6

Functional methods in nonequilibrium QFT

One of the major goals in the establishment of a quantum field theory for nonequilibrium systems is to study dynamical problems, following the evolution of the expectation value of a physical variable with respect to an *in* state. This is different from a scattering problem characterized by the transition amplitude between the *in* and *out* states, as is treated in every textbook on quantum field theory. This problem is usually coupled with how one could identify a relevant sector of the theory as the system (light fields vs. heavy fields, slow modes vs. fast modes, long wavelength modes vs. short ones, etc.) and determine the effect of its other sectors as the environment on this system, as we have discussed in the last chapter.

Given a classical nonequilibrium system, described, for example, by a Langevin equation, there are essentially three possible strategies to follow. One may attempt to solve it, usually numerically. In the quantum field case, this gets difficult beyond the linear case. Second, one may try to transform it into an equation for the evolution of a probability distribution function in the system's configuration space. In the quantum field case the relevant object is the reduced density matrix, and the relevant equation is the Liouville–von Neumann equation. This is also infeasible beyond the linear case, unless under restrictive approximations (such as Gaussianity) which in fact reduce this approach to the third and coarsest. The third approach is to use the Langevin equation to obtain equations of motion for the expectation value $\langle x \rangle$ of the system variable x and its correlations. In the linear case, the relevant equation of motion is Heisenberg's equation (Ehrenfest's theorem). In the nonlinear case, the equation of motion for $\langle x \rangle$ will necessarily couple to higher correlation functions $\langle x^n \rangle$, and we will have a hierarchy of equations, just as in the BBGKY–Boltzmann paradigm. Then we will have to truncate the hierarchy, slave the higher correlations, etc., according to the particular set-up of the problem to extract the information accessible to a particular class of observers. We have given an introductory discussion of these issues in the context of the Boltzmann equation in Chapters 1 and 2.

The third approach is the one we will follow in the bulk of the book. Our immediate concern is to show that the (truncated) equations of motion may be obtained from variational principles of increasing complexity. At the simplest level, where we only seek an equation of motion for the expectation value ϕ of the Heisenberg picture field operator Φ, this equation follows from the variation of the Schwinger–Keldysh (CTP) effective action (EA). At the next level of a more

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comprehensive approach we seek coupled equations for ϕ and the "propagators" $G \sim \langle(\Phi - \phi)^2\rangle$. The relevant action functional is the two-particle irreducible (2PI) effective action. Each higher order truncation of the hierarchy has its proper action functional, which are particular cases of the so-called "master" effective action (MEA).

Therefore the task at hand is to develop the techniques to compute these objects, and to learn to read the physics coded into their structure. In particular, we shall see that the CTPEA has the structure of a Feynman–Vernon influence action, and we shall simply borrow the physical insight gained from the study of quantum open systems. The analogy is less straightforward for the higher order truncations, but this approach remains essentially applicable.

Since field expectation values and propagators are going to be the main subjects of our discussion, it is appropriate that we start by gaining some insight into the different two-point functions and the information they contain. Thus, let us begin our discussion with a review of some basic scalar quantum field theory.

6.1 Propagators

A good deal of our discussion will revolve around the different properties of the propagators of the theory, that is, the expectation values of binary products of field operators with respect to the initial state. Since field operators at different locations do not generally commute, we have several different propagators according to the ordering of the field operators within the expectation value. In particular, we shall consider eight different propagators, namely:

The four basic propagators

Feynman

$$G_{\mathrm{F}} \equiv \langle T\left(\Phi\left(x\right)\Phi\left(x'\right)\right)\rangle \tag{6.1}$$

Dyson

$$G_{\mathrm{D}} \equiv \langle \tilde{T}\left(\Phi\left(x\right)\Phi\left(x'\right)\right)\rangle \tag{6.2}$$

positive frequency

$$G^{+} \equiv \langle \Phi\left(x\right)\Phi\left(x'\right)\rangle \tag{6.3}$$

and negative frequency

$$G^{-} \equiv \langle \Phi\left(x'\right)\Phi\left(x\right)\rangle \tag{6.4}$$

where T stands for time ordering and $\tilde{T}$ stands for anti-time ordering:

$$\begin{aligned} T\left[F\left(t\right)G\left(t'\right)\right] &= F\left(t\right)G\left(t'\right)\theta\left(t-t'\right) + G\left(t'\right)F\left(t\right)\theta\left(t'-t\right) \\ \tilde{T}\left[F\left(t\right)G\left(t'\right)\right] &= G\left(t'\right)F\left(t\right)\theta\left(t-t'\right) + F\left(t\right)G\left(t'\right)\theta\left(t'-t\right) \end{aligned} \tag{6.5}$$

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The Feynman and Dyson propagators are even. We also have $G_{\mathrm{F}} = G_{\mathrm{D}}^{*}$; $G^{-} = G^{+*}$; $G^{-}(x, x') = G^{+}(x', x)$. Finally we have the identity $G_{\mathrm{F}} + G_{\mathrm{D}} = G^{+} + G^{-}$, which follows from the time ordering constraints.

