_3 to be the sides of a Euclidean triangle, namely

$$r_1 \leq r_2 + r_3, \quad r_2 \leq r_1 + r_3, \quad r_3 \leq r_1 + r_2.$$

The proof of necessity is elementary, and is given in §3.1. Sufficiency is deeper. Only an outline of the argument is given in §3.2 and §3.3; for details the reader is referred to the literature. It follows from standard results (see for example [MFK, Theorem 8.3]) that $M_{\mathbf{r}}$ is canonically homeomorphic to a weighted (by $\mathbf{r}$) complex analytic quotient of the n-fold product $\prod_1^n \mathbb{CP}^m$. For $m = 1$, *i.e.,* for spatial polygons, the connection between weighted analytic quotients of $\mathbb{CP}^1$ and $M_{\mathbf{r}}$ was found independently in [KM96] and [Kly92]. The weighted quotient is nonempty if and only if there exists a weighted semistable configuration of points on $\mathbb{CP}^m$. If $\mathbf{r}$ satisfies the strong triangle inequalities, then any n-tuple of points on $\mathbb{CP}^m$ in general position (these always exist) is semistable for the weights $\mathbf{r}$.

In the Euclidean case, it was shown in [KM96] that $M_{\mathbf{r}}$ is smooth if there is no polygon contained in a line. The corresponding statement in our case is that $M_{\mathbf{r}}$ is smooth if there is no *decomposable* polygon. A polygon is decomposable, roughly speaking, if there are closed subpolygons contained in orthogonal subspaces of $\mathbb{C}^{m+1}$. The side lengths $\mathbf{r}$ for which a decomposable polygon exists lie on certain hyperplane sections of the cone $C(n, m+1)$ called *walls*. A connected component of the complement of the walls is a *chamber*. We prove that $M_{\mathbf{r}}$ and $M_{\mathbf{r}'}$ are diffeomorphic if $\mathbf{r}, \mathbf{r}'$ lie in the same chamber. The topology of $M_{\mathbf{r}}$ will change as $\mathbf{r}$ crosses a wall (see for example [Hu] and [Goldin]).

Turning now to the bending flows, we remark first that the action variables, or Hamiltonians generating the "bending", are a natural generalization of the Euclidean case. There, one has $\mathbb{R}^3 \equiv \mathfrak{su}(2)$; a diagonal A of a polygon is identified with an element $\hat{A} \in \mathfrak{su}(2)$, and the length $\|A\|$ is just the positive eigenvalue of $\hat{A}$. In our generalization, the bending Hamiltonians are also the eigenvalues, λ_{ij}, $j = 1, \ldots, m+1$, of the diagonals $A_i = e_1 + \cdots + e_{i+1}$, and they generate 2π-periodic flows. These flows again leave part of the polygon fixed and conjugate the other part by $\exp(\sqrt{-1}\, tE_{ij})$, where E_{ij} is the spectral projection for λ_{ij}. This is a kind of bending with "internal degrees of freedom". Enough of the λ_{ij} are functionally independent to give action coordinates (only on a dense open set, however). The eigenvalues of A_{i+1} and A_i interlace; this is a direct consequence of the Weinstein–Aronszajn formula from perturbation theory. The interlacing property can be pictured by a triangular organization of the λ_{ij}, starting with the eigenvalue r_1 of $e_1 = r_1 w_1 \otimes w_1^*$ and working up:

$$(*) \qquad \begin{array}{ccccc} \lambda_{21} & & \lambda_{22} & & \lambda_{23} \\ & \lambda_{11} & & \lambda_{12} & \\ & & r_1 & & \end{array}$$

This is called a *Gel'fand–Tsetlin* pattern. As long as these inequalities are respected, the λ_{ij} can be prescribed arbitrarily. Gel'fand–Tsetlin patterns were introduced in the Euclidean context by Hausmann and Knutson [HK97]. They showed, and this extends easily to our setting, that the bending flows and the Gel'fand–Tsetlin flows of Guillemin and Sternberg [GS83] are dual to each other (via the Gel'fand–MacPherson duality [GGMS]).

The angle variables also extend the Euclidean case (dihedral angles), but in a rather more subtle way. For vectors w, x, y, z in $\mathbb{C}^{m+1}$, one defines the *four-point function* [BeSch]

$$F(w, x, y, z) = (w, x)(x, y)(y, z)(z, w).$$

It is independent of the phases of its arguments, and so $\arg F$ is well defined on $\mathbb{CP}^m$. The arguments of F in our setting will be the w_i that define the edges $e_i = r w_i \otimes w_i^*$, and eigenvectors u_{ij} corresponding to λ_{ij}. These are only defined up to phase, but using $\arg F(w_{i+1}, u_{ij}, w_{i+2}, u_{i,j+1})$ we get global angle variables (on a dense open set again). In the Euclidean case, this amounts to a rather complicated way of expressing a dihedral angle. The proof of the Poisson bracket relations, $\{\text{angle}, \text{angle}\} = 0$, etc., takes up all of §7.

The connection between bending flows and representation theory follows the ideas of Guillemin and Sternberg [GS83]. In Bohr–Sommerfeld quantization, one asks that the action variables take on integral values. If the r_i, λ_{ij}, and $\left(\sum_i r_i\right)/(m+1)$ are integers, the interlacing property of the λ_{ij} reproduces the Pieri formula for the decomposition of tensor products of symmetric powers

$$(**) \qquad \bigotimes S^{r_i}(\mathbb{C}^{m+1})$$

of the basic representation of U(m+1) on $\mathbb{C}^{m+1}$.

The Hausmann–Knutson duality also has a representation-theoretic meaning. Gel'fand–Tsetlin patterns were invented to parameterize bases for vector spaces carrying representations of unitary (and general linear) groups. The possible patterns $(*)$ built from (integer) eigenvalues of successive diagonals index vectors of weight $\mathbf{r} = (r_1, \dots, r_n)$ in the (Grassmannian) representation of U(n) with highest weight

$$(***) \qquad (\underbrace{\Lambda, \dots, \Lambda}_{m+1}, 0, \dots, 0).$$

We conclude that the multiplicity of the one-dimensional representation $\det^{\Lambda}$ in the tensor product $(**)$ of U(m+1) representations equals the multiplicity of the weight $\mathbf{r}$ in the representation $(***)$ of U(n).

It must be said that these results are based on counting lattice points in convex polytopes and comparing their number with multiplicities known in representation theory. We do not construct actual representation spaces by any quantization method.

It is our hope that there are analogous results for all symplectic quotients of products of flag manifolds. In general for such products, one can find integrable systems that reduce to ours in the case of projective space, but it appears very hard to construct an explicit family of Hamiltonians *with periodic flows*, *i.e.*, action variables. If such a construction could be carried out and the associated momentum polytope could be computed, then by counting lattice points in the momentum polytope one could find information on decomposing tensor products of irreducible representations. Many deep connections are now known between tensor product decompositions and convex polyhedra; these, however, do not seem to arise as images of momentum mappings. One of the main motivations for our paper is that the special case treated here is probably the only case where everything can be worked out with simple explicit formulas.

We conclude by noting that the spaces $M_{\mathbf{r}}$ studied in this paper were the subject of the book, [DO]. The study of the spaces $M_{\mathbf{r}}$ in [DO] was from the point of view of algebraic geometry and combinatorics, necessitating the restriction to the case in which the r_i's were integral (moreover the authors assumed that all the r_i's were equal). There appear to be interesting relations between our work and theirs.

2 The Moduli Space of Polygons in $\mathcal{H}_{m+1}$

In this section, we collect the notation used throughout and in particular, introduce the moduli space of polygons with which we will be concerned.

2.1 Coadjoint Orbits

(1) Let $\mathcal{H}_{m+1}$ be the vector space of $(m+1)\times(m+1)$ Hermitean matrices. We identify it with the dual of the Lie algebra $\mathfrak{u}(m+1)$ via the pairing $\langle \xi, X\rangle = \operatorname{Im}\operatorname{Tr}\xi X$, for $\xi \in \mathfrak{u}(m+1), X \in \mathcal{H}_{m+1}$.

(2) $\mathcal{H}^0_{m+1} = \{X \in \mathcal{H}_{m+1} \mid \operatorname{Tr} X = 0\}$. It is the dual of $\mathfrak{su}(m+1)$.

(3) The gradient $\nabla f(X) \in \mathfrak{u}(m+1)$ of a smooth function f on $\mathcal{H}_{m+1}$ is defined by

$$\operatorname{Im}\operatorname{Tr}(\nabla f(X)Y) = \frac{d}{dt}\Big|_{t=0} f(X+tY), \quad \text{for all } Y \in \mathcal{H}_{m+1}.$$

(4) A U(m+1)-orbit $\mathcal{O} \subset \mathcal{H}_{m+1}$ carries the Kostant–Kirillov symplectic form ω_{KK} defined by

$$\omega_{KK}(X)([\xi, X], [\eta, X]) = \operatorname{Im}\operatorname{Tr}(X[\xi, \eta]).$$

The Lie–Poisson bracket is

$$\{f, g\}(X) = \operatorname{Im}\operatorname{Tr}(X\,[\nabla f(X), \nabla g(X)]), \tag{2.1.1}$$

and Hamilton's equations have the form

$$\dot{X} = [\nabla f(X), X]. \tag{2.1.2}$$

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In §7.1, we use the identification between $\mathcal{H}_2^0$ with bracket (2.1.1), and Euclidean space $\mathbb{R}^3$ with its standard Poisson bracket.

(5) Let $w \in \mathbb{C}^{m+1}$ be a unit vector. Define $w \otimes w^* \in \mathcal{H}_{m+1}$ by $w \otimes w^*(v) = (v, w)w$; it is a rank-one projection. The matrices $rw \otimes w^*$, $r > 0$, form an orbit $\mathcal{O}_r$ of U(m+1). They will be "edges" of polygons, and are denoted by the letter e. Given $e \in \mathcal{O}_r$, the unit vector w is determined up to multiplication by a complex number of modulus one. Hence $\mathcal{O}_r$ is diffeomorphic to $\mathbb{CP}^m$. As symplectic manifolds, they are related by $\omega_{KK} = r\omega_{FS}$, where ω_{FS} is the Fubini–Study form on $\mathbb{CP}^m$.

Remark 2.1.1 For completeness, we verify the last assertion about the symplectic forms. The Fubini–Study metric is the $U(m+1)$-invariant Kähler metric normalized so that the holomorphic sectional curvature is 1. To determine the scalar multiple relating two $U(m+1)$ invariant symplectic forms on $\mathbb{CP}^m$, it suffices to compute the period of each over a (complex linearly) embedded projective line $\mathbb{CP}^1 \subset \mathbb{CP}^m$. For the Fubini–Study metric, a projective line has curvature 1, and is therefore isometric to a sphere of radius 1 and area 4π. We may obtain such a $\mathbb{CP}^1$ by embedding $\mathcal{H}_2^0 \subset \mathcal{H}_{m+1}$ into the principal 2×2 block. Thanks to the identification of $\mathcal{H}_2^0$ with $\mathbb{R}^3$, it now suffices to relate the Kostant–Kirillov form on an orbit of $\mathfrak{su}(2)$ on $\mathbb{R}^3$, *i.e.,* on a sphere S_r^2 of radius r, to the usual area form dA. The calculation in [MR, p. 460] shows that $\omega_{KK}(S_r^2) = dA/r$. The symplectic area is $4\pi r$, which agrees with the area given by the scaled Fubini–Study form, $r\omega_{FS}$, on the embedded $\mathbb{CP}^1 \subset \mathcal{O}_r$.

2.2 The Space of Closed Polygons

Let $\mathbf{r} = (r_1, r_2, \dots, r_n)$ be an n-tuple of positive numbers. We define a *(closed) polygon with side-lengths* $\mathbf{r}$ to be an n-tuple $\mathbf{e} = (e_1, e_2, \dots, e_n)$ such that for all i, $1 \le i \le n$ we have

(a) $e_i \in \mathcal{O}_{r_i}$,
(b) $\sum_1^n e_i = \Lambda \mathbb{I}$.

Note that $\Lambda = \frac{1}{m+1}\sum_1^{m+1} r_i$ follows from equality of traces in (b). We call the matrices e_i the *edges* of the polygon $\mathbf{e}$ and r_i the *length* of the edge e_i. Condition (b) says that the polygon $\mathbf{e}$ is closed, modulo the center of $\mathcal{H}_{m+1}$.

(1) When $\mathbf{r}$ is given, Λ always stands for $\frac{1}{m+1}\sum r_i$. Sometimes the notation $\Lambda_\mathbf{r}$ is used to emphasize the dependence of Λ on $\mathbf{r}$.
(2) Given $\mathbf{r}$, define $\widetilde{N}_\mathbf{r}$ to be the product symplectic manifold $\prod_1^n \mathcal{O}_{r_i}$. The diagonal action of U(m+1) on $\widetilde{N}_\mathbf{r}$ is Hamiltonian with momentum map $\mu_\mathbf{r}$ given by

$$\mu_\mathbf{r}(\mathbf{e}) = \sum_1^n e_i.$$

We refer to elements of $\widetilde{N}_\mathbf{r}$ as *linkages*; they may or may not be closed.

(3) Given $\mathbf{r}$, let

$$\widetilde{M}_\mathbf{r} = \mu_\mathbf{r}^{-1}(\Lambda\mathbb{I}) = \left\{ \mathbf{e} \in \widetilde{N}_\mathbf{r} \;\middle|\; \sum_{i=1}^n e_i = \Lambda\mathbb{I} \right\}.$$

The elements of $\widetilde{M}_{\mathbf{r}}$ are the *polygons*, or *closed polygons* for emphasis. The unitary group acts diagonally on $\widetilde{M}_{\mathbf{r}}$. We let $\mathbf{i}\colon \widetilde{M}_{\mathbf{r}} \to \widetilde{N}_{\mathbf{r}}$ be the inclusion.

(4) Finally, we define the moduli space, $M_{\mathbf{r}}$, of polygons (with side-lengths $\mathbf{r}$) to be the quotient of $\widetilde{M}_{\mathbf{r}}$ by the diagonal action of U(m+1).

Because the stabilizer of the scalar matrix $\Lambda\mathbb{I}$ is all of U(m+1), we obtain:

Lemma 2.2.1 *$M_{\mathbf{r}}$ is the symplectic quotient of $\widetilde{N}_{\mathbf{r}}$ corresponding to the (one-point) orbit $\Lambda\mathbb{I} \in \mathcal{H}_{m+1}$ under the diagonal action of U(m+1).*

3 Nonemptiness of the Moduli Spaces

A simple set of inequalities on the side-lengths r_i is necessary and sufficient for the moduli space to be nonempty. The elementary proof of necessity is given first. Sufficiency is deeper and is based on the interpretation of polygons as weighted sets of points in projective space. For the sake of completeness, that argument will be summarized in §3.2 with references to literature where details may be found.

3.1 Necessity of the Triangle Inequalities

Theorem 3.1.1 *The moduli space $M_{\mathbf{r}}$ is nonempty if and only if $\mathbf{r}$ satisfies the system of inequalities*

$$mr_i \le r_1 + r_2 + \cdots + \widehat{r_i} + \cdots + r_n,\ 1 \le i \le n.$$

Here $\widehat{r_i}$ means that r_i has been omitted in the summation.

Remark 3.1.2 We will call this system of inequalities (together with the inequalities $r_i \ge 0$, $1 \le i \le n$) the *strong triangle inequalities of weight m*. When $m = 1$, they give the familiar conditions $r_1 \le r_2 + r_3$, $r_2 \le r_1 + r_3$, $r_3 \le r_1 + r_2$ on the side lengths of a planar triangle. We will omit reference to the weight m when it is clear from the context. Note that if we define $\rho = \sum_i r_i$ then the i-th inequality above is equivalent to

$$r_i \le \frac{1}{m+1}\,\rho. \tag{3.1.1}$$

We now turn to the proof of the necessity of the triangle inequalities.

Definition 3.1.3 Let $X \in \mathcal{H}^0_{m+1}$. Say that X is *maximally singular* if X is conjugate to a diagonal matrix with eigenvalues $(r, -\frac{r}{m}, \dots, -\frac{r}{m})$. We note that the orbit $\mathcal{O}^0_r$ under U(m+1) of such an X is the projection onto tracefree matrices of the orbit $\mathcal{O}_r$ through $\operatorname{diag}(r, 0, \dots, 0)$.

Lemma 3.1.4 *Suppose $X_1, X_2 \in \mathcal{H}^0_{m+1}$ are distinct, maximally singular, and satisfy $\operatorname{Tr}(X_j^2) = 1$. Then $\operatorname{Tr}(X_1X_2) \ge -1/m$, with equality if and only if X_1 and X_2 commute.*

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Proof We may write

$$X_j = \sqrt{\frac{m+1}{m}}\left(w_j \otimes w_j^* - \frac{1}{m+1}\mathbb{I}\right),$$

where $\|w_j\| = 1,\ j = 1, 2$. Then

$$\begin{aligned}\operatorname{Tr} X_1X_2 &= \frac{m+1}{m}\operatorname{Tr}\left[\left(w_1 \otimes w_1^* - \frac{1}{m+1}\mathbb{I}\right)(w_2 \otimes w_2^*)\right]\\ &= \frac{m+1}{m}\left[|(w_1, w_2)|^2 - \frac{1}{m+1}\right] \geq \frac{m+1}{m}\cdot - \frac{1}{m+1}\\ &= -\frac{1}{m}.\end{aligned}$$

Clearly we have equality if and only if $(w_1, w_2) = 0$ if and only if X_1 and X_2 commute. ■

Proposition 3.1.5 *Suppose that $M_{\mathbf{r}}$ is nonempty. Then* $\mathbf{r}$ *satisfies the strong triangle inequalities of weight m.*

Proof Choose $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}$. Then $e_1 + \cdots + e_n = \Lambda\mathbb{I}$ is equivalent to $r_1X_1 + \cdots + r_nX_n = 0$, where the matrices

$$X_j = \sqrt{\frac{m+1}{m}}\left(w_j \otimes w_j^* - \frac{1}{m+1}\mathbb{I}\right)$$

satisfy the hypotheses of Lemma 3.1.4. Alternatively,

$$r_iX_i = -r_1X_1 - \cdots - \widehat{r_iX_i} - \cdots - r_nX_n.$$

Multiply each side by X_i and take the trace to obtain

$$r_i^2 = -\sum_{j(\neq i)} r_ir_j \operatorname{Tr}(X_jX_i) \leq \frac{1}{m}\sum_{j(\neq i)} r_ir_j.$$

Now divide both sides by r_i to obtain the result. ■

The generalized triangle inequalities define a cone in $(\mathbb{R}_+)^n$:

Definition 3.1.6

$$C(n, m+1) = \{\mathbf{r} \in (\mathbb{R}_+)^n \mid M_{\mathbf{r}} \neq \varnothing\}.$$

The intersection of $C(n, m+1)$ with the hyperplane $\sum r_i = m+1$ is known in the literature as the *hypersimplex*. Side lengths $\mathbf{r}$ for which the moduli space $M_{\mathbf{r}}$ is singular will be shown to lie on certain hyperplane sections of $C(n, m+1)$.

We next give the outline of the proof that $M_{\mathbf{r}}$ is nonempty when $\mathbf{r} \in C(n, m+1)$. The method is based on the identification of an edge $r_iw_i \otimes w_i^*$ with a point $\mathbb{C}w_i \in \mathbb{CP}^m$ of weight r_i. The nonemptiness of the moduli space of such weighted points follows from a comprehensive general theory.

3.2 Semistability and Sufficiency of the Triangle Inequalities

3.2.1 Analytic Quotients and Symplectic Quotients

In [Sj95] and [HL94], the authors constructed the analytic quotient of a (not necessarily projective) compact Kähler manifold M by the action of a complex reductive group G. It is assumed that some maximal compact subgroup $K \subset G$ acts in a Hamiltonian fashion on M with momentum map μ. In their theory, a point $m \in M$ is defined to be *semistable* if the closure of the orbit $G \cdot m$ intersects the subset $\mu^{-1}(0)$ of M. The set of semistable points is denoted by M^{sst}; it is open in M. A point of M is defined to be *nice semistable* if the orbit itself intersects $\mu^{-1}(0)$. Define an equivalence relation, called *extended orbit equivalence*, by declaring two points to be related if their orbit closures intersect. (That this is indeed an equivalence relation follows from a theorem asserting that each equivalence class of semistable points contains a unique nice semistable orbit). We emphasize that the notion of semistability *depends on the symplectic structure*, *i.e.,* on $\mathbf{r}$ in our case.

The *analytic quotient* of M by G, denoted $M//G$, is then defined to be the quotient of M^{sst} by extended orbit equivalence. It is a hard theorem of [Sj95] and [HL94] that the resulting quotient topological space is Hausdorff and compact and in fact has a canonical structure of a complex analytic space.

Since, by definition any point in $\mu^{-1}(0)$ is (nice) semistable, there is an induced map from the symplectic quotient $\mu^{-1}(0)/K$ to the analytic quotient. [Sj95] and [HL94] prove that this map is a homeomorphism.

These results were proved earlier for smooth quotients and when the quotient has only orbifold singularities (*i.e.,* the stabilizer of every $x \in \mu^{-1}(0)$ is finite) in [Ki, Theorem 7.5] and also for general point stabilizers if M is a smooth complex projective variety and the symplectic form represents the (dual of the) hyperplane section class in [Ki, Remark 8.14]. In our setting, this amounts to assuming that either $\mathbf{r}$ is not on a wall or that $\mathbf{r}$ is integral (it is probable that some weakening of this condition will still result in an integral Kähler class). In this case the analytic quotient $M//G$ is a complex projective variety.

We have seen that our space $M_{\mathbf{r}}$ is a symplectic quotient of $(\mathbb{CP}^m)^n$ by U(m+1), where the ith factor is given the symplectic structure which is r_i times the usual Fubini–Study form. In the next subsection we will describe the corresponding *analytic* quotient of $(\mathbb{CP}^m)^n$ by GL(m+1,$\mathbb{C}$) in the sense indicated above. In particular, when $\mathbf{r}$ is integral then $M_{\mathbf{r}}$ will have a canonical structure of a complex projective variety.

3.2.2 Weighted Semistable Configurations on $\mathbb{CP}^m$

In this subsection we describe the semistable configurations on $(\mathbb{CP}^m)^n$ equipped with the $\mathbf{r}$-dependent symplectic structure just described. Set $[n] = \{1, 2, \ldots, n\}$.

Definition 3.2.1 A configuration of n points on $\mathbb{CP}^m$ is a map f from $[n]$ to $\mathbb{CP}^m$.

Let $\nu_{\mathbf{r}}$ be the measure on $[n]$ that assigns mass r_i to the point i. Also, recall that we have defined $\rho = \sum r_i$.

The proof of the following theorem is left to the reader. For the case of integral weights it is one of the standard results in Geometric Invariant Theory. For example, when all weights are $\frac{1}{n+1}$ (so $\rho = 1$), it is proved in [MFK, Definition 3.7/Proposition 3.4].

Theorem 3.2.2 *A configuration f on $\mathbb{CP}^m$ is semistable if and only if*

$$f_* \nu_{\mathbf{r}}(L) \leq \frac{\dim L + 1}{m+1}\, \rho$$

for any linear subspace L of $\mathbb{CP}^m$

Remark 3.2.3 In this inequality, the left side is the mass of the closed subset $L \subset \mathbb{CP}^m$ for the push-forward measure $f_* \nu_{\mathbf{r}}$. Intuitively, these semistability inequalities say that not too many points can coincide, not too many can lie on a line, not too many on a plane, etc.

The configuration f is said to be "in general position" if no two points coincide, at most two points lie on a line, at most three on a plane, ..., at most $k+1$ lie in a projective subspace of dimension k. The set of configurations in general position is a nonempty Zariski-dense open subset of $(\mathbb{CP}^m)^n$. In particular, such configurations exist. Thus the result that $\mathbf{r}$ satisfies strong triangle inequalities implies $M_{\mathbf{r}}$ is nonempty is an immediate consequence of the following

Proposition 3.2.4 *Suppose that f is in general position. Then f is weighted semistable if and only if $\mathbf{r}$ satisfies the generalized triangle inequalities.*

Proof Since any subset of k points in general position spans a projective subspace of projective dimension $k-1$, it is an immediate consequence of Theorem 3.2.2 that f is semistable if and only if for all $I \subset \{1, 2, \ldots, n\}$

$$\sum_{i \in I} r_i \leq \frac{|I|}{m+1}\, \rho.$$

Clearly, the resulting system of inequalities contains, and is implied by, the subset in which $|I| = 1$:

$$r_i \leq \frac{1}{m+1} \rho.$$

We have already noted (3.1.1) that this system is equivalent to the system of strong triangle inequalities. ∎

Since configurations in general position always exist, we have the missing implication in Theorem 3.1.1.

Corollary 3.2.5 *$M_{\mathbf{r}}$ is nonempty if $\mathbf{r}$ satisfies the strong triangle inequalities.*

4 Smoothness of the Moduli Spaces

In this section we give a sufficient condition in terms of $\mathbf{r}$ for the space $M_{\mathbf{r}}$ to be smooth.

4.1 Decomposable Polygons

For $m = 1$, it was shown in [KM96] that $M_{\mathbf{r}}$ will have singularities if and only if the index set $\{1, \dots, n\}$ can be partitioned into proper subsets I, J so that

$$\sum_{i \in I} r_i = \sum_{j \in J} r_j. \tag{4.1.1}$$

Then there exists a polygon (in Euclidean space) with the given side lengths $\mathbf{r}$ that is contained in a line segment; such a polygon was called *degenerate*. It was further proved in [KM96] that a polygon is a singular point of $M_{\mathbf{r}}$ if and only if it is degenerate. We need analogs of (4.1.1) and of the notion of degenerate polygon for the case $m \geq 2$.

Definition 4.1.1 For $1 \leq k \leq m$ and $I \cup J$ a proper partition of $\{1, \dots, n\}$, define the hyperplane

$$H_{I,J,k} = \left\{ \mathbf{r} \in \mathbb{R}^n_+ \;\middle|\; k \sum_{i \in I} r_i = (m - k + 1) \sum_{j \in J} r_j \right\}.$$

(Note that this reduces to (4.1.1) when $m = 1$). The *wall* corresponding to this hyperplane is the intersection

$$W_{I,J,k} = H_{I,J,k} \cap C(n, m + 1)$$

(cf. Definition 3.1.6).

Notation 4.1.2 We will write $I = \{i_1, \dots, i_p\}$, $J = \{j_1, \dots, j_k\}$, $p + q = n$, and take I and J to be ordered, $i_1 < i_2 < \cdots$, $j_1 < j_2 < \cdots$. Set $\mathbf{r}_I = (r_{i_1}, \dots, r_{i_p})$, and likewise for J. Let $\rho_I = \sum_{i \in I} r_i$ (similarly for ρ_J), and define

$$\Lambda_I = \rho_I/(m - k + 1), \quad \Lambda_J = \rho_J/k,$$

by analogy with $\Lambda = \rho/(m + 1)$.

Lemma 4.1.3 *Suppose that $\mathbf{r} \in W_{I,J,k}$. Then $\mathbf{r}_I$ (resp., $\mathbf{r}_J$) satisfies the strong triangle inequalities with weight $m - k$ (resp., $k - 1$). Explicitly:*

$$r_i \leq \frac{1}{m - k + 1} \rho_I, \quad \textit{for all } i \in I$$
$$r_j \leq \frac{1}{k} \rho_J, \quad \textit{for all } j \in J.$$

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Proof We show that $\mathbf{r}_I$ satisfies the strong triangle inequalities with weight $m-k$. According to Definition 4.1.1, $k\rho_I = (m-k+1)\rho_J$. Obviously, $\rho_I + \rho_J = \sum r_i = \rho$. Solving these two equations for ρ_I, we get $(m+1)\rho_I = (m-k+1)\rho$, or

$$\frac{\rho_I}{m-k+1} = \frac{\rho}{m+1}.$$

Since $\mathbf{r}$ already satisfies the strong triangle inequalities of weight m, (3.1.1) shows the right side to be greater than r_i. This gives the desired inequality for ρ_I; the proof for ρ_J is similar. ■

Lemma 4.1.4 *If* $\mathbf{r} \in W_{I,J,k}$*, then* $\Lambda_I = \Lambda_J = \Lambda$.

Proof Since $\mathbf{r} \in W_{I,J,k}$, we have $k\rho_I = (m-k+1)\rho_J$ which implies $\Lambda_I = \Lambda_J$. Furthermore,

$$k\rho = k\rho_I + k\rho_J = (m-k+1)\rho_J + k\rho_J = (m+1)\rho_J,$$

whence $\Lambda = \rho/(m+1) = \rho_J/k = \Lambda_J$. ■

We will see that if $\mathbf{r}$ does not lie on a wall, then $M_{\mathbf{r}}$ is smooth. To this end, we need the analog of the degenerate polygon. It is the "decomposable polygon", in which the edges indexed by I and J act in orthogonal subspaces.

Let $\mathbf{r} \in W_{I,J,k}$. Choose an orthogonal decomposition $\mathbb{C}^{m+1} = V_1 \oplus V_2$ with $\dim V_1 = m-k+1$ and $\dim V_2 = k$. Let $\mathcal{H}_i$ denote the set of Hermitean endomorphisms of V_i. We have inclusions $\alpha_i \colon \mathcal{H}_i \to \mathcal{H}_{m+1}$ given by

$$\alpha_i(X) = \iota_{V_i} \circ X \circ \pi_{V_i}$$

where ι_{V_i} is the inclusion of V_i into V and π_{V_i} is the orthogonal projection from V to V_i. Note that if $w \in V_i$, then $w \otimes w^*$ is in the image of α_i.

We wish to define a map

$$\iota_{I,J,V_1,V_2} \colon \widetilde{M}_{\mathbf{r}_I}(\mathcal{H}_1) \times \widetilde{M}_{\mathbf{r}_J}(\mathcal{H}_2) \to \widetilde{N}_{\mathbf{r}}.$$

Let σ be the p,q shuffle permutation given by

$$\sigma(k) = i_k, \quad 1 \le k \le p,$$
$$\sigma(p+l) = j_l, \quad 1 \le l \le q.$$

Choose polygons

$$\mathbf{e}^{(1)} = (e_1^{(1)}, \dots, e_p^{(1)}) \in \widetilde{M}_{\mathbf{r}_I}(\mathcal{H}_1),\ \mathbf{e}^{(2)} = (e_1^{(2)}, \dots, e_q^{(2)}) \in \widetilde{M}_{\mathbf{r}_I}(\mathcal{H}_2).$$

Such polygons exist, since $\mathbf{r}_I$ and $\mathbf{r}_J$ satisfy the strong triangle inequalities (Lemma 4.1.3).

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Then we define $\bar{\mathbf{e}} := \iota_{I,J,V_1,V_2}(\mathbf{e}^{(1)}, \mathbf{e}^{(2)})$ by

$$\bar{e}_{i_k} = \alpha_1(e_k^{(1)}), \quad 1 \le k \le p,$$
$$\bar{e}_{j_l} = \alpha_2(e_l^{(2)}), \quad 1 \le l \le q.$$

We then have $\bar{\mathbf{e}} = \bar{\mathbf{e}}_I \oplus \bar{\mathbf{e}}_J$, and it follows from Lemma 4.1.4 that $\bar{\mathbf{e}} \in \widetilde{M}_{\mathbf{r}}$. Therefore the image of ι_{I,J,V_1,V_2} lies in $\widetilde{M}_{\mathbf{r}}$, *i.e.*, consists of closed polygons.

Definition 4.1.5 We say that $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}$ is *decomposable* if it lies in the image of the map ι_{I,J,V_1,V_2} for some choice of I, J, V_1, V_2 as above.

Proposition 4.1.6 *$\widetilde{M}_{\mathbf{r}}$ contains a decomposable polygon if and only if $\mathbf{r}$ lies on a wall.*

Proof Suppose that $\mathbf{e}$ is decomposable. We use the notation above. Since $\mathbf{e}_I$ is closed, $\sum_I e_i = \Lambda_I \mathbb{I}_1$, where $\mathbb{I}_1$ is the identity in $\mathrm{End}(V_1)$. Similarly $\sum_J e_j = \Lambda_J \mathbb{I}_2$. Because $\bar{\mathbf{e}}$ is closed, we have $\sum_{k=1}^n \bar{e}_k = \Lambda \mathbb{I}$ ($\mathbb{I}$ is the identity on $\mathbb{C}^{m+1}$). Clearly this last sum is also of block form $\Lambda_I \mathbb{I}_1 \oplus \Lambda_J \mathbb{I}_2$ in $\mathrm{End}(\mathbb{C}^{m+1}) = \mathrm{End}(V_1 \oplus V_2)$. Hence $\Lambda_I = \Lambda_J = \Lambda$. This implies $\mathbf{r} \in W_{I,J,k}$.

Conversely, suppose that $\mathbf{r} \in W_{I,J,k}$. By Lemma 4.1.3 and Theorem 3.1.1 there exist a closed p-gon $\mathbf{e}^{(1)} \in \mathcal{H}_1$ with side-lengths $\mathbf{r}_I$ and a closed q-gon $\mathbf{e}^{(2)} \in \mathcal{H}_2$ with side lengths $\mathbf{r}_J$. Then $\iota_{I,J,V_1,V_2}(\mathbf{e}^{(1)}, \mathbf{e}^{(2)})$ is a decomposable polygon with side-lengths $\mathbf{r}$. ■

The next few lemmas are well-known, but we will state and prove them because some of them will play a critical role later. As above, $\mathcal{H}_j$ denotes the Hermitean endomorphisms of a subspace $V_j \subset \mathbb{C}^{m+1}$ and $\alpha_j \colon \mathcal{H}_j \to \mathcal{H}_{m+1}$ is the natural inclusion.

For $\mathbf{e} \in \widetilde{N}_{\mathbf{r}}$, let $Z(e_i)$ denote the centralizer of e_i in $\mathcal{H}_{m+1}^0$, and let $Z(\mathbf{e}) = \bigcap_i Z(e_i)$.