Hadamard and Jordan propagators

The Hadamard propagator

$$G_1 = G^{+} + G^{-} \equiv \langle \{\Phi(x), \Phi(x')\} \rangle \tag{6.6}$$

is real and even. The Jordan propagator

$$G = G^{+} - G^{-} \equiv \langle [\Phi(x), \Phi(x')] \rangle \tag{6.7}$$

is imaginary and odd.

Advanced and retarded propagators

The advanced and retarded propagators are the fundamental solutions for the equations of motion for linear fluctuations in the field.

Each propagator conveys some specific information. For example, for the free scalar field G, G_1, G^{+} and G^{-} are solutions to the homogeneous Klein–Gordon equation, while G_{F}, G_{D}, G_{ret} and G_{adv} are fundamental solutions. The retarded and advanced propagators may be obtained from consideration of the dynamics alone; they have no information on the state. The same can be said of the Jordan propagator, since for linear fields the commutator is a c-number. Indeed, consideration of the respective Cauchy data shows that we must have the identities (the Cauchy data for the Jordan propagator are prescribed by the equal time canonical commutation relations)

$$\begin{aligned} G_{\mathrm{adv}}(x, x') &= -\frac{i}{\hbar} G(x, x')\, \theta(t' - t) \\ G_{\mathrm{ret}}(x, x') &= G_{\mathrm{adv}}(x', x) = \frac{i}{\hbar} G(x, x')\, \theta(t - t') \\ G(x, x') &= (-i\hbar)\left[G_{\mathrm{ret}}(x, x') - G_{\mathrm{adv}}(x, x')\right] \end{aligned} \tag{6.8}$$

or else

$$G_{\mathrm{ret}} = \frac{i}{\hbar}\left(G_F - G^{-}\right) \tag{6.9}$$

$$G_{\mathrm{adv}} = \frac{-i}{\hbar}\left(G_D - G^{-}\right) \tag{6.10}$$

Therefore the state information is coded primarily in the remaining propagators, most of all in Hadamard's. Knowledge of the Hadamard and Jordan propagators determines all others

$$G^{\pm}(x, x') = \frac{1}{2}\left[G_1(x, x') \pm G(x, x')\right] \tag{6.11}$$

$$\begin{aligned} G_{\mathrm{F,D}}(x, x') &= \frac{1}{2}\left[G_1(x, x') \pm G(x, x')\, \mathrm{sign}(t - t')\right] \\ &= \frac{1}{2}\left[G_1(x, x') \mp i\hbar\left(G_{\mathrm{ret}}(x, x') + G_{\mathrm{ret}}(x', x)\right)\right] \end{aligned} \tag{6.12}$$

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As a warm up, we shall compute the propagators (that is, the expectation values of products of fields) in the Minkowski vacuum $|0\rangle$. Let us begin with the positive and negative frequency propagators. The negative frequency propagator $\Delta^{-}(x,x')$ is given in Chapter 5, equation (5.21), and

$$\Delta^{+}(x,x') = \Delta^{-}(x',x) \tag{6.13}$$

All other propagators may be found as linear combinations of these. For example, their difference gives the commutator or Jordan propagator, which for free fields is both independent of the state and of the particle model

$$\Delta(x,x') = \langle [\Phi(x), \Phi(x')] \rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-x')} \mathrm{sign}\left(k^0\right) 2\pi\hbar\delta\left(k^2+m^2\right) \tag{6.14}$$

The sum of the positive and negative frequency propagators gives the anticommutator or Hadamard propagator

$$\Delta_1(x,x') = \langle \{\Phi(x), \Phi(x')\} \rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-x')} 2\pi\hbar\delta\left(k^2+m^2\right) \tag{6.15}$$

The four propagators introduced so far are homogeneous solutions of the Klein–Gordon equation. The retarded propagator

$$\Delta_{\mathrm{ret}}(x,x') = \frac{i}{\hbar}\Delta(x,x')\,\theta(t-t') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{(k+i\varepsilon)^2+m^2} \tag{6.16}$$

$$(k+i\varepsilon)^2 = -\left(k^0+i\varepsilon\right)^2 + \mathbf{k}^2 \tag{6.17}$$

is the (only) solution to the equation

$$\left[\nabla^2 - m^2\right]\Delta_{\mathrm{ret}}(x,x') = -\delta(x,x') \tag{6.18}$$

with causal boundary conditions. We also have the advanced propagator

$$\Delta_{\mathrm{adv}}(x,x') = \Delta_{\mathrm{ret}}(x',x) = -\frac{i}{\hbar}\Delta(x,x')\,\theta(t'-t) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{(k-i\varepsilon)^2+m^2} \tag{6.19}$$

which is the fundamental solution with advanced boundary conditions. Finally, there are the Feynman and Dyson propagators, given in Chapter 5, equation (5.20).