Lemma 4.1.7 *Suppose that $X \in Z(e_i)$, and let $\mathbb{C}^{m+1} = \bigoplus_{j=1}^l V_j$ be the eigenspace decomposition of X. Then there exists j such that $e_i \in \alpha_j(\mathcal{H}_j)$.*

Proof If $e_i = r_i w_i \otimes w_i^*$, we have $\mathbb{C}w_i = \ker(e_i - r_i \mathbb{I})$. Since X and e_i commute, w_i is also an eigenvector of X. Hence $w_i \in V_j$ for some j and $r_i w_i \otimes w_i^* \in \alpha_j(\mathcal{H}(V_j))$. ■

Lemma 4.1.8 *Let $\mathbf{e} \in \widetilde{N}_{\mathbf{r}}$. Then $\mathrm{Im}\, T_{\mathbf{e}}\mu_{\mathbf{r}} \subset \mathcal{H}_{m+1}^0$.*

Proof If $w(t)$ is a smooth curve in $\mathbb{C}^{m+1}$, with $\|w(t)\| \equiv 1$, then $\mathrm{Tr}\, w(t) \otimes w(t)^* \equiv 1$ implies $\mathrm{Tr}\, \frac{d}{dt}(w(t) \otimes w(t)^*) \equiv 0$. Hence the derivative of the momentum map $\mu_{\mathbf{r}} \colon \mathbf{e} \mapsto \sum_i r_i w_i \otimes w_i^*$ maps into $\mathcal{H}_{m+1}^0$. ■

Lemma 4.1.9 *Let $^\perp$ denote orthogonal complement in $\mathcal{H}_{m+1}^0$. Then, again for $\mathbf{e} \in \widetilde{N}_{\mathbf{r}}$,*

$$(\mathrm{Im}\ T_{\mathbf{e}}\mu)^\perp = Z(\mathbf{e}).$$

Proof Indeed, the derivative $T_e\mu_{\mathbf{r}}$ will be onto if and only if

$$T_{e_1}(\mathcal{O}_{r_1}) + \cdots + T_{e_n}(\mathcal{O}_{r_n}) = \mathcal{H}^0_{m+1}.$$

Thus $T_e\mu_{\mathbf{r}}$ is onto if and only if

$$T_{e_1}(\mathcal{O}_{r_1})^\perp \cap \cdots \cap T_{e_n}(\mathcal{O}_{r_n})^\perp = \{0\}.$$

But $T_{e_i}(\mathcal{O}_{r_i})^\perp = \{[e_i, X] \mid X \in \mathcal{H}^0_{m+1}\}^\perp = Z(e_i)$, and the lemma follows. ■

Corollary 4.1.10 *Let* $\mathbf{e} \in \widetilde{N}_{\mathbf{r}}$. *Then*

$$T_{\mathbf{e}}\mu_{\mathbf{r}} \colon T_{\mathbf{e}}(\widetilde{N}_{\mathbf{r}}) \to \mathcal{H}^0_{m+1} \text{ is not onto } \iff Z(\mathbf{e}) \neq \{0\}.$$

The next lemma relates surjectivity of the moment map to indecomposability.

Lemma 4.1.11 *Now suppose that* $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}$. *Then*

$$T_{\mathbf{e}}\mu_{\mathbf{r}} \text{ is onto } \iff \mathbf{e} \text{ is not decomposable.}$$

Proof Suppose that $T_{\mathbf{e}}\mu_{\mathbf{r}}$ is not onto. Choose a nonzero $X \in Z(\mathbf{e})$. Suppose that X has ℓ distinct eigenvalues, so that $\mathbb{C}^{m+1}$ is the orthogonal sum of the corresponding eigenspaces W_j. For each $e_i = r_i w_i \otimes w_i^*$, $w_i \in W_{j_i}$ for some j_i, by Lemma 4.1.7. Now set $V_1 = W_1 + \cdots + W_{\ell-1}$, $V_2 = W_\ell$. Define $I = \{i \mid w_i \in V_1\}$, $J = \{j \mid w_j \in V_2\}$. It follows that $\mathbf{e}$ lies in the image of the map ι_{I,J,V_1,V_2} and so is decomposable.

Now suppose that $\mathbf{e}$ is decomposable. Then there exists an orthogonal splitting $V = V_1 \oplus V_2$ and a partition $\{1, \ldots, n\} = I \cup J$ such that $\mathbf{e} = \mathbf{e}_I + \mathbf{e}_J$ is in the image of the map ι_{I,J,V_1,V_2}. Let $X = \mathbb{I}_1 \oplus \mathbf{0}$. Then $X \in Z(\mathbf{e}) = (\operatorname{Im} T_{\mathbf{e}}\mu_{\mathbf{r}})^\perp$. ■

Let $\widetilde{\Sigma}_{\mathbf{r}} \subset \widetilde{M}_{\mathbf{r}}$ denote the set of decomposable polygons. It is invariant under U(m+1); let $\Sigma_{\mathbf{r}}$ be the image of $\widetilde{\Sigma}_{\mathbf{r}}$ in $M_{\mathbf{r}}$.

Theorem 4.1.12

(i) $\widetilde{M}_{\mathbf{r}} - \widetilde{\Sigma}_{\mathbf{r}}$ *is a smooth manifold.*

(ii) *The group SU(m+1) acts freely on* $\widetilde{M}_{\mathbf{r}} - \widetilde{\Sigma}_{\mathbf{r}}$, *hence the quotient* $M_{\mathbf{r}} - \Sigma_{\mathbf{r}}$ *is a smooth manifold.*

Proof Part (i) follows from one implication in Lemma 4.1.11: if $\mathbf{e}$ is not decomposable, then $\Lambda\mathbb{I}$ is a regular value of $\mu_{\mathbf{r}}$.

For (ii), we need to check that if $\mathbf{e}$ is not decomposable, then the stabilizer of $\mathbf{e}$ under the action of U(m+1) is trivial. The argument in Lemma 4.1.11 still works because we deal with matrix groups. If $\kappa\mathbf{e}\kappa^{-1} = \mathbf{e}$, we write $\mathbb{C}^{m+1}$ as a sum of eigenspaces of κ and proceed as before. ■

Corollary 4.1.13 *If* $\mathbf{r}$ *does not lie on a wall, then* $M_{\mathbf{r}}$ *is a smooth manifold of dimension* $2m(n - m - 2)$.

Proof The smoothness of $M_{\mathbf{r}}$ follows from the theorem. To compute the dimension of $M_{\mathbf{r}}$, we note that $M_{\mathbf{r}}$ is the symplectic quotient of $(\mathbb{CP}^m)^n$ by the projective unitary group $PU(m+1)$. Thus we obtain

$$\dim_{\mathbb{R}} M_{\mathbf{r}} = 2mn - 2[(m+1)^2 - 1] = 2mn - 2m^2 - 4m. \qquad \blacksquare$$

4.2 The Critical Sidelengths of Closed Polygons

In this subsection, we study the map $\bar{\mathbf{s}}$ that maps a closed polygon $\mathbf{e}$ to the vector $\mathbf{r}$ of its side lengths and show in Theorem 4.2.7 that its critical values are the union of the walls (Definition 4.1.1).

Notation 4.2.1 The space of not necessarily closed linkages in $\mathcal{H}_{m+1}$ with rank 1 positive semi-definite edges and *arbitrary (positive) side lengths* is denoted by

$$\mathrm{Pol} = \{\mathbf{e} \mid \mathbf{e} \in \widetilde{N}_{\mathbf{r}} \text{ for some } \mathbf{r}\}.$$

We will use μ to denote the restriction of the momentum map μ for the diagonal action of $U(m+1)$ on $\mathcal{H}^n_{m+1}$ to Pol. (Recall that $\mu_{\mathbf{r}}$ is the moment map on linkages, closed or not, with given side lengths $\mathbf{r}$.) Further, introduce the subset of *closed* polygons with arbitrary side lengths,

$$\mathrm{CPol} = \{\mathbf{e} \mid \mathbf{e} \in \widetilde{M}_{\mathbf{r}} \text{ for some } \mathbf{r}\} \subset \mathrm{Pol}\,.$$

The idea of the argument is this. If $\mathbf{e}(t) = (r_1(t)w_1(t) \otimes w_1(t)^*, \dots)$ is a curve in CPol through $\mathbf{e}(0) = \mathbf{e}$, $\mathbf{r}(0) = \mathbf{r}$, the derivative of the moment map μ has the form

$$T_{\mathbf{e}}\mu = \sum \dot{r}_i(0)e_i + \sum r_i(w_i \otimes \dot{w}_i(0)^* + \dot{w}_i(0) \otimes w_i^*).$$

The first sum is a linear combination of edges, while the second sum is in the image of the moment map $\mu_{\mathbf{r}}$. Since $T_{\mathbf{e}}\bar{\mathbf{s}} = \dot{\mathbf{r}}(0)$, the span of the edges relates surjectivity of $T_{\mathbf{e}}\mu$ with surjectivity of $T_{\mathbf{e}}\bar{\mathbf{s}}$.

Let $\mathbf{E} \subset \mathcal{H}_{m+1}$ be the span of the edges e_i of the linkage $\mathbf{e}$.

Lemma 4.2.2 *Suppose that* $\mathbf{e} \in \mathrm{CPol}$. *Then*

$$Z(\mathbf{e}) \subset \mathbf{E}.$$

Proof Let $X \in Z(\mathbf{e})$, with eigenvalues λ_i, $i = 1, \dots, l$, and let $\mathbb{C}^{m+1} = \bigoplus_{i=1}^l V_i$ be the eigenspace splitting of $\mathbb{C}^{m+1}$ under X. Then by Lemma 4.1.7, for each i, $1 \le i \le n$, there exists j_i such that $e_i \in \alpha_{j_i}(\mathcal{H}_{j_i})$. Hence, $\mathbf{e}$ is decomposable with respect to this splitting. Thus there exists a permutation σ such that

$$\mathbf{e} = \sigma\big(e_1^{(1)}, \dots, e_{p_1}^{(1)}, \dots, e_1^{(l)}, \dots, e_{p_l}^{(l)}\big)$$

with $e_j^{(i)} = r(i)_j w(i)_j \otimes (w_j^{(i)})^*$, and $w_j^{(i)} \in V_i$, $1 \le i \le l$, $1 \le j \le p_i$.

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Since $\mathbf{e}$ is closed, we have $\sum_i e_i = \Lambda \mathbb{I}$. As a consequence we have

$$\sum_{j=1}^{p_i} e_j^{(i)} = \Lambda \prod_i$$

where $\prod_i$ is the projection on V_i. Thus the $\prod_i$ lie in the span $\mathbf{E}$. But since the splitting of $\mathbb{C}^{m+1}$ is the eigenspace decomposition of X, we have $X = \sum_{i=1}^{l} \lambda_i \prod_i$. Thus $X \in \mathbf{E}$. ■

We do not need the next corollary in what follows but have included it for completeness. We let $\mathfrak{t}$ denote the abelian Lie subalgebra consisting of the diagonal matrices in $\mathcal{H}_{m+1}$.

Corollary 4.2.3 *There exists $k \in U(m+1)$ such that*

$$\mathrm{Ad}_k(Z(\mathbf{e})) \subset \mathfrak{t}.$$

Proof It suffices to prove that $Z(\mathbf{e})$ is abelian. To this end let $X, Y \in Z(\mathbf{e})$. By the lemma we may write X as a linear combination of the edges of $\mathbf{e}$. But by definition Y centralizes all the edges of $\mathbf{e}$. ■

Lemma 4.2.4

$$\mathrm{Im}\ T_{\mathbf{e}}\mu = \mathrm{Im}\ T_e\mu_r + \mathbf{E}.$$

Proof Define an action of $(\mathbb{R}_+)^n$ on Pol by $\mathbf{a} \cdot \mathbf{e} = (a_1e_1, a_2e_2, \dots, a_ne_n)$, where $\mathbf{a} = (a_1, \dots, a_n)$. It is immediate that $\widetilde{N}_{\mathbf{r}}$ is a cross-section for this action and consequently we have Pol $\cong (\mathbb{R}_+)^n \times \widetilde{N}_{\mathbf{r}}$. Now let $\mathbf{e} \in$ Pol. Then

$$\mu(\mathbf{a} \cdot \mathbf{e}) = a_1e_1 + a_2e_2 + \cdots + a_ne_n.$$

The lemma follows upon differentiating this identity with respect to $\mathbf{e}$ and the action of $(\mathbb{R}_+)^n$. ■

The following proposition will play a critical role in our analysis of the diagram below.

Proposition 4.2.5 *Suppose that* $\mathbf{e} \in$ CPol. *Then $T_{\mathbf{e}}\mu$ maps onto $\mathcal{H}_{m+1}$.*

Proof We again use the fact that $(\mathrm{Im}\ T_{\mathbf{e}}\mu_{\mathbf{r}})^{\perp} = Z(\mathbf{e})$. By Lemma 4.2.2 the directions coming by changing the side lengths (the action of $(\mathbb{R}_+)^n$) contain $Z(\mathbf{e})$. The result now follows from Lemma 4.2.4. ■

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Let $\mathbf{s}\colon \operatorname{Pol} \to \mathbb{R}^n$ and $\bar{\mathbf{s}}\colon \operatorname{CPol} \to \mathbb{R}^n$ denote the side-length maps, and let $\mathbf{j}\colon \operatorname{CPol} \to \operatorname{Pol}$ and $\mathbf{k}\colon \widetilde{N}_{\mathbf{r}} \to \operatorname{Pol}$ be the inclusions. It is immediate (by using the action of $(\mathbb{R}_+)^n$) that $T_{\mathbf{e}}\mathbf{s}$ maps onto $\mathbb{R}^n$.

We will need the subspaces of Pol and CPol obtained by fixing the sums of the side lengths (but not the side lengths themselves). For $\lambda \in \mathbb{R}_+$, define

$$\operatorname{Pol}_\lambda = \Big\{ \mathbf{e} \in \operatorname{Pol} \;\Big|\; \sum_{i=1}^{n} ||e_i|| = \lambda \Big\}$$

$$\operatorname{CPol}_\lambda = \operatorname{CPol} \cap \operatorname{Pol}_\lambda .$$

We observe that $T_{\mathbf{e}}\mu$ maps $T_{\mathbf{e}} \operatorname{Pol}_\lambda$ into $\mathcal{H}^0_{m+1}$, and moreover it is an immediate consequence of Proposition 4.2.5 that if $\mathbf{e}$ is closed, then this map is onto.

Remark 4.2.6 In what follows we will use the fact that if $f\colon X \to Y$ is a real-analytic map of real analytic sets with $f(x) = y$, then the sequence

$$T_x(f^{-1}(y)) \to T_x(X) \to T_y(Y)$$

is exact at $T_x(X)$ (the second arrow is $T_x(f)$).

Theorem 4.2.7 *The set of critical values of* $\bar{\mathbf{s}}$ *is the union of the walls.*

Proof We have seen that $\mathbf{r}$ lies on a wall if and only if $\Lambda_{\mathbf{r}}\mathbb{I}$ is a critical value of $\mu_{\mathbf{r}}\colon \widetilde{N}_{\mathbf{r}} \to \mathcal{H}_{m+1}$. Let $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}$. The result will follow once we prove that $T_{\mathbf{e}}\mu_{\mathbf{r}}$ is onto if and only if $T_{\mathbf{e}}\bar{\mathbf{s}}$ is onto. This follows from a diagram chase in the following commutative diagram. Recall that $\rho = \sum_{i=1}^{n} r_i$. Let $\mathbb{R}^n_0$ denote the subspace of n-tuples with sum 0. It is clear that $T_{\mathbf{e}}\mathbf{s}$ maps $T_{\mathbf{e}} \operatorname{Pol}_\rho$ onto $\mathbb{R}^n_0$.

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_{\mathbf{e}}\widetilde{M}_{\mathbf{r}} & \xrightarrow{T_{\mathbf{e}}\mathbf{i}} & T_{\mathbf{e}}\widetilde{N}_{\mathbf{r}} & \xrightarrow{T_{\mathbf{e}}\mu_{\mathbf{r}}} & \mathcal{H}^0_{m+1} \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle T_{\mathbf{e}}\mathbf{k}} & & \big\downarrow{\scriptstyle \mathbb{I}} \\
0 & \longrightarrow & T_{\mathbf{e}} \operatorname{CPol}_\rho & \xrightarrow{T_{\mathbf{e}}\mathbf{j}} & T_{\mathbf{e}} \operatorname{Pol}_\rho & \xrightarrow{T_{\mathbf{e}}\mu} & \mathcal{H}^0_{m+1} \\
\big\downarrow & & {\scriptstyle T_{\mathbf{e}}\bar{\mathbf{s}}}\big\downarrow & & \big\downarrow{\scriptstyle T_{\mathbf{e}}\mathbf{s}} & & \big\downarrow \\
0 & \longrightarrow & \mathbb{R}^n_0 & \xrightarrow{\mathbb{I}} & \mathbb{R}^n_0 & \longrightarrow & 0
\end{array}$$

We will perform the diagram chase that proves $T_{\mathbf{e}}\bar{\mathbf{s}}$ onto $\Longrightarrow$ $T_{\mathbf{e}}\mu_{\mathbf{r}}$ onto. To this end let $y \in \mathcal{H}^0_{m+1}$. We will construct $x \in T_{\mathbf{e}}\widetilde{N}_{\mathbf{r}}$ with $T_{\mathbf{e}}\mu_{\mathbf{r}}(x) = y$. Indeed, since $T_{\mathbf{e}}\mu$ is onto, there exists $z \in T_{\mathbf{e}} \operatorname{Pol}_\rho$ with $T_{\mathbf{e}}\mu(z) = y$. By our assumption that $T_{\mathbf{e}}\bar{\mathbf{s}}$ is onto, there exists $w \in T_{\mathbf{e}} \operatorname{CPol}_\rho$ with $T_{\mathbf{e}}\bar{\mathbf{s}}(w) = T_{\mathbf{e}}\mathbf{s}(z)$. Then $T_{\mathbf{e}}\mathbf{s}(z - T_{\mathbf{e}}\mathbf{j}(w)) = 0$.

Since the next to last vertical sequence is exact at $T_{\mathbf{e}} \operatorname{Pol}$ by Remark 4.2.6, there exists $x \in T_{\mathbf{e}}\widetilde{N}_{\mathbf{r}}$ with $T_{\mathbf{e}}\mathbf{k}(x) = z - T_{\mathbf{e}}\mathbf{j}(w)$. Then $T_{\mathbf{e}}\mu_{\mathbf{r}}(x) = y$ as required. ■

This theorem is a critical first step towards finding the topologies of the moduli spaces $M_{\mathbf{r}}$. We define a *chamber* of the polyhedral cone $C(n, m+1)$ to be a connected component of the complement of the union of the walls. Then we have the following corollary of the previous theorem.

Corollary 4.2.8 *The sidelength map* $\mathbf{s}$ *is a (trivial) fiber bundle over each chamber of* $C(n, m+1)$*. Hence if* $\mathbf{r}$ *and* $\mathbf{r}'$ *lie in the same chamber, the moduli spaces* $M_{\mathbf{r}}$ *and* $M_{\mathbf{r}'}$ *are diffeomorphic.*

Proof The map $\mathbf{s}$ is a proper submersion over each chamber, hence by the Ehresmann Fibration Theorem, [BrJa, p. 84], it is a fiber bundle (necessarily trivial since chambers are contractible). ∎

In fact we have to replace Pol_ρ by its quotient by $SU(m+1)$ in the argument above, but the reader will check that the critical set of the map induced by $\bar{\mathbf{s}}$ on this quotient remains the same.

Remark 4.2.9 It is possible to implement wall-crossing techniques in order to compute the topologies ofthe moduli spaces $M_{\mathbf{r}}$ (see[Hu] or [Goldin] for cohomology computations).

5 Bending Hamiltonians

Kapovich and Millson ([KM96]) studied an integrable Hamiltonian system on $\widetilde{M}_{\mathbf{r}}$ in the case $m = 1$ and $e_2 = -e_1$, *i.e.*, in $\mathcal{H}_2^0 \equiv \mathfrak{su}(2)$, which is isomorphic to Euclidean space $\mathbb{E}^3$. In §1 we introduced the diagonals $A_0 = e_1$ and $A_i = e_1 + \cdots + e_{i+1}, i = 1, \dots, n-2$. In this case, a closed polygon has $A_{n-1} = \Lambda\mathbb{I}$. It was shown that the functions $f_i(\mathbf{e}) = \|A_i\|$ Poisson commute; $\|A_i\|$ is the positive eigenvalue of A_i. The diagonal A_i divides the polygon into two “flaps”, and the flow generated by f_i is 2π-periodic, consisting of a rigid rotation of one flap about the diagonal.

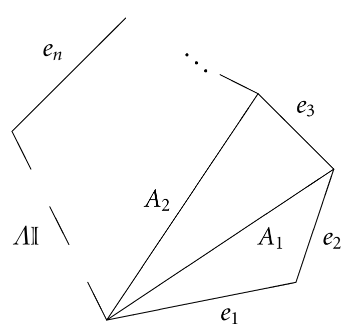

Figure 1: A polygon in $\mathfrak{u}(m+1)$

The analogs of "bending Hamiltonians" for $m > 1$ are again the eigenvalues of the diagonals. Now, however, $A_{n-1} = \varLambda\mathbb{I}$, indicated by a dashed line in Figure 1; it would be absent in $\mathfrak{su}(m+1)$.

Notation 5.1 The eigenvalues of A_i are denoted by λ_{ij} in decreasing order, $\lambda_{i1} \geq \cdots \geq \lambda_{i,m+1}$.

We note that $A_{n-2} = \varLambda\mathbb{I} - e_n$, has eigenvalues $\varLambda$ (multiplicity m) and $\varLambda - r_n$, and those are fixed. Thus only the λ_{ij} for $1 \leq i \leq n-3$ are of possible interest. Furthermore, it will be seen in §6.1 that off submanifolds of $\widetilde{M}_{\mathbf{r}}$ of lower dimension, the nontrivial λ_{ij} (those not identically 0 or $\varLambda$) are simple. In that case, they will be smooth functions of $\mathbf{e}$, which is assumed throughout the present section.

5.1 Bending Flows

We want to calculate the Hamiltonian vector fields and flows generated by the λ_{ij}. By analogy with the case of $\mathbb{E}^3$, we call them "bending flows".

On a product of orbits, the Poisson bracket is the sum of the orbit brackets, and the next formula is evident from (2.1.2):

Proposition 5.1.1 *Suppose $f\colon \widetilde{M}_{\mathbf{r}} \to \mathbb{C}$ is smooth and depends only on $e_1, \dots, e_{i+1}$. Then the Hamiltonian system generated by f is*

$$\dot{e}_k = \begin{cases} [\nabla_k f(e_1, \dots, e_{i+1}), e_k] & \text{if } 1 \leq k \leq i+1, \\ 0 & \text{if } i+1 < k \leq n, \end{cases} \tag{5.1.1}$$

where ∇_k denotes gradient with respect to e_k, all other e_j being held fixed.

To solve these equations when $f = \lambda_{ij}$, we recall a standard lemma from perturbation theory.

Lemma 5.1.2 *Let λ be an isolated eigenvalue of $A \in \mathcal{H}_{m+1}$, with unit eigenvector u. Then $\nabla\lambda(A) = \sqrt{-1}\, u \otimes u^*$.*

Proof For A' sufficiently close to A, the eigenvalue $\lambda(A')$ and (with proper choice of phase) normalized eigenvector $u(A')$ vary analytically in a neighborhood of λ, u. Take a curve $A(s)u(s) = \lambda(s)u(s)$ and take the inner product with the unit length $u(s)$ to get $\lambda(s) = (A(s)u(s), u(s))$. Differentiate and set $s = 0$, and use $(Au, \dot{u}(0)) + (A\dot{u}(0), u) = \lambda((u, \dot{u}(0)) + (\dot{u}(0), u)) = 0$, resulting in

$$\dot{\lambda}(0) = (\dot{A}(0)u, u) = \operatorname{Im}\operatorname{Tr}(\sqrt{-1}\, u \otimes u^* \dot{A}(0)),$$

as was to be shown. ■

Write $E_j(A)$ for the spectral projection onto the λ_j eigenspace of A; the lemma thus states that $\nabla\lambda_j(A) = \sqrt{-1}\, E_j(A)$. We compute the Hamiltonian flows generated by the functions $\lambda_{ij}\colon \mathbf{e} \mapsto \lambda_j(A_i)$ on $\widetilde{N}_{\mathbf{r}}$. These functions are invariant under the U(m+1) action, and hence descend to the symplectic quotient $M_{\mathbf{r}}$. The λ_{ij} are smooth when they are simple eigenvalues.

Proposition 5.1.3 *For $i = 1, \dots, n-3$ and $j = 1, \dots, m+1$, λ_{ij} is the Hamiltonian for the system*

$$\dot{e}_k = \begin{cases} \sqrt{-1}\,[E_j(e_1 + \cdots + e_{i+1}), e_k] & \text{if } 1 \le k \le i+1, \\ 0 & \text{if } i+1 < k \le n. \end{cases} \tag{5.1.2}$$

The Hamiltonian flow $\phi^t_{ij}(\mathbf{e}) = \mathbf{e}(t)$ is given by

$$e_k(t) = \begin{cases} \big(\mathrm{Ad}\exp(\sqrt{-1}\,tE_j(A_i))\big)(e_k) & \text{if } 1 \le k \le i+1, \\ e_k & \text{if } i+1 < k \le n. \end{cases} \tag{5.1.3}$$

Proof To obtain the system (5.1.2) we wish to apply Proposition 5.1.1. It is necessary to relate the partial gradients $\nabla_k\lambda_{ij}$ to the full gradient $\nabla\lambda_{ij} = \sqrt{-1}\, E_j(A_i)$. According to Lemma 5.1.2, the former are found by computing

$$\dot{A}_i(s) = (e_1 + \cdots + e_k(s) + \cdots + e_{i+1})^{\cdot} = \dot{e}_k(s),$$

but because $\dot{A}_i(0)$ is tangent to $\mathcal{O}_{r_k}$, this only determines $\nabla_k\lambda_{ij}$ up to a vector normal to the orbit:

$$\nabla_k\lambda_{ij}(A_i) = \sqrt{-1}\, E_j(A_i) + \xi_k, \quad [\xi_k, e_k] = 0.$$

Then $[\nabla_k\lambda_{ij}, e_k] = \sqrt{-1}\,[E_j(A_i), e_k]$, and (5.1.2) follows.

Next, add the equations (5.1.2) for $1 \le k \le i+1$ to find

$$\dot{A}_i(t) = \sqrt{-1}\,[E_j(A_i(t)), A_i(t)].$$

Since A_i commutes with its own spectral projections, we get $\dot{A}_i(t) = 0$ and $A_i(t) = A_i$. With constant A_i, the solution of (5.1.2) is immediate. ∎

Corollary 5.1.4 *The flows ϕ_{ij} have period 2π in t.*

Proof If P is a projection, then $P^2 = P$. Consequently,

$$\exp(\sqrt{-1}\,tP) = \mathbb{I} + (\exp(\sqrt{-1}\,t) - 1)P,$$

which has period 2π. ∎

5.2 Involutivity

It is not *a priori* clear from the formulas for ϕ_{ij} that these flows commute. This is a short calculation; we again work only with simple eigenvalues of the A_i on $\widetilde{N}_{\mathbf{r}}$, and the flows will also commute on $M_{\mathbf{r}}$.

Proposition 5.2.1 $\{\lambda_{ij}, \lambda_{k\ell}\} = 0$ *for* $1 \le i, k \le n-3$ *and* $1 \le j, \ell \le m+1$.

Proof By Proposition 5.1.1 and the proof of Proposition 5.1.3,

$$\{\lambda_{ij}, \lambda_{k\ell}\}(\mathbf{e}) = \sum_{s=1}^{i+1} \operatorname{Im}\operatorname{Tr}\Big(e_s[\sqrt{-1}\,E_j(A_i) + \xi_s, \sqrt{-1}\,E_\ell(A_k) + \eta_s]\Big),$$

where again ξ_s, η_s commute with e_s. The ad-invariance of the trace form produces $[\xi, e_s]$ and $[\eta, e_s]$, which are zero. This leaves

$$\begin{aligned}\{\lambda_{ij}, \lambda_{k\ell}\}(\mathbf{e}) &= -\sum_{s=1}^{i+1} \operatorname{Im}\operatorname{Tr}\Big([e_s, E_j(A_i)]E_\ell(A_k)\Big)\\ &= -\operatorname{Im}\operatorname{Tr}\Big([A_i, E_j(A_i)]E_\ell(A_k)\Big)\\ &= 0.\end{aligned}$$

■

Remark 5.2.2 The proof works more generally, if instead of A_i and A_k, one has $\sum_I e_i$ and $\sum_J e_j$ with $I \subset J$. Thus, for example, the eigenvalues of $e_2 + e_3$ and $e_1 + \cdots + e_5$ are in involution. On the other hand, if λ, μ are eigenvalues of $e_1 + e_2$ and $e_2 + e_3$, respectively, then

$$\{\lambda, \mu\}(\mathbf{e}) = -\operatorname{Im}\operatorname{Tr}\Big(e_2[E_\lambda(e_1 + e_2), E_\mu(e_2 + e_3)]\Big),$$

which need not be zero. See [KM01] for more information.

6 A Complete Set of Bending Flows

The eigenvalues $\lambda_{ij}(\mathbf{e})$ have been shown to Poisson commute and to generate 2π-periodic flows. If there were $\frac{1}{2}\dim M_{\mathbf{r}}$ eigenvalues and if they were smooth, they would constitute a set of action variables on $M_{\mathbf{r}}$. Smoothness everywhere cannot be achieved, but there are $\frac{1}{2}\dim M_{\mathbf{r}}$ that are smooth and functionally independent on a dense open submanifold of $M_{\mathbf{r}}$. This section presents the proof.

6.1 The Weinstein–Aronszajn Formula

The diagonal A_i is a rank-one perturbation of A_{i-1}, and because of this, the eigenvalues λ_{ij} and $\lambda_{i-1,j}$ are related in a special way. This connection is the simplest instance of the Weinstein–Aronszajn formula [Kato, Ch. 4, §6]. We describe the formula and two consequences that will be used later.

Let A be an $(m+1)\times(m+1)$ Hermitean matrix with eigenvalues $\lambda_1,\dots,\lambda_{m+1}$ and let $u_1,\dots,u_{m+1}$ be corresponding orthonormal eigenvectors. (If an eigenvalue has multiplicity >1, which is now permitted, the choice of its eigenvectors is irrelevant). Let $w\in\mathbb{C}^{m+1}$ be a unit vector and let $r\in\mathbb{R}$. Set $L=A+rw\otimes w^*$ and call its eigenvalues $\nu_1,\dots,\nu_{m+1}$. Finally, define $\alpha_1,\dots,\alpha_{m+1}\in\mathbb{C}$ by $w=\sum_{j=1}^{m+1}\alpha_j u_j$.

Proposition 6.1.1

$$\frac{\det(z\mathbb{I}-L)}{\det(z\mathbb{I}-A)}=1-r\sum_{j=1}^{m+1}\frac{|\alpha_j|^2}{z-\lambda_j}. \tag{6.1.1}$$

.

Proof Write $R_z=(z\mathbb{I}-A)^{-1}$ for the resolvent of A. The left side of (6.1.1) is

$$\begin{aligned}&=\det((z\mathbb{I}-A)^{-1}(z\mathbb{I}-A-rw\otimes w^*))\\&=\det(\mathbb{I}-R_z(rw\otimes w^*))\\&=\det(\mathbb{I}-r(R_zw)\otimes w^*).\end{aligned}$$

Now, $\det(\zeta\mathbb{I}-r(R_zw)\otimes w^*)$ is the characteristic polynomial of a rank-one matrix, and so has an m-fold root at $\zeta=0$ and a simple root at $\zeta=r(R_zw,w)$. Setting $\zeta=1$ we get

$$\det(\mathbb{I}-r(R_zw)\otimes w^*)=1-r(R_zw,w). \tag{6.1.2}$$

The lemma now follows by expanding w in (6.1.2) in the basis u_j. ■

It is convenient to write (6.1.1) more explicitly:

$$\frac{(z-\nu_1)\cdots(z-\nu_{m+1})}{(z-\lambda_1)\cdots(z-\lambda_{m+1})}=1-r\sum_{j=1}^{m+1}\frac{|\alpha_j|^2}{z-\lambda_j}. \tag{6.1.3}$$

Corollary 6.1.2 *The $|\alpha_j|^2$ are rational functions of ν_k,λ_ℓ, $1\le k,\ell\le m+1$.*

Finally, we show that the eigenvalues of A and L interlace. This will play a basic role below.

Proposition 6.1.3 *If $r>0$, then $\nu_1\ge\lambda_1\ge\nu_2\ge\cdots\ge\nu_{m+1}\ge\lambda_{m+1}$. If $r<0$, we have $\lambda_1\ge\nu_1\ge\dots$ instead.*

Proof Suppose $r>0$. It suffices to prove the proposition for a dense set of w, so that we may assume $|\alpha_j|^2>0$ for all j. Let $R(z)$ be the rational function on the right side of (6.1.3). Since $\lim_{z\to\infty}R(z)=1$ and $\lim_{z\downarrow\lambda_1}=-\infty$, R has a zero in (λ_1,∞). Likewise, because $\lim_{z\uparrow\lambda_j}=+\infty$ and $\lim_{z\downarrow\lambda_{j+1}}=-\infty$, R has a zero in $(\lambda_{j+1},\lambda_j)$. This provides $m+1$ zeros of R which must coincide with the zeros ν_j of the left side of (6.1.3). ■

6.2 Gel'fand–Tsetlin Patterns

Let $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}$. We will arrange the eigenvalues of $A_0 = e_1, A_1, \dots, A_{n-1}$ in a triangle with vertex at the bottom. The eigenvalues of A_k are written in row k of the triangle, along with some space-filling zeros. For $0 \leq k \leq m$, the rank of A_k is at most $k+1$, so zero must be at least an $(m-k)$-fold eigenvalue of A_k. Those zeros are not recorded. When $k > m$, there are $m+1$ eigenvalues, potentially nonzero; these are recorded *along with* $k-m$ *zeros*. Figure 2 shows the case $m = 2, n = 6$. Note that entries of successive rows are offset to reflect the interlacing property deduced in Proposition 6.1.3. This diagram is called a *Gel'fand–Tsetlin pattern*, or *GTs pattern* for short. It is denoted by $\Gamma(\mathbf{e})$. The extra zeros will be explained in §8, see Remark 8.5.

$$
\begin{array}{ccccccccccc}
\Lambda & & \Lambda & & \Lambda & & 0 & & 0 & & 0 \\
& d_1 & & d_2 & & d_3 & & 0 & & 0 & \\
& & c_1 & & c_2 & & c_3 & & 0 & & \\
& & & b_1 & & b_2 & & b_3 & & & \\
& & & & a_1 & & a_2 & & & & \\
& & & & & r_1 & & & & &
\end{array}
$$

Figure 2

Since $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}$, there are additional restrictions on the entries of $\Gamma(\mathbf{e})$. Row $n-1$ must consist of $m+1$ Λ's (because $e_1 + \cdots + e_n = \Lambda \mathbb{I}$) and $(n-m-1)$ zeros. The interlacing property forces the first m entries of row $n-2$ to be Λ, so in Figure 2, $d_1 = d_2 = \Lambda$. Likewise, $c_1 = \Lambda$. It becomes apparent that the extra zeros remind one that (for example) the eigenvalues $d_3 = \lambda_{4,3}$ and $c_3 = \lambda_{3,3}$ must be non-negative.