6.1.1 Interacting fields

The basic property of the full propagators for an interacting field, that is, the expectation values of binary products of field operators with respect to the vacuum state, is Poincaré invariance. In particular, these propagators are translation invariant, which allows us to describe them in terms of their Fourier transforms, namely, any propagator G may be represented as

$$G(x,x') = \int \frac{d^4p}{(2\pi)^4}\, e^{ipu} G(p) \tag{6.20}$$

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with $u = x - x'$. Some important properties of the propagators actually follow from their definition as time-ordered products of field operators. For example, since the Feynman and Dyson propagators are even and $G_{\mathrm{F}} = G_{\mathrm{D}}^{*}$, $G^{-} = G^{+*}$ and $G^{-}(x, x') = G^{+}(x', x)$, G_{F} and $G_{\mathrm{D}}(p)$ are even functions of momentum, while $G^{-}(p) = G^{+}(-p)$. Moreover, G^{-} and $G^{+}(p)$ are real, and $G_{\mathrm{D}}(p)^{*} = G_{\mathrm{F}}$.

The Hadamard propagator G_1 is real and even and therefore also is $G_1(p)$. The Jordan propagator G is imaginary and odd, and so $G(p)$ is odd but real. The Jordan and retarded propagators are related through equation (6.8)

$$G(p) = (-i\hbar)\left[G_{\mathrm{ret}}(p) - G_{\mathrm{ret}}(-p)\right] = 2\hbar \mathrm{Im}\, G_{\mathrm{ret}}(p) \tag{6.21}$$

where we have used the fact that $G_{\mathrm{adv}}(p) = G_{\mathrm{ret}}(p)^{*}$. Also observe that $G_{\mathrm{ret}}(x, x')$ is real, so $G_{\mathrm{ret}}(-p) = G_{\mathrm{ret}}(p)^{*}$.

Since the retarded propagator is causal, it satisfies the equation $G_{\mathrm{ret}} = \theta(t - t') G_{\mathrm{ret}}$, and the real and imaginary parts of its transform are Hilbert transforms of each other

$$\begin{aligned} G_{\mathrm{ret}}(p) &= \frac{i}{2\pi}\int \frac{d\omega}{p^0 - \omega + i\varepsilon} G_{\mathrm{ret}}(\omega, \mathbf{p}) \\ &= \frac{1}{2} G_{\mathrm{ret}}(p) + \frac{i}{2\pi} PV \int \frac{d\omega}{p^0 - \omega} G_{\mathrm{ret}}(\omega, \mathbf{p}) \end{aligned} \tag{6.22}$$

$$\mathrm{Re} G_{\mathrm{ret}}(p) = \frac{1}{\pi} PV \int \frac{d\omega}{\omega - p^0} \mathrm{Im} G_{\mathrm{ret}}(\omega, \mathbf{p}) \tag{6.23}$$

These are the so-called Kramers–Kronig relations.

For further properties of the propagators, such as their Lehmann representation, we refer the reader to the literature on QFT; some classic textbooks are listed [Rom69, LanLif72, BjoDre64, BjoDre65, ItzZub80, Ram80, Hua98, PesSch95, LeB91, Zin93, Wei95, GrReBr96].

6.2 Functional methods

Before we confront the full nonequilibrium problem, it is instructive to review the more familiar case of finding the expectation values under equilibrium conditions. We will then be in a better position to judge whether a nonequilibrium formalism is a straightforward generalization of the equilibrium one, or where some new insights are required.

So let us begin by asking what is the expectation value of the field operator at a given spacetime event. In a theory such as $\lambda\Phi^4$, which is symmetric under the inversion $\Phi \to -\Phi$, one is tempted to say that the expectation value must vanish, by symmetry. However, this is not necessarily so; the quantum state of the field may have a lesser symmetry than the Hamiltonian, supporting a nonzero expectation value or condensate. In this case we say the symmetry is spontaneously broken. This is even more so if the theory is not even invariant under inversion, for example, for a potential $V[\Phi] = g\Phi^3/6$.