Moreover,

$$\text{(6.2.1)} \qquad \operatorname{Tr} A_k = \operatorname{Tr}(e_1 + \cdots + e_{k+1}) = r_1 + \cdots + r_{k+1},$$

which is a linear constraint on the rows of $\Gamma(\mathbf{e})$. In Figure 2, that leaves c_2, b_1, b_2, a_1 as potentially independent commuting Hamiltonians, and indeed $\dim_{\mathbb{R}} M_{\mathbf{r}} = 8$ in this case.

We summarize this discussion.

Definition 6.2.1 Let $m, n, \mathbf{r}$ be fixed. We write $\mathbf{P}$ for the convex polytope of GTs patterns satisfying the following conditions.

(1) There are n rows numbered $0, \dots, n-1$ (starting at the bottom);
(2) Row $n-1$ consists of $m+1$ Λ's and $n-m-1$ zeros;
(3) The sum of the entries of row k is $\sum_{i=0}^{k} r_{i+1}$.
(4) The interlacing property $\lambda_{ij} \geq \lambda_{i-1,j} \geq \lambda_{i,j+1}$ holds.

Proposition 6.2.2 $\dim \mathbf{P} = (n-m-2)m = \frac{1}{2} \dim_{\mathbb{R}} M_{\mathbf{r}}$.

Proof There are two cases: (1) $n \geq 2(m+1)$ and (2) $n \leq 2m+1$. The difference comes from the position of row m corresponding to the eigenvalues of $A_m = e_1 + \cdots + e_{m+1}$. Generically, this matrix will have full rank. In case (2), some of its eigenvalues are forced by interlacing to be Λ. In case (1), all the automatic Λ's have been "exhausted". (Figure 2 falls into the latter category). Let us sketch the counting.

Case (1): Unconstrained λ_{ij} can appear in rows $i = 1, \ldots, n-3$. Break this index set into three parts: $S_1 = \{1, \ldots, m\}$, $S_2 = \{m+1, \ldots, n-m-2\}$, $S_3 = \{n-m-1, \ldots, n-3\}$. If $n = 2(m+1)$ (as in Figure 2), then $S_2 = \varnothing$. The numbers of unconstrained λ_{ij} for the corresponding A_k are:

- in S_1, $1, \ldots, m$;
- in S_2, $m, \ldots, m$;
- in S_3, $m-1, \ldots, 1$.

Adding, we obtain

$$\frac{m(m+1)}{2} + (n - (2(m+1))m + \frac{m(m-1)}{2} = (n-m-2)m.$$

Case (2): We set $S_1 = \{1, \ldots, n-m-2\}$, $S_2 = \{n-m-1, \ldots, m\}$, $S_3 = \{m+1, \ldots, n-3\}$ (if $m = 1, 2$, then $S_3 = \varnothing$). The numbers of unconstrained λ_{ij} are:

- in S_1, $1, \ldots, n-m-2$;
- in S_2, $n-m-2, \ldots, n-m-2$;
- in S_3, $n-m-3, \ldots, 1$.

Now add. ■

6.3 Constructing a Polygon with Given GTs Pattern

In the last section, we saw that $\Gamma(M_{\mathbf{r}}) \subset \mathbf{P}$. We now prove the converse.

Theorem 6.3.1

(i) $\Gamma(M_{\mathbf{r}}) = \mathbf{P}$.
(ii) *There are* $\frac{1}{2}\dim M_{\mathbf{r}}$ *functionally independent* λ_{ij}*'s.*

Proof Let $\mathcal{S}_{m+1} \subset \mathcal{H}_{m+1}$ denote the space of real symmetric matrices, and let $\widetilde{M}_{\mathbf{r}}(\mathcal{S}_{m+1})$ be the set of polygons in $\widetilde{M}_{\mathbf{r}}$ with each $e_i \in \mathcal{S}_{m+1}$. The obvious inclusion $\widetilde{M}_{\mathbf{r}}(\mathcal{S}_{m+1}) \hookrightarrow \widetilde{M}_{\mathbf{r}}$ is the analog of the inclusion $\widetilde{M}_{\mathbf{r}}(\mathbb{R}^2) \hookrightarrow \widetilde{M}_{\mathbf{r}}(\mathbb{R}^3)$ used in [KM96]. We will see later that elements of $\mathcal{S}_{m+1}(\widetilde{M}_{\mathbf{r}})$ can be thought of as "unbent" polygons; these will be important in our proof of the involutivity of the angle variables in the next section. We now show that

$$\Gamma(\widetilde{M}_{\mathbf{r}}(\mathcal{S}_{m+1})) = \mathbf{P}. \tag{6.3.1}$$

Since $\Gamma\colon \widetilde{M}_{\mathbf{r}}(\mathcal{S}_{m+1}) \to \mathbf{P}$ is continuous (though not differentiable), the image of Γ is closed and it suffices to prove that the image of Γ contains the interior $\mathbf{P}^o$ of $\mathbf{P}$.

Thus, choose a GTs pattern γ in which all unconstrained inequalities are strict; we are to find $\mathbf{e} \in \Gamma(\widetilde{M}_{\mathbf{r}}(\mathcal{S}_{m+1}))$ such that $\Gamma(\mathbf{e}) = \gamma$.

Set $A_0 = r_1 w_1 \otimes w_1^*$, where w_1 is an arbitrary real unit vector. Assuming that a real symmetric A_{k-1} with a given spectrum has been found, we want $w_{k+1} \in \mathbb{R}^{m+1}$ so that

$$A_k = A_{k-1} + r_{k+1} w_{k+1} \otimes w_{k+1}^* \tag{6.3.2}$$

has the required next spectrum.

We carry out the induction step for Case (1) in the terminology of Proposition 6.2.2. First, let $k \in S_1$. Thus

$$A_{k-1} = \sum_{j=1}^{k} r_j w_j \otimes w_j^*;$$

it has spectrum $\{\lambda_1, \dots, \lambda_k, 0, \dots, 0\}$ with $\lambda_1 > \cdots > \lambda_k > 0$, and $\sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} r_i$. We are further given ν_i with

$$\nu_1 > \lambda_1 > \nu_2 > \cdots > \lambda_k > \nu_{k+1} > 0,$$

and $\sum_{i=1}^{k+1} \nu_i = \sum_{i=1}^{k+1} r_i$.

Let $u_1, \dots, u_k, u$ be normalized (real) eigenvectors of A_{k-1} corresponding to $\lambda_1, \dots, \lambda_k, 0$ and seek w_{k+1} in the form

$$w_{k+1} = \sum_{j=1}^{k} \alpha_j u_j + \alpha u$$

with α_i, α real.

Now solve for $|\alpha_j|^2$, $1 \le j \le k$ and $|\alpha|^2$ in equation (6.1.3) which takes the special form

$$\frac{(z-\nu_1)\cdots(z-\nu_{k+1})z^{m-k}}{(z-\lambda_1)\cdots(z-\lambda_k)z^{m-k+1}} = 1 - r_{k+1}\Bigl(\sum_{j=1}^{k} \frac{|\alpha_j|^2}{z-\lambda_j} + \frac{|\alpha|^2}{z}\Bigr).$$

Clearly one can take α_i, α real. Taking traces in equation (6.3.2), we get

$$\sum_{j=1}^{k+1} r_j = \sum_{j=1}^{k+1} \nu_j = \sum_{j=1}^{k} r_j + r_{k+1} \|w_{k+1}\|^2,$$

whence $\|w_{k+1}\| = 1$.

The same procedure works in the remaining subcases as well; for $k \in S_2$ the eigenvalues λ_j and ν_j are simple, while for $k \in S_3$, account must be taken of the multiplicity of Λ. ■

Remark 6.3.2 The proof shows that if w_{k+1} is not required to be real, each term $\alpha_j u_j$ is determined only up to a multiple $\exp(\sqrt{-1}\,\theta_{k+1,j})$. Thus, the possible polygons $\mathbf{e}$ corresponding to a given pattern γ lie on a torus. The angle coordinates are studied in the next section.

We conclude by making a choice of functionally independent action variables.

Definition 6.3.3 Let $\mathcal{I}$ be the set of pairs (i, j) satisfying $1 \le i \le n-1, 1 \le j \le i$ which index eigenvalues λ_{ij} such that λ_{ij} is not forced to be 0 or Λ, with the further property that $\lambda_{i,j+1}$ is not forced to be 0 (this last condition says that in each row we throw away the right-most j such that λ_{ij} is not forced to be 0).

Corollary 6.3.4 *The set $\mathcal{I}$ indexes a functionally independent set of action variables λ_{ij}.*

Proof Indeed, these action variables map onto a polyhedron of dimension equal to the cardinality of $\mathcal{I}$. ■

Remark 6.3.5 For general coadjoint orbits, one can define a complete set of constants of motion that reduce to the α_j in the rank one case; the construction also makes use of Gel'fand–Tsetlin patterns. Action variables which generate 2π-periodic flows are not known, however.

7 Angle Variables and Four-Point Functions

In this section, we construct angle variables θ_{ij} conjugate to the action variables λ_{ij} discussed thus far. The angles are implicit in Corollary 5.1.4 and Remark 6.3.2; what we now find is a global description.

7.1 Four-Point Functions and Polygons

The geometric picture in [KM96] serves as model. For the moment, think of the sides e_j as vectors in $\mathbb{R}^3$. The action variables are the lengths of the diagonals $A_i = e_1 + \cdots + e_{i+1}$ of the polygon. The corresponding conjugate angle is the *oriented* dihedral angle between the two triangles spanned, respectively, by A_{i-1}, e_{i+1}, A_i and A_i, e_{i+2}, A_{i+1}. By this we mean the oriented angle between the two normal vectors to the triangles. These two vectors are elements of the plane orthogonal to A_i. We orient this plane so that a positively oriented basis for the plane followed by A_i is a positively oriented basis for $\mathbb{R}^3$.

Remark 7.1.1 In an oriented plane Π equipped with a positive definite inner product $U \cdot V$, we can define the oriented angle $\angle(U, V)$ for a pair of vectors U and V in Π as follows. First we say that two unit vectors U, V make an angle of ninety degrees if $U \cdot V = 0$ and the basis $\{U, V\}$ is positively oriented. We let J be the operation of rotation by ninety degrees. We make Π into a complex vector space by defining $\imath V := JV$. Then the unit circle in $\mathbb{C}$ acts simply-transitively on the oriented lines in Π. We define $\angle(U, V) = \theta$ if $\exp(\imath\theta)U$ is a positive real multiple of V. If $\theta = \angle(U, V)$

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then we have

$$\cos\theta = \frac{U \cdot V}{\|U\|\|V\|}$$
$$\sin\theta = \frac{JU \cdot V}{\|U\|\|V\|}.$$

For the case at hand, the oriented angle θ_i is given by

$$\cos\theta_i = \frac{(A_i \times e_{i+1}) \cdot (A_i \times e_{i+2})}{\|A_i \times e_{i+1}\| \, \|A_i \times e_{i+2}\|} \tag{7.1.1}$$

$$\sin\theta_i = \frac{(A_i \times e_{i+1}) \times (A_i \times e_{i+2}) \cdot A_i}{\|A_i \times e_{i+1}\| \, \|A_i \times e_{i+2}\| \, \|A_i\|}. \tag{7.1.2}$$

Note that $\theta_i = 0$ when the triangles are coplanar, so that the collection of planar polygons forms a reference cross-section for the angle variables.

We now transfer (7.1.1) and (7.1.2) back to our Lie algebra $\mathcal{H}_2^0$ of tracefree Hermitean 2×2 matrices. Define $f \colon \mathbb{R}^3 \to \mathcal{H}_2^0$ by

$$f \colon \mathbf{x} = (x_1, x_2, x_3) \mapsto \hat{\mathbf{x}} = \frac{1}{2}\begin{pmatrix} x_1 & x_2 + \sqrt{-1}\,x_3 \\ x_2 - \sqrt{-1}\,x_3 & -x_1 \end{pmatrix}. \tag{7.1.3}$$

Then $\widehat{\mathbf{x} \times \mathbf{y}} = \sqrt{-1}\,[\hat{\mathbf{x}}, \hat{\mathbf{y}}]$, $\mathbf{x} \cdot \mathbf{y} = 2\operatorname{Tr}\hat{\mathbf{x}}\hat{\mathbf{y}}$, and a vector in the $x_3 = 0$ plane corresponds to a real symmetric matrix. (Thus, a planar polygon is represented by a symmetric matrix, *cf.* Theorem 6.3.1).

We return to identifying vectors with matrices via (7.1.3).

Let $\lambda > 0$ and $-\lambda$ be the eigenvalues of A_i, with orthonormal eigenvectors u, v, so that $A_i = \lambda(u \otimes u^* - v \otimes v^*)$. Write, for notational simplicity,

$$e_{i+1} = r_1 w_1 \otimes w_1^* - (r_1/2)\mathbb{I}, \quad e_{i+2} = r_2 w_2 \otimes w_2^* - (r_2/2)\mathbb{I}.$$

Then the numerator of (7.1.1) becomes (since $\mathbb{I}$ does not contribute)

$$2\operatorname{Tr}\left(\sqrt{-1}\,[A_i, r_1 w_1 \otimes w_1^*]\sqrt{-1}\,[A_i, r_2 w_2 \otimes w_2^*]\right), \tag{7.1.4}$$

and the numerator of (7.1.2) becomes

$$2\|A_i\| \operatorname{Tr}\left(\sqrt{-1}\,A_i[r_1 w_1 \otimes w_1^*, r_2 w_2 \otimes w_2^*]\right). \tag{7.1.5}$$

Definition 7.1.2 ([BeSch]) Let $a, b, c, d \in \mathbb{C}^{m+1}$. Define the *four-point function* by

$$F_4(a, b, c, d) = \frac{(a, b)(b, c)(c, d)(d, a)}{\|a\|^2\|b\|^2\|c\|^2\|d\|^2}$$

where $(\cdot\,, \cdot)$ is the usual Hermitean inner product.

Two properties of F_4 are important:

(1) $F_4(a,b,c,d)$ may be thought of as a function on $(\mathbb{CP}^m)^4$; in particular, F_4 is independent of the phases of its arguments.

(2) $\overline{F_4(a,b,c,d)} = F_4(a,d,c,b)$ (plus other such symmetries).

A longish calculation, using property (2), gives the following.

Proposition 7.1.3 *Expression* (7.1.4) *reduces to*

$$16\lambda^2 r_1 r_2 \operatorname{Re} F_4(w_1,u,w_2,v).$$

Expression (7.1.5) *reduces to*

$$16\lambda^2 r_1 r_2 \operatorname{Im} F_4(w_1,u,w_2,v).$$

The denominator in (7.1.1) *and* (7.1.2) *becomes*

$$16\lambda^2 r_1 r_2 |F_4(w_1,u,w_2,v)|.$$

Thus, the oriented dihedral angle is $\theta = \arg F_4(w_1,u,w_2,v)$.

This formula, suitably adapted, will be shown to define the conjugate angles in the more general case as well.

We mention, as an aside, that the argument of the four-point function has an interesting geometric description.

Theorem 7.1.4 *Let* a_j, $j = 1,\dots,4$ *be four points in* $\mathbb{C}^{m+1}$ *defining points* $p_j \in \mathbb{CP}^m$. *Construct a geodesic quadrilateral* π *in* $\mathbb{CP}^m$ *with vertices at the* p_j. *Let* σ *be a two-chain with boundary* π *and let* ω *be the Kähler form on* $\mathbb{CP}^m$. *Then*

$$\arg F_4(a_1,a_2,a_3,a_4) = -\int_\sigma \omega. \tag{7.1.6}$$

Proof Draw a geodesic segment (a diagonal of the quadrilateral) from p_1 to p_3. The analogue of (7.1.6) for triangles was proved in [HM], see also [Go, Ch. 7]. Now choose σ to be the union of two two-chains each of which has as boundary one of the two triangles created by drawing the diagonal p_1p_3. Combining (7.1.6) for the triangles gives the equation for the quadrilateral. ■

7.2 Construction of Angle Variables

We will define the angle variables as in Proposition 7.1.3, via the four-point function of the w's associated with two consecutive edges and eigenvectors of the diagonal between them. These vectors all involve a choice of phase, and the first goal will be to remove the ambiguity.