The problem is enormously simplified if we still assume that Poincaré invariance will not be even spontaneously broken. In this case, the expectation value ϕ of the field operator $\Phi(x)$ will be Poincaré invariant; in particular, it will be space and time independent.

In order to find the equilibrium value of the expectation value, it is convenient to work in two stages. First we assume that the system is constrained, by some external means, to a state where the expectation value of the field operator takes a preassigned value ϕ; we then pick the value of ϕ leading to greatest stability. To solve the first half of the problem, we must find the (properly normalized) state which minimizes the energy while having the correct expectation value of the field operators. To enforce these constraints (there is one at every event) we introduce a Lagrange multiplier J. Thus the object to be minimized is

$$\langle | H_J + \int d^3\mathbf{x}\; J\phi \, | \rangle \tag{6.24}$$

where

$$H_J = H - J \int d^3\mathbf{x}\; \Phi(x) \tag{6.25}$$

Let $|J\rangle$ be the state that minimizes this operator. It will be a proper vector of the operator H_J with proper value E_J

$$H_J |J\rangle = E_J |J\rangle \tag{6.26}$$

Because the state is assumed to be homogeneous and energy is an extensive quantity, the energy E_J will be proportional to the "volume" V of space, with a finite energy density E_J/V. First-order perturbation theory shows that

$$\frac{\delta}{\delta J}\left(\frac{E_J}{V}\right) = -\frac{1}{V}\left\langle J \left| \int d^3\mathbf{x}\; \Phi(x) \right| J \right\rangle = -\phi \tag{6.27}$$

If we introduce the Legendre transform of E_J/V, the so-called "effective potential" $V[\phi]$,

$$V[\phi] = \frac{E_J}{V} + \frac{1}{V}\int d^3\mathbf{x}\; J\phi. \tag{6.28}$$

then

$$\frac{d}{d\phi} V[\phi] = J \tag{6.29}$$

This equation determines ϕ if J is known. The most stable state is the one which does not require external intervention ($J = 0$); the true equilibrium expectation values are the extrema of the effective potential.

By the way, we see what is effective in the effective potential: it is not really the value of the energy density, but rather a Legendre transform thereof. The external source J and the expectation value ϕ are analogous, respectively, to the applied magnetic field B and the magnetization M in a model for ferromagnetism. Also keep in mind that we are describing equilibria at a prescribed temperature (prescribed to be zero). We may think of the effective potential as the thermodynamic potential whose critical points are the equilibria at constant temperature and ϕ, and therefore as a free energy (as opposed to the internal energy). However, so far we have not said how we intend to compute the effective potential, so that the full approach takes on meaning. We turn now to this important issue.

6.2.1 The generating functional and the effective action

We shall begin from the observation that the state $|J\rangle$ minimizes the operator H_J. Let $|\alpha\rangle$ be any state, and let it evolve from *Euclidean* time $\tau = -t_\mathrm{E}$ to $\tau = t_\mathrm{E}$ (subscript E under t denotes Euclidean time) adopting H_J as the Hamiltonian. Then $|\alpha\rangle$ evolves into

$$|\alpha, t_\mathrm{E}\rangle = e^{-2\hbar^{-1}H_J t_\mathrm{E}}\,|\alpha\rangle = e^{-2\hbar^{-1}E_J t_\mathrm{E}}\,|J\rangle\,\langle J\,|\alpha\rangle + \delta\,|\alpha, t_\mathrm{E}\rangle \tag{6.30}$$

where $\delta\,|\alpha, t_\mathrm{E}\rangle$ decays faster than $e^{-2\hbar^{-1}E_J t_\mathrm{E}}$. It follows that, given any other state $|\beta\rangle$, and as long as $\langle J\,|\alpha\rangle \neq 0$ and $\langle J\,|\beta\rangle \neq 0$, then

$$E_J = -\lim_{t_\mathrm{E}\to\infty}\frac{\hbar}{2t_\mathrm{E}}\ln\langle\beta|\,e^{-2\hbar^{-1}H_J t_\mathrm{E}}\,|\alpha\rangle \tag{6.31}$$

Using a path integral representation for the evolution operator, we find

$$E_J = -\lim_{t_\mathrm{E}\to\infty}\frac{\hbar}{2t_\mathrm{E}}\ln\int D\varphi\; e^{\hbar^{-1}\left[-S_\mathrm{E}+\int d^4x\; J\varphi(x)\right]} \tag{6.32}$$

where S_E stands for the Euclidean action

$$S_\mathrm{E} = \int d^4x\,\left\{\frac{1}{2}\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + \frac{1}{2}\left(\nabla\varphi\right)^2 + V\left[\varphi\right]\right\} \tag{6.33}$$