Let $M^0_{\mathbf{r}}$ be the open subset of $M_{\mathbf{r}}$ on which the interlacing inequalities $\lambda_{ij} > \lambda_{i-1,j} > \lambda_{i,j+1}$ are strict and let $\widetilde{M}^0_{\mathbf{r}}$ be its inverse image in $\widetilde{M}_{\mathbf{r}}$. We consider only polygons in $\widetilde{M}^0_{\mathbf{r}}$, so that the (unconstrained) eigenvalues and eigenvectors may be taken to be locally smooth functions of $\mathbf{e}$.

Let ϕ^t be one of the λ_{ik}-flows defined in Proposition 5.1.3. We will follow the transformed polygon $\phi^t(\mathbf{e})$. Its ℓth edge, $r_\ell w^t_\ell \otimes (w^t_\ell)^*$, and the normalized λ_{ij}-eigenvector, u^t_{ij}, of the diagonal $\phi^t(A_i)$, will depend on time t. They may be taken to be locally smooth on $M^0_{\mathbf{r}}$, but will depend on an initial choice while the polygon $\phi^t(\mathbf{e})$ itself is well defined.

Definition 7.2.1 Make smooth local choices of w_ℓ and u_{ij}. Here u_{ij} is a (choice of) unit length eigenvector belonging to the eigenvalue λ_j of A_i. Define

$$\alpha_{ij} : \widetilde{M}^0_{\mathbf{r}} \to \mathbb{C}, (i,j) \in \mathfrak{I}, \text{ by } \alpha_{ij} \colon \mathbf{e} \mapsto (w_{i+1}(\mathbf{e}), u_{ij}(\mathbf{e}))(u_{ij}(\mathbf{e}), w_{i+2}(\mathbf{e}));$$

this depends on the phases of w_{i+1}, w_{i+2}. (We will usually drop the argument $\mathbf{e}$). Set

$$\beta_{ij} = F_4(w_{i+1}, u_{ij}, w_{i+2}, u_{i,j+1}) = \alpha_{ij}\overline{\alpha_{i,j+1}}.$$

The β_{ij} are *independent* of all phase choices. Finally, we define the angle variables $\theta_{ij}, (i,j) \in \mathfrak{I}$, by

$$\theta_{ij} = \arg \beta_{ij}.$$

Clearly the number of four-point functions β_{ij} is the same as the number of independent, unconstrained λ_{ij}'s, since for every i there is one more λ_{ij} than β_{ij} and there are no β_{ij}'s corresponding to the eigenvalues 0 and Λ. Thus we obtain the correct formal count of angle variables. We now prove that the angle variables are well-defined on $M^0_{\mathbf{r}}$.

Lemma 7.2.2

(1) *All* $|\alpha_{ij}|^2$ *are constant under all bending flows* $\phi_{k\ell}$.

(2) *All* $|\alpha_{ij}|^2$ *are nonzero on* $\widetilde{M}^0_{\mathbf{r}}$.

In particular, $\arg \beta_{ij} = \arg \alpha_{ij}\overline{\alpha_{i,j+1}}$ *is defined.*

Proof The first statement follows from Proposition 6.1.1 and Corollary 6.1.2. Indeed,

$$A_{i-1} = A_i - r_{i+1} w_{i+1} \otimes w^*_{i+1}.$$

Hence $|(w_{i+1}, u_{ij})|^2$, being a rational function of action variables, is a constant of motion. Likewise,

$$A_{i+1} = A_i + r_{i+2} w_{i+2} \otimes w^*_{i+2}$$

implies that $|(w_{i+2}, u_{ij})|^2$ is a constant of motion.

To prove the second statement we apply the Weinstein–Aronszajn formula to obtain

$$\frac{\det(z\mathbb{I} - A_{i-1})}{\det(z\mathbb{I} - A_i)} = 1 + r_{i+1} \sum_{j=1}^{m+1} \frac{|(w_{i+1}, u_{ij})|^2}{z - \lambda_{ij}}. \tag{7.2.1}$$

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If $|\alpha_{ij}| = 0$ then either $|(w_{i+1}, u_{ij})| = 0$ or $|(w_{i+2}, u_{ij})| = 0$. Assume first that $|(w_{i+1}, u_{ij})| = 0$. From the Weinstein–Aronszajn formula we see that it follows that λ_{ij} is not a pole, so the $(z - \lambda_{ij})$ in the denominator of the left-hand side must cancel with one of the terms in the numerator. Hence one of the interlacing inequalities between the ith and $(i-1)$st rows is not strict, contradicting the assumption that $\mathbf{e} \in \widetilde{M}^0_{\mathbf{r}}$. Similarly, $(w_{i+2}, u_{ij}) \neq 0$. ∎

Lemma 7.2.3 *The β_{ij} are invariant under conjugation.*

Proof Let $g \in \mathrm{U(m+1)}$ and consider the conjugated polygon $g\mathbf{e}g^{-1}$. Its ℓth edge is $r_\ell(gw_\ell) \otimes (gw_\ell)^*$. However, the choice $w_\ell(g\mathbf{e}g^{-1})$ made in Definition 7.2.1 may not coincide with gw_ℓ. If they differ, it is by a multiple of modulus one. The four-point function β_{ij} is not affected by such a factor. ∎

In the following we will make essential use of

Remark 7.2.4 In view of the proof of Lemma 7.2.3, we may replace $w_\ell(g\mathbf{e}g^{-1})$ by gw_ℓ in calculations involving β_{ij}, and for the same reason, $u_{ij}(g\mathbf{e}g^{-1})$ by gu_{ij}.

We will now compute the Poisson brackets of the action variables with the angle variables.

Lemma 7.2.5

$$\{\lambda_{il}, \theta_{ij}\} = \begin{cases} 1, & l = j \\ -1, & l = j+1 \\ 0, & l \neq j, j+1 \end{cases}$$

Proof We will verify, using (5.1.3), that

$$\beta_{ij}(\phi^t_{il}(\mathbf{e})) = \begin{cases} \beta_{ij}(\mathbf{e}), & l \neq j, j+1, \\ \exp(\sqrt{-1}\,t)\beta_{ij}(\mathbf{e}), & l = j, \\ \exp(-\sqrt{-1}\,t)\beta_{ij}(\mathbf{e}), & l = j+1. \end{cases}$$

Note from (5.1.3) that the ith diagonal A_i of $\mathbf{e}$ and the $(i+2)$nd edge are fixed under ϕ^t_{il}. Hence the normalized eigenvectors u_{ij} of A_i are also fixed. Now abbreviate $g_t = \exp(\sqrt{-1}\,tE_l(A_i))$, and as explained in Remark 7.2.3, make the replacement

$$w_{i+1}(\phi^t_{il}(\mathbf{e})) = w_{i+1}(g_t\mathbf{e}g_t^{-1}) \rightsquigarrow g_t w_{i+1}(\mathbf{e}).$$

We obtain

$$\begin{aligned}\beta_{ij}(\phi^t_{il}(\mathbf{e})) &= (g_t w_{i+1}, u_{ij})(u_{ij}, w_{i+2})(w_{i+2}, u_{i,j+1})(u_{i,j+1}, g_t w_{i+1}) \\ &= (w_{i+1}, g_t^{-1}u_{ij})(u_{ij}, w_{i+2})(w_{i+2}, u_{i,j+1})(g_t^{-1}u_{i,j+1}, w_{i+1}).\end{aligned}$$

Since $E_l(A_i)u_{ij} = \delta_{jl}u_{ij}$ the lemma follows by definition of g_t. ∎

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Lemma 7.2.6

$$\{\lambda_{ij}, \theta_{kl}\} = 0, i \neq k.$$

Proof

If $i < k$ then the kth diagonal, the $(k+1)$st edge, and the $(k+2)$nd edge are fixed by the bending flow ϕ^t_{ij}, and hence θ_{kl} is unchanged.

If $i > k$, then the kth diagonal, the $(k+1)$st edge and the $(k+2)$nd edge are rigidly moved by the g_t under the bending flow ϕ^t_{ij}, and hence θ_{kl} is unchanged. (Note that Remark 7.2.3 is used once more). ■

To remove the redundancy in the λ_{ij}, we define new action variables μ_{ij} by the formula

$$\mu_{ij} = \sum_{k=1}^{j} \lambda_{ik}. \tag{7.2.2}$$

As a consequence of the two preceding lemmas we obtain:

Proposition 7.2.7 *The action variables $\{\mu_{ij}\}$ and the angle variables $\{\theta_{ij}\}$ are conjugate*

$$\{\mu_{ij}, \theta_{kl}\} = \begin{cases} 1 & i = k, j = l, \\ 0 & otherwise. \end{cases}$$

We deduce two corollaries.

Corollary 7.2.8 *The angle variables are functionally independent.*

Corollary 7.2.9 *The Hamiltonian flows of the new action variables $\{\mu_{ij}\}$ permute the simultaneous level sets $\{\theta_{ij} = c_{ij}, (i, j) \in \mathcal{I}\}$ transitively.*

We now begin the proof that

$$\{\theta_{ij}, \theta_{kl}\} = 0.$$

Recall that $\mathcal{S}_{m+1}$ is the space of real symmetric $(m+1) \times (m+1)$ matrices. Let $\sigma\colon \mathcal{H}_{m+1} \to \mathcal{H}_{m+1}$ be complex conjugation. Then $\mathcal{S}_{m+1}$ is the fixed subspace of σ. The following lemma is immediate from (2.1.1):

Lemma 7.2.10 *The involution σ is anti-Poisson (a Poisson isomorphism from $\mathcal{H}_{m+1}$ equipped with the Lie Poisson tensor to $\mathcal{H}_{m+1}$ equipped with the negative of the Lie Poisson tensor).*

We obtain

Corollary 7.2.11 *If f and g are constant on $\mathcal{S}_{m+1}$, then $\{f, g\}$ vanishes on $\mathcal{S}_{m+1}$.*

Proof Let $\pi(\cdot,\cdot)$ be the Lie Poisson bivector considered as a skew-symmetric bilinear form on the cotangent bundle of $\mathcal{H}_{m+1}$. For $x \in \mathcal{S}_{m+1}$ and u, v cotangent vectors at x, the Lemma gives $\pi_x(u,v) = -\pi_x(\sigma u, \sigma v)$. If u and v are conormal covectors at x then they are in the (-1)-eigenspace for σ, and therefore $\pi_x(u,v) = 0$. But if f and g are constant on $\mathcal{S}_{m+1}$, then df_x and dg_x are conormal at x. ■

As an immediate consequence we have

Lemma 7.2.12 *If f and g are constant on $M_{\mathbf{r}}(\mathcal{S}_{m+1})$, then $\{f,g\}$ vanishes on $M_{\mathbf{r}}(\mathcal{S}_{m+1})$.*

Our next goal is to prove that the simultaneous zero level set of the angle variables is $M_{\mathbf{r}}(\mathcal{S}_{m+1})$. In order to obtain this we will need two technical lemmas to handle the regions S_1 and S_3 (in the notation of Proposition 6.2.2). The first lemma will be used to deal with the region S_3.

Lemma 7.2.13 *Let $V_i = \ker(A_i - \Lambda \mathbb{I})$, $n-m-2 \le i \le n-1$. Then*

$$V_{n-1} \supset V_{n-2} \supset \cdots \supset V_{n-m-2} = \{0\}.$$

Moreover (recalling $A_i = A_{i-1} + r_{i+1} w_{i+1} \otimes w_{i+1}^*$) we have

$$V_{i-1} = \{v \in V_i : (v, w_{i+1}) = 0\}.$$

Proof Let $v \in V_{i-1}$ and $\|v\| = 1$. Then

$$\Lambda = (A_{i-1}v, v) = (A_i v, v) - r_{i+1}|(w_{i+1}, v)|^2.$$

But Λ is the largest eigenvalue of A_i so $(A_i v, v) \le \Lambda$. Hence the above equation can hold if and only if

$$(A_i v, v) = \Lambda (\text{so } v \in V_i) \quad \text{and} \quad (w_{i+1}, v) = 0. \qquad ■$$

Corollary 7.2.14 *Let w_{i+1}^{Λ} be the orthogonal projection of w_{i+1} on the Λ-eigenspace of A_{i-1}. Then*

$$w_{i+1}^{\Lambda} = 0.$$

The next lemma will be used to deal with the region S_1.

Lemma 7.2.15 *Let $U_i = \ker A_i$, $1 \le i \le m$. Then*

$$U_1 \supset U_2 \supset \cdots \supset U_m = \{0\}.$$

Moreover

$$U_i = \{u \in U_{i-1} : (u, w_{i+1}) = 0\}.$$

Proof Suppose $A_i u = 0$. Then

$$0 = (A_i u, u) = (A_{i-1}u, u) + r_{i+1}|(w_{i+1}, u)|^2.$$

But A_{i-1} is positive semidefinite and $r_{i+1} > 0$. Hence $u \in \ker A_{i-1}$ and $(u, w_{i+1}) = 0$. ■

Corollary 7.2.16 *Let w^0_{i+1} be the projection of w_{i+1} on $\ker A_i$. Then*

$$w^0_{i+1} = 0.$$

Now we can prove the result we need. Let $Z(\Theta)$ be the simultaneous zero level set of the angle variables $\{\theta_{ij}\}$.

Proposition 7.2.17

$$Z(\Theta) = M_{\mathbf{r}}(\mathcal{S}_{m+1}).$$

Proof The inclusion

$$M_{\mathbf{r}}(\mathcal{S}_{m+1}) \subset Z(\Theta)$$

is obvious (all the edges and diagonals are real, so the eigenvectors are real, so the β_{ij} are real). The point is to prove the reverse inclusion. We will assume $n \geq 2(m+1)$ and leave the case $n \leq 2m+1$, which is similar, to the reader.

Given a polygon $\mathbf{e}$ with all $\theta_{ij} = 0$. We wish to show that a sequence of conjugations of $\mathbf{e}$ by elements of U(m+1) will make all sides e_k real symmetric, or equivalently, all the w_k real. The proof is by descending induction starting with the last diagonal $A_{n-1} = e_1 + \cdots + e_n = \Lambda\mathbb{I}$, which is of course real symmetric. First, conjugate $\mathbf{e}$ by $g \in$ U(m+1) (without changing A_{n-1}) to arrange that A_{n-2} is diagonal, hence real. This moves all the w_k to gw_k, but in the sequel we do not need to keep track of those changes. Now we know that A_{n-3} has the form

$$A_{n-3} = A_{n-2} - r_{n-1}w_{n-1} \otimes w^*_{n-1},$$

and we want to show that we can move w_{n-1} to a real vector. We have

$$\ker(A_{n-2} - \Lambda\mathbb{I}) = \{\epsilon_1, \ldots, \epsilon_m\},$$

where $\{\epsilon_1, \ldots, \epsilon_{m+1}\}$ is the standard basis for $\mathbb{C}^{m+1}$. Suppose $A_{n-2}\epsilon_{m+1} = \mu\epsilon_{m+1}$, $\mu = \Lambda - r_n$.

Write w_{n-1} in the form $w_{n-1} = w^{\Lambda}_{n-1} + w^{\perp}_{n-1}$, where w^{Λ}_{n-1} is the orthogonal projection of w_{n-1} onto $\ker(A_{n-2} - \Lambda\mathbb{I})$. Hence there exists $z \in \mathbb{C}$ such that $w^{\perp}_{n-1} = z\epsilon_{m+1}$. Since w_{n-1} is defined only up to a complex multiple of unit length, we may multiply w_{n-1} by an element of S^1 in order to arrange that z be real. Let $c = \|w^{\Lambda}_{n-1}\|$. Now choose $g \in$ U(m+1) such that $g\epsilon_{m+1} = \epsilon_{m+1}$ and $gw^{\Lambda}_{n-1} = c\epsilon_m$. Then $gA_{n-2}g^{-1} = A_{n-2}$ (because $g\epsilon_{m+1} = \epsilon_{m+1}$ and $gw_{n-1} = c\epsilon_m + z\epsilon_{m+1}$). We change $\mathbf{e} = (e_1, \ldots, e_n)$ to $g\mathbf{e}g^{-1} = (ge_1g^{-1}, \ldots, ge_ng^{-1})$.

Next, we show how to find a conjugation $g\mathbf{e}g^{-1}$ that keeps A_{n-2}, A_{n-3} and w_{n-1} real and also makes gw_{n-2} real. This step exhibits the general pattern.

By Lemma 7.2.13,

$$\begin{aligned}\ker(A_{n-3} - \Lambda\mathbb{I}) &= \{v \in \ker(A_{n-2} - \Lambda\mathbb{I}) : (v, w_{n-1}) = 0\} \\ &= \operatorname{span}\{\epsilon_1, \dots, \epsilon_{m-1}\}.\end{aligned}$$

The matrix A_{n-3} has two new eigenvalues (in addition to Λ); let their eigenvectors be $u_{n-3,m+1}, u_{n-3,m}$. There is one angle variable

$$\theta_{n-3,m+1} = \arg\Big[(w_{n-2}, u_{n-3,m})(u_{n-3,m}, w_{n-1}) \\ (w_{n-1}, u_{n-3,m+1})(u_{n-3,m+1}, w_{n-2})\Big]$$

We have seen that A_{n-3} is real symmetric, hence $u_{n-3,j}$ can be chosen to be real for all $1 \le j \le m+1$. Since w_{n-1} is real, we may normalize $u_{n-3,m}$ and $u_{n-3,m+1}$ so that $(w_{n-1}, u_{n-3,m}) > 0$ and $(w_{n-1}, u_{n-3,m+1}) > 0$. Since, by assumption, $\theta_{n-3,m+1} = 0$, we have

$$\arg(w_{n-2}, u_{n-3,m+1}) = \arg(w_{n-2}, u_{n-3,m}).$$

Hence by multiplying w_{n-2} by an element in S^1 we may assume that $(w_{n-2}, u_{n-3,m+1})$ and $(w_{n-2}, u_{n-3,m})$ are real. Now we may write

$$w_{n-2} = w_{n-2}^{\Lambda} + w_{n-2}^{\perp},$$

where

$$w_{n-2}^{\Lambda} \in \ker((A_{n-3} - \Lambda\mathbb{I}) = \operatorname{span}\{\epsilon_1, \dots, \epsilon_{m-1}\}$$

and

$$w_{n-2}^{\perp} \in \operatorname{span}\{u_{n-3,m}, u_{n-3,m+1}\} = \operatorname{span}\{\epsilon_m, \epsilon_{m+1}\}.$$

We have arranged for $w_{n-2}^{\perp}$ to be real. Choose $g \in \mathrm{U(m+1)}$ with $g\epsilon_m = \epsilon_m$ and $g\epsilon_{m+1} = \epsilon_{m+1}$ such that

$$gw_{n-2}^{\Lambda} = c'\epsilon_{m-1},$$

with $c' = \|w_{n-2}^{\Lambda}\|$ as in the preceding step. Now change $\mathbf{e}$ to $g\mathbf{e}g^{-1}$ and proceed to w_{n-3}.

We continue in this way until $\ker(A_k - \Lambda\mathbb{I}) = 0$ and we enter the region S_2. The argument for this region is simpler and is left to the reader. Note that the vanishing of the angle variables says that *all* the coordinates $(w_k, u_{k-1,j})$ in the eigenvector basis of A_{k-1} have a common phase which can be eliminated by multiplication by an element of S^1; no conjugation is needed, so the preceding edges all remain real symmetric. However, the zero eigenvalue, which is unavoidable when we enter region S_1, causes new problems, and Lemma 7.2.15 is required.

Suppose then we have proved that A_m is real (note that A_m has rank m). We want to prove that A_{m-1} is real. We know that

$$A_m = A_{m-1} + r_{m+1} w_{m+1} \otimes w_{m+1}^*,$$

and since $\ker A_m = \{0\}$, we have enough angle variables to prove that all coordinates of w_{m+1} have a common phase. We clear this phase as before and move on to A_{m-2}. We have $A_{m-1} = A_{m-2} + r_m w_m \otimes w_m^*$, and wish to prove that one can make w_m real without destroying reality of $w_{n-1}, \dots, w_{m+1}$. Write $w_m = w_m^\perp + w_m^0$ with $A_{m-1} w_m^0 = 0$ and $w_m^\perp$ orthogonal to $\ker A_{m-2}$ (the latter has dimension 2). By the corollary to Lemma 7.2.15, we have $w_m^0 = 0$. Also, we have enough angle variables to conclude that the coordinates of $w_m^\perp$ relative to the eigenvectors of A_{m-1} orthogonal to $\ker A_{m-2}$ have a common phase. Thus, no conjugations are required to make w_m real, and all preceding edges remain real symmetric. Now continue. ■

We remark that the proof could equally well be done by ascending induction; in that case, region S_1 would be the one requiring conjugations, while an overall scaling would do in S_2, S_3.

We are now ready to prove:

Proposition 7.2.18

$$\{\theta_{ij}, \theta_{kl}\} = 0.$$

Proof Let $\mathbf{e} \in M_{\mathbf{r}}$ be given. By Corollary 7.2.9, the bending deformations flows permute the level sets of the θ_{ij}'s transitively. Hence we may apply a bending ϕ to move $\mathbf{e}$ into $Z(\Theta)$. Since ϕ is symplectic and the Hamiltonian vector fields of the θ_{ij} are invariant under bending, we have

$$\{\theta_{ij}, \theta_{kl}\}(\mathbf{e}) = \{\theta_{ij}, \theta_{kl}\}(\phi\mathbf{e}).$$

But by Proposition 7.2.17

$$Z(\Theta) = M_{\mathbf{r}}(\mathcal{S}_{m+1}).$$

Hence by Lemma 7.2.12

$$\{\theta_{ij}, \theta_{kl}\} = 0.$$ ■

8 The Duality Between the Bending Systems and the Gel'fand–Tsetlin Systems on Grassmannians

In this section we use Gel'fand–MacPherson duality, following [HK97] for the case of $m = 1$, to show that the bending system is equivalent to the Gel'fand–Tsetlin integrable system (as defined in [GS83]) on a torus quotient of the Grassmannian $G(m+1, \mathbb{C}^n)$. This equivalence will explain the appearance and form of the Gel'fand–Tsetlin patterns in §8.

Our first goal is to construct a symplectomorphism Φ from $M_{\mathbf{r}}$ to a symplectic quotient of $G(m+1, \mathbb{C}^n)$ by the n-torus T of diagonal matrices in U(n). This is the symplectic version of Gel'fand–MacPherson duality.

Let $\mathcal{M}$ denote the vector space of $(m+1) \times n$ complex matrices. We give $\mathcal{M}$ the Hermitean form $(\cdot, \cdot)$ defined by $(X, Y) = 2\operatorname{Tr}(X^*Y)$, and thus $\mathcal{M}$ is a symplectic vector space. The product group U(m+1) $\times$ U(n) acts isometrically and symplectically on $\mathcal{M}$. Denote the ith row (resp., jth column) of $N \in \mathcal{M}$ by R_i (resp., C_j).

Proposition 8.1 *The action of U(n) has momentum map*

$$\mu_{U(n)}\colon \mathcal{M} \to \mathcal{H}_n, \quad \mu_{U(n)}\colon N \mapsto N^*N.$$

In particular, the momentum map for the T-action is

$$\mu_T\colon N \mapsto (\|C_1\|^2, \dots, \|C_n\|^2).$$

The momentum map for the U(m+1) action is

$$\mu_{U(m+1)}\colon \mathcal{M} \to \mathcal{H}_{m+1}, \quad \mu_{U(m+1)}\colon N \mapsto NN^*.$$

Note that

$$\mu_{\mathrm{U(m+1)}}(N) = \sum_{i-1}^{n} C_i \otimes C_i^*.$$

This will provide the connection with polygons.

We construct the desired symplectomorphism by computing the symplectic quotient corresponding to the μ_T-level $\mathbf{r}$ and the $\mu_{\mathrm{U(m+1)}}$ level $\Lambda\mathbb{I}$ in two different orders. If we first quotient with respect to T with momentum level $\mathbf{r}$ and then with respect to U(m+1) with momentum level $\Lambda\mathbb{I}$, we get the space $M_{\mathbf{r}}$. In order to see this, we note that the (left) action of $\prod_1^n$ U(m+1) on $\mathcal{M}$ (acting on the columns) commutes with the (right) action of T (in fact one obtains a dual pair in the sense of Howe, see [KKS78]). We first compute the symplectic quotient by T.

Lemma 8.2

(1) *The momentum map $\mu_{(U(m+1))^n}$ induces an embedding of the symplectic quotient $\mu_T^{-1}(\mathbf{r})/T$ into $\Pi_1^n\mathcal{H}_{m+1}$, with image $\prod_1^n \mathcal{O}_{r_i}$.*
(2) *The form on $\mu_T^{-1}(\mathbf{r})/T$ induced by reducing the form $2\,\mathrm{Im}\,X^*Y$ when carried over to $\prod_1^n \mathcal{O}_{r_i}$ agrees with the Kostant–Kirillov form ω_{KK}.*

Proof The first statement follows because it is a general feature of dual pairs (see [KKS78]) that the momentum map for one action embeds the symplectic quotient of the other as an orbit in (the dual of) the Lie algebra of the first group. This principle, applied to the pair $(\mathrm{U(m+1)})^n \times T$, implies the first statement in the lemma. The second follows from a straight-forward computation. ∎

Thus we have identified the quotient by T with the correct product of rank one orbits in $\mathcal{H}_{m+1}$. Clearly, after taking the symplectic quotient of this product by the diagonal action of U(m+1) (at momentum level $\Lambda\mathbb{I}$), we obtain $M_{\mathbf{r}}$.

Suppose instead we first quotient with respect to U(m+1) and momentum level $\Lambda\mathbb{I}$. We get the Grassmannian $G(m+1, \mathbb{C}^n)$ with a certain U(n)-invariant symplectic structure.

Lemma 8.3 *The momentum map $\mu_{U(n)}$ induces an embedding of the symplectic quotient $\mu_{U(m+1)}^{-1}(\Lambda\mathbb{I})/U(m+1)$ into $\mathcal{H}_n$, with image the U(n)-orbit $\mathcal{O}_\Lambda$ consisting of those matrices that have eigenvalue Λ with multiplicity $m+1$ and eigenvalue 0 with multiplicity $n-m-1$.*

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Proof The argument is analogous to the previous case, only this time we use the dual pair U(m+1) × U(n). ∎

Denote the torus quotient at momentum level $\mathbf{r}$ of the Grassmannian with the Kostant–Kirillov symplectic structure corresponding to Λ by $\mathcal{M}_\Lambda$. We have now obtained the desired symplectomorphism Φ from $M_{\mathbf{r}}$ to $\mathcal{M}_\Lambda$.

Of course this symplectomorphism gives a Poisson isomorphism between the Poisson algebras of smooth functions of $M_{\mathbf{r}}$ and $\mathcal{M}_\Lambda$. However, we want to make this more explicit and to localize it. Let $\mathcal{M}_{\mathbf{r},\Lambda}$ be the subset of $\mathcal{M}$ consisting of matrices N such that $\|C_j\|^2 = r_j$ and $N^*N = \Lambda\mathbb{I}$. Thus we have U(m+1) $\times T$ quotient mappings $\pi_1 : \mathcal{M}_{\mathbf{r},\Lambda} \to M_{\mathbf{r}}$ (first quotient by T then by U(m+1)) and $\pi_2 : \mathcal{M}_{\mathbf{r},\Lambda} \to \mathcal{M}_\Lambda$ (first quotient by U(m+1) then by T). We use the mappings π_1 and π_2 to realize (and localize) the Poisson isomorphism Φ from above. Let f be a function which is smooth on an open subset of $M_{\mathbf{r}}$. Use π_1 to pull f back to a U(m+1) $\times$ T-saturated open subset of $\mathcal{M}_{\mathbf{r},\Lambda}$. Since π_2 is a quotient map and $\pi_1^* f$ is invariant under U(m+1), we can first descend it to to a T-saturated open subset of the Grassmannian, then to the torus quotient of that open set, which is an open subset of $\mathcal{M}_\Lambda$. We note that Φ is determined by the equation

$$\Phi(\pi_1(N)) = \pi_2(N).$$

We now briefly review the Gel'fand–Tsetlin integrable system (for the details see [GS83]). As before, we identify the space $\mathcal{H}_n$ of $n \times n$ Hermitean matrices with the dual of the Lie algebra of U(n). We now construct $n(n+1)/2$ Poisson commuting functions on $\mathcal{H}_n$ which are smooth on a dense open subset. Let $X \in \mathcal{H}_n$. Let $\beta_i(X)$ be the principal $i \times i$ diagonal block. Define γ_{ij} on $\mathcal{H}_n$ by

$$\gamma_{ij}(X) = \lambda_j(\beta_i(X)),$$

where λ_j is the jth eigenvalue of the block. As usual, we assume that the eigenvalues of the ith block are arranged in nonincreasing order. It is proved in [GS83] that the γ_{ij} Poisson commute. We note that the γ_{nj} are Casimirs. The restrictions of the remaining Gel'fand–Tsetlin Hamiltonians to generic orbits are functionally independent and give rise to integrable systems on such orbits. The eigenvalues of the blocks interlace and can be arranged in a "Gel'fand–Tsetlin" pattern, shown here for $n = 6$.

$$\begin{array}{ccccccccccc}
\gamma_{61} & & \gamma_{62} & & \gamma_{63} & & \gamma_{64} & & \gamma_{65} & & \gamma_{66} \\
& \gamma_{51} & & \gamma_{52} & & \gamma_{53} & & \gamma_{54} & & \gamma_{55} & \\
& & \gamma_{41} & & \gamma_{42} & & \gamma_{43} & & \gamma_{44} & & \\
& & & \gamma_{31} & & \gamma_{32} & & \gamma_{33} & & & \\
& & & & \gamma_{21} & & \gamma_{22} & & & & \\
& & & & & \gamma_{11} & & & & &
\end{array}$$

Figure 3

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Since we are dealing with a degenerate orbit here (the Grassmannian), many of the γ_{ij} (at the ends of the rows) will be zero (see Remark 8.5 below, and Figure 2 above). The next proposition, combined with the earlier sections, shows how to extract a functionally independent set of Gel'fand–Tsetlin Hamiltonians and obtain angle variables for the Gel'fand–Tsetlin system on the Grassmannian.

Proposition 8.4 $\Phi^*\gamma_{ij} = \lambda_{ij}$.

Proof Let $\mathbb{I}_k$ be the diagonal n by n matrix whose first k eigenvalues are equal to 1 and last $n-k$ eigenvalues are equal to 0. We use $\mathbb{I}_k$ to "truncate" N, N^*N and NN^*. Put $N_k := N\mathbb{I}_k$. Then

$$\mu_{\mathrm{U(n)}}(N_k) = \mathbb{I}_k N^* N \mathbb{I}_k$$
$$\mu_{\mathrm{U(m+1)}}(M_k) = N\mathbb{I}_k\mathbb{I}_k N^*.$$

The matrix on the first line is $\beta_k(N^*N)$, the principal k by k block of the $n \times n$ matrix N^*N, and the matrix on the second line is the diagonal $A_{k-1} = C_1C_1^* + C_2C_2^* + \cdots + C_kC_k^*$. The matrices $\mathbb{I}_k N^* N \mathbb{I}_k$ and $N\mathbb{I}_k\mathbb{I}_k N^*$ have the same nonzero eigenvalues. But the eigenvalues of the second matrix are the bending Hamiltonians λ_{kj}, and the eigenvalues of the first matrix are the Gel'fand–Tsetlin Hamiltonians γ_{kj}. Finally we observe that

$$\begin{aligned}\gamma_{ij}\big(\Phi(\pi_1(M))\big) &= \gamma_{ij}(\pi_2(M)) = \lambda_j(\beta_i(\pi_M)) \\ &= \lambda_j\big(A_i(\pi_1(M))\big) = \lambda_{ij}(\pi_1(M)).\end{aligned}$$ ■

We conclude this section with three remarks.

Remark 8.5 Proposition 8.4 explains the appearance of Gel'fand–Tsetlin patterns in connection with the bending Hamiltonians. The appearance of the zeroes at the end of the rows in our patterns is explained because the Gel'fand–Tsetlin system in question is defined on a subset of the Hermitean matrices of rank at most $m+1$. Hence $\gamma_{ij} = 0$, $j > m+1$.

Remark 8.6 The reconstruction process in §9 may be interpreted as saying that the class of patterns introduced in §8 is precisely the class corresponding to Hermitean matrices of the form N^*N, where N is as above.

Remark 8.7 Fixing the row sums in the patterns in §8 to be partial sums of the r_j corresponds to taking the quotient of the Grassmannian by T (at level $\mathbf{r}$).

9 Pieri's Formula and the Duality at the Quantum Level

In this section we will assume that the r_i's are (positive) integers and that $\Lambda = (r_1 + \cdots + r_n)/(m+1)$ is an integer.The orbit $\mathcal{O}_{r_i}$ then corresponds under geometric quantization to the irreducible representation $\mathcal{S}^{r_i}(V)$ of U(m+1), where V denotes the standard (or vector) representation of U(m+1) on $\mathbb{C}^{m+1}$ and $\mathcal{S}^{r_i}(V)$ the r_ith symmetric power.

The (classical) duality result of the last section should have a quantum version. We note that the duality connected an integrable system (bending) on a symplectic quotient of $\prod_1^n \mathcal{O}_{r_i}$ by the diagonal action of U(m+1) and an integrable system (Gel'fand–Tsetlin) on a torus quotient of the Grassmannian $G(m+1, n)$. Thus, according to geometric quantization, at the quantum level we would expect a relation between an n-fold tensor product multiplicity for GL(m+1,ℂ) and a weight multiplicity for a Cartan power of the the $(m+1)$st exterior power of GL(n,ℂ). The bending system provides a (singular) real polarization of the space $M_{\mathbf{r}}$, the symplectic quotient (at level $\Lambda\mathbb{I}$) of $\prod_i \mathcal{O}_{r_1}$. Thus the number of lattice points in the momentum polyhedron $\mathbf{P}$ for bending should be equal to the multiplicity of the the 1-dimensional representation $(\det)^{\Lambda}$ in $\bigotimes_1^n \mathcal{S}^{r_i}(V)$. But on the other hand, the Gel'fand–Tsetlin system is a real polarization of the torus quotient of the Grassmannian (at level $\mathbf{r}$) where the Grassmannian is given the symplectic structure which corresponds to the orbit of U(n) through the diagonal matrix with $m+1$ Λ's and $n-m-1$ zeroes. Thus the above number of lattice points should also be the multiplicity of the $\mathbf{r}$th weight space in $C^{\Lambda} \bigwedge^{m+1} W$, the Λth Cartan power of the $m+1$st exterior power of the vector representation W of $U(n)$. (We recall that if W^{ν} is a representation with highest weight ν, then the pth Cartan power $C^p W^{\nu}$ is the irreducible representation with highest weight $p\nu$). This equality of multiplicities predicted is in fact true and will be proved below.

Remark 9.1 It is unfortunate that the theory of geometric quantization using a real polarization is not sufficiently well developed to allow one to deduce theorems in representation theory from equalities of numbers of lattice points in momentum polyhedra. At this time we can only regard such equalities as predictions of theorems in representation theory.

We first note how the interlacing of the spectra of the perturbed matrix and the unperturbed matrix (see §8)from the Weinstein–Aronszajn formula predicts Pieri's formula in representation theory.

9.1 The Weinstein–Aronszajn and Pieri Formulas

We recall Pieri's formula for tensoring an irreducible polynomial representation of U(m+1) with a symmetric power of the vector representation [FH, §A.1].

Theorem 9.1.1 (Pieri's Formula) *Let $\lambda = (\lambda_1, \ldots, \lambda_{m+1})$ be the highest weight of the polynomial representation $V(\lambda_1, \ldots, \lambda_{m+1})$ of U(m+1). Let k be a positive integer. Then*

$$V(\lambda_1, \ldots, \lambda_{m+1}) \otimes \mathcal{S}^k(V) = \oplus V(\nu_1, \ldots, \nu_{m+1})$$

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where the sum is taken over all dominant $\nu = (\nu_1, \ldots, \nu_n)$ *satisfying*

$$\nu_1 \geq \lambda_1 \geq \nu_2 \geq \cdots \geq \nu_{m+1} \geq \lambda_{m+1} \geq 0$$

and

$$\sum_{i=1}^{m+1} \nu_i = \sum_{i=1}^{m+1} \nu_i + k.$$

This is of course Proposition 6.1.3 restricted to integer eigenvalues. If $A \in \mathcal{O}_\lambda$, then the spectrum of the rank-one perturbation $A + k\, w \otimes w^*$ satisfies the interlacing and row sum conditions of the Pieri formula.

9.2 Duality at the Quantum Level

In this subsection we prove the theorem from representation theory that is predicted by the equality (of the numbers of lattice points) of the momentum polyhedra for bending and Gel'fand–Tsetlin. The required facts from representation theory can be found in [FH, Ze]. In what follows V will denote the standard representation of $U(m+1)$ (on $\mathbb{C}^{m+1}$) and W will denote the standard representation of U(n).

Theorem 9.2.1 *The multiplicity of the* 1*-dimensional representation* $(\det)^\Lambda$ *of* $U(m+1)$ *in* $\bigotimes_1^n \mathcal{S}^{r_j}(V)$ *is equal to the multiplicity of the weight* $\mathbf{r}$ *in the irreducible representation* $C^\Lambda \bigwedge^{m+1} W$ *of U(n). This common multiplicity is in fact equal to the number of lattice points in* $\mathbf{P}$.

The theorem will be a consequence of the next three lemmas. We will need:

Definition 9.2.2 Let λ be an l-tuple of positive integers and μ be a partition. Then the Kostka number $K_{\mu\lambda}$ is the number of ways to fill in the Young diagram corresponding to μ with λ_1 1's, λ_2 2's, ..., λ_l l's so that the rows are weakly increasing and the columns are strongly increasing.

By applying Pieri's formula iteratively one gets [FH, (A.9)]:

Lemma 9.2.3

$$\mathcal{S}^{r_1}(V) \otimes \mathcal{S}^{r_2}(V) \otimes \cdots \otimes \mathcal{S}^{r_n}(V) = \bigoplus_\mu K_{\mu\mathbf{r}} V(\mu).$$

We obtain

Corollary 9.2.4 *The multiplicity of the* 1*-dimensional representation* $(\det)^\Lambda$ *of* $U(m+1)$ *in* $\bigotimes_1^n \mathcal{S}^{r_j}(V)$ *is equal to the Kostka number* $K_{\Lambda(1^{m+1})\mathbf{r}}$.

Here the symbol $\Lambda(1^{m+1})$ means the partition $(\Lambda, \Lambda, \ldots, \Lambda)$ (there are $m+1$ Λ's). The corresponding Young diagram has $m+1$ rows and Λ columns.

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In order to compare $K_{\Lambda(1^{m+1})\mathbf{r}}$ with the multiplicity of the weight $\mathbf{r}$ in the irreducible representation $C^{\Lambda} \bigwedge^{m+1} W$ of U(n) we recall there is a basis for an irreducible representation of U(n) labeled by semistandard Young tableaux. Suppose the highest weight of the representation is μ. We also use μ to denote the Young diagram associated to μ. A *semistandard filling* of μ is an assignment of the integers between 1 and n to the boxes of μ such that the rows are weakly increasing and the columns are strongly increasing. The associated basis is a weight basis and the weight of of the basis vector corresponding to a semistandard tableau is $(k_1, \dots, k_n)$, where k_i is the number of i's in the tableau. Thus we have proved:

Lemma 9.2.5 *$K_{\Lambda(1^{m+1})\mathbf{r}}$ is also the multiplicity of the weight $\mathbf{r}$ in $C^{\Lambda} \wedge^{m+1} W$ of U(n).*

It still remains to prove that the number of lattice points in $\mathbf{P}$ is the common multiplicity.

To see this we recall that there is an orthonormal basis (the Gel'fand–Tsetlin basis) for the irreducible representation $C^{\Lambda} \wedge^{m+1} V$ indexed by Gel'fand–Tsetlin patterns whose top row consists of $m + 1$ Λ's and $n - m - 1$ zeroes. Moreover, this basis is a weight basis and the weight of a basis vector corresponding to a Gel'fand–Tsetlin pattern is given by the differences in the row sums starting with the bottom entry in the pattern. Thus we have:

Lemma 9.2.6 *The number of lattice points in $\mathbf{P}$ is equal to the dimension of the $\mathbf{r}$th weight space in $C^{\Lambda} \bigwedge^{m+1} W$.*

It follows that the count of lattice points in $\mathbf{P}$ gives the correct answer for both multiplicities.

Remark 9.2.7 One might ask whether there is a direct combinatorial argument to establish the last lemma above, *i.e.,* that the number of semistandard Young tableaux of weight $\mathbf{r}$ is equal to the number of Gel'fand–Tsetlin patterns of weight $\mathbf{r}$. In fact, there is a one-to-one weight preserving correspondence between semistandard Young tableaux and Gel'fand–Tsetlin patterns, see [GZ86].

A Appendix: Bending and Hitchin Hamiltonians

In [AHH], the authors constructed a duality between integrable systems on certain pairs of finite dimensional coadjoint orbits of loop groups. The spaces considered in §8 above belong to their family. However we shall show below that the Hamiltonians considered in [AHH] are different from ours even for the case of polygons in $\mathbb{R}^3$ *i.e.,* $m = 1$. The point is that our Hamiltonians depend on a triangulation of the polygon by diagonals (equivalently, the Gelfand–Tsetlin decomposition into increasing principal diagonal blocks). This introduces an asymmetry which is not present in the theory of [AHH]. We will not review the theory of [AHH] here but will instead review how one obtains integrable systems on $M_{\mathbf{r}}$ by associating a matrix $A(z)$ with entries which are polynomials in a complex variable z to a point $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}$. The construction of [AHH] is more general but reduces to the above for the case in hand.

We will refer to the resulting Hamiltonians as Hitchin Hamiltonians although the construction we are about to describe antedated and is a very special case of Hitchin's construction of integrable systems associated to Higgs fields. We will follow the notation of [Hi, pp. 46–52].

A.1 Hitchin Hamiltonians

We will describe the construction of Hitchin Hamiltonians for the case considered in this paper. So let $\mathbf{e} \in \widetilde{M}_{\mathbf{r}}$. Choose n points $\alpha_1, \alpha_2, \dots, \alpha_n$. Put $p(z) = \prod_{i=1}^{n}(z - \alpha_i)$ and define a matrix-valued polynomial $A(z)$ by

$$A(z) = p(z) \sum_{i=1}^{n} \frac{e_i}{z - \alpha_i}.$$

We then consider the characteristic polynomial $\det(w\mathbb{I} - A(z))$ of $A(z)$. The coefficients of the characteristic polynomial (considered as functions on $M_{\mathbf{r}}$) are the Hitchin Hamiltonians. In the case in which $m = 1$, it suffices to consider the functions

$$H_j(\mathbf{e}) = \sum_{i \neq j} \frac{e_i \cdot e_j}{\alpha_i - \alpha_j}, 1 \leq j \leq n.$$

We will now examine when the bending flows leave invariant the Hitchin Hamiltonians for the special case $m = 1$. We note that if f is a smooth function of $M_{\mathbf{r}}$, in order to compute its derivative along a bending flow we may choose any lift of f to $\widetilde{M}_{\mathbf{r}}$ and any lift of the bending flow to $\widetilde{M}_{\mathbf{r}}$.

A.2 Bending Hamiltonians Are Not Hitchin Hamiltonians

In this section we will prove

Proposition A.2.1 *If $\alpha_1, \dots, \alpha_n$ are distinct and $n \geq 5$, then the diagonal $d_{13} := \|e_1 + e_2\|$ does not belong to the Hitchin system.*

The proposition will follow from the next two lemmas. Let ϕ_t be the bending flow associated to diagonal d_{13}. We will lift d_{13} to $\widetilde{M}_{\mathbf{r}}$ and lift ϕ_t to $\widetilde{M}_{\mathbf{r}}$ so that the edges e_1 and e_2 are rotated around the diagonal $e_1 + e_2$ and the remaining edges are left fixed. We will continue to use ϕ_t for the lifted flow.

Lemma A.2.2 *Let $i > 2$ and c be a constant. Then the function $g(\mathbf{e}) := c\, e_1 \cdot e_i$ is invariant under the flow $\phi_t \Leftrightarrow c = 0$.*

Proof We have

$$\dot{e}_1 = e_2 \times e_1, \quad \dot{e}_2 = e_1 \times e_2 \quad \text{and} \quad \dot{e}_i = 0,\ i > 2.$$

Hence, if $i > 2$ we have

$$\frac{d}{dt} c e_1 \cdot e_i = c(e_2 \times e_1) \cdot e_i \equiv 0.$$

But there exist polygons such that the scalar triple $(e_2 \times e_1) \cdot e_i$ is nonzero. Consequently $c = 0$. ■

In the proof of the next two lemmas we will use the following notation. If f and g are two smooth functions on $\widetilde{M}_{\mathbf{r}}$ we will write $f \equiv g$ if f and g differ by a function that is invariant under ϕ_t.

Recall $H_n(e)$ is the Hitchin Hamiltonian defined by

$$H_n(e) = \sum_{i=1}^{n-1} \frac{e_i \cdot e_n}{\alpha_i - \alpha_n}.$$

Lemma A.2.3 *H_n is invariant under $\phi_t \Leftrightarrow \alpha_1 = \alpha_2$.*

Proof We have, since $e_i \cdot e_n$ is invariant under ϕ_t for $i > 2$ (because e_i is),

$$\begin{aligned} H_n(e) &\equiv \frac{e_1 \cdot e_n}{\alpha_1 - \alpha_n} + \frac{e_2 \cdot e_n}{\alpha_2 - \alpha_n} \\ &= \frac{e_1 \cdot e_n}{\alpha_1 - \alpha_n} + \frac{((e_1 + e_2) - e_1) \cdot e_n}{\alpha_2 - \alpha_n} \\ &\equiv \Big(\frac{1}{\alpha_1 - \alpha_n} - \frac{1}{\alpha_2 - \alpha_n} \Big) e_1 \cdot e_n. \end{aligned}$$

The last line follows because $e_1 + e_2$ is invariant under ϕ_t.

By the previous lemma we find that $H_n(e)$ is invariant if and only if

$$\frac{1}{\alpha_1 - \alpha_n} - \frac{1}{\alpha_2 - \alpha_n} = 0$$ ■

There remains the possibility that all the bending Hamiltonians could be Hitchin Hamiltonians for a singular curve. We will now show that this cannot happen (for the case $n = 5$). Suppose then $n = 5$. Let ϕ_t be the bending flow along the diagonal $e_1 + e_2$ and ψ_t be the bending flow along the diagonal $e_1 + e_2 + e_3$. We will lift ϕ_t (resp., ψ_t) to $\widetilde{M}_{\mathbf{r}}$ so that only the first two (resp., three) edges are moved. We will now use the notation $f \equiv g$ to denote that f and g differ by a function that is invariant under both flows.

Lemma A.2.4 *Suppose the Hitchin Hamiltonian H_5 is invariant under both bending flows ϕ_t and ψ_t. Then the first three α_i's are equal.*

Proof Since e_4 and e_5 are invariant under both flows we have

$$\begin{aligned} H_5(e) &\equiv \frac{e_1 \cdot e_5}{\alpha_1 - \alpha_5} + \frac{e_2 \cdot e_5}{\alpha_2 - \alpha_5} + \frac{e_3 \cdot e_5}{\alpha_3 - \alpha_5} \\ &= \frac{e_1 \cdot e_5}{\alpha_1 - \alpha_5} + \frac{e_2 \cdot e_5}{\alpha_2 - \alpha_5} + \frac{((e_1 + e_2 + e_3) - (e_1 + e_2)) \cdot e_5}{\alpha_3 - \alpha_5} \\ &\equiv \Big(\frac{1}{\alpha_1 - \alpha_5} - \frac{1}{\alpha_3 - \alpha_5} \Big) e_1 \cdot e_5 + \Big(\frac{1}{\alpha_2 - \alpha_5} - \frac{1}{\alpha_3 - \alpha_5} \Big) e_2 \cdot e_5. \end{aligned}$$

But by Lemma A.2.3 we have

$$\frac{1}{\alpha_1-\alpha_5}=\frac{1}{\alpha_2-\alpha_5}.$$

Consequently we have

$$H_5(e)\equiv\Big(\frac{1}{\alpha_1-\alpha_5}-\frac{1}{\alpha_3-\alpha_5}\Big)(e_1+e_2)\cdot e_5.$$

We put c equal to the coefficient of $(e_1+e_2)\cdot e_5$ act by ψ_t and differentiate to obtain

$$\frac{d}{dt}c(e_1+e_2)\cdot e_5=c((e_1+e_2+e_3)\times(e_1+e_2))\cdot e_5=0.$$

But there exist polygons such that the number $((e_1+e_2+e_3)\times(e_1+e_2))\cdot e_5$ is nonzero. Consequently $c=0$.

Hence $\alpha_1=\alpha_3$ and the first three α_i's are equal. ■

Now we prove that if $H_1(e)$ is also invariant under ϕ_t and ψ_t then all the α_i's are equal.

Lemma A.2.5 *Suppose the Hitchin Hamiltonian H_1 is invariant under both bending flows ϕ_t and ψ_t. Then the last three α_i's are equal.*

Proof Lift ψ_t from $M_{\mathbf{r}}$ to $\widetilde{M}_{\mathbf{r}}$ so that it rotates the triangle with edges $e_1+e_2+e_3, e_4$ and e_5 around the diagonal $e_1+e_2+e_3$ (so ψ_t leaves e_1, e_2 and e_3 fixed). Repeat the argument in Lemma A.2.3 to find $\alpha_4=\alpha_5$. Now lift ϕ_t to $\widetilde{M}_{\mathbf{r}}$ so that it rotates the quadrilateral with edges e_5, e_4, e_3 and e_1+e_2 around the diagonal e_1+e_2 (so ϕ_t leaves e_1 and e_2 fixed). Repeat the argument of Lemma A.2.4 to find the last three α_i's are equal. ■

We can now show for pentagons that the bending systems and the Hitchin systems never coincide.

Theorem A.2.6 *For the case $n=5$, the Hitchin system never coincides with the bending system.*

Proof If the systems coincide, then H_1 and H_5 are invariant under both bending flows and all the α_i's are equal. But in this case we obtain

$$A(z)=\sum_{i=1}^{5}\frac{e_i}{z-\alpha_i}=\frac{1}{z-\alpha_1}\sum_{i=1}^{5}e_i=0.$$

Hence $\det(w\mathbb{I}-A(z))=w^2$ and there are no nontrivial Hitchin Hamiltonians. ■

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