This path integral representation displays the close connection between E_J and the Euclidean generating functional for connected Green functions $W_\mathrm{E}\left[J\right]$

$$e^{\hbar^{-1}W_\mathrm{E}[J]} = \int D\varphi\; e^{\hbar^{-1}\left[-S+\int d^4x\; J(x)\varphi(x)\right]} \tag{6.34}$$

If the source J is spacetime independent, then

$$W_\mathrm{E}\left[J\right] \sim -2t_\mathrm{E}E_J \tag{6.35}$$

The Legendre transform of $W_\mathrm{E}\left[J\right]$ is the Euclidean effective action (EA) $\Gamma_\mathrm{E}\left[\phi\right]$. As we all know, if we Taylor expand the EA in powers of the background field ϕ, the coefficients are given by the sum of all one-particle irreducible Feynman graphs. These are the graphs that are connected, and remain so if any internal line is cut. This method is not efficient as a practical tool, but the observation that the EA could be computed this way is at the base of a much better strategy, the loop expansion, which we shall discuss below. For the time being, simply recall that

$$\Gamma_\mathrm{E}\left[\phi\right] = W_\mathrm{E}\left[J\right] - \int d^4x\; J\left(x\right)\phi\left(x\right) \tag{6.36}$$

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So, if the background field $\phi(x)$ is constant, then

$$\Gamma_{\mathrm{E}}[\phi] = -(2t_{\mathrm{E}}V)\, V[\phi] \tag{6.37}$$

and the relationship between mean fields and sources is

$$\frac{\delta \Gamma_{\mathrm{E}}[\phi]}{\delta \phi(x)} = -J(x) \tag{6.38}$$

In particular, the true equilibria are the Poincaré invariant extrema of the effective action. For further discussion, we refer the reader to Coleman's Lectures [Col85]; see also [JacKer79].

6.2.2 Not quite beyond equilibrium

Although only Poincaré invariant extrema are meaningful, the effective action may be constructed for arbitrary field configurations. However, equation (6.38) does not provide an off-equilibrium equation of motion for the mean field. This is an important point, and we must be sure we understand it before we carry on.

There is of course the observation that equation (6.38) applies to Euclidean field configurations. However, this difficulty is easily overcome. Define the Lorentzian or in-out generating functional $W_{\textit{in-out}}[J]$

$$e^{i\hbar^{-1}W_{\textit{in-out}}[J]} = \int D\varphi\, e^{i\hbar^{-1}\left[S+\int d^4x\; J(x)\varphi(x)\right]} \tag{6.39}$$

where S is the physical action. Then define

$$\frac{\delta W_{\textit{in-out}}[J]}{\delta J} = \tilde{\phi}(x) \tag{6.40}$$

Performing the Legendre transform

$$\Gamma_{\textit{in-out}}\left[\tilde{\phi}\right] = W_{\textit{in-out}}[J] - \int d^4x\; J(x)\,\tilde{\phi}(x) \tag{6.41}$$

$W_{\textit{in-out}}[J]$ is the generating functional for connected graphs, and $\Gamma_{\textit{in-out}}$ generates one-particle irreducible graphs. Moreover, $\tilde{\phi}$ satisfies

$$\frac{\delta \Gamma_{\textit{in-out}}\left[\tilde{\phi}\right]}{\delta \tilde{\phi}(x)} = -J(x) \tag{6.42}$$

However, in a truly off-equilibrium situation it is impossible to identify $\tilde{\phi}(x)$ with the expectation value of the field operator, and in any case equation (6.42) is unsuitable as an equation of motion. This important point is best appreciated with a concrete example, to which we turn [HarHu79, DeW67].

6.2.3 Trouble in the $g\phi^3$ theory

To be concrete, let us assume the potential

$$V[\varphi] = \frac{1}{2}m^2\varphi^2 - \frac{1}{6}g\varphi^3 - h\varphi \tag{6.43}$$

The linear term is included to enforce the constraint

$$\int D\varphi\, e^{iS}\varphi(x) = 0 \tag{6.44}$$

In spite of some formal drawbacks (for example, equation (6.43) cannot really hold for all values of the field operator, as the theory would have no stable ground state if it did), this model is appealing because of its simplicity, and it is actually a useful model for unstable quantum systems (such as a strongly underdamped Josephson junction).

We shall be concerned only with the small oscillations of the mean field around $\phi = 0$, which, given equation (6.44), is a solution of equation (6.42) by design when $J = 0$. To find the linearized "equations of motion," we need the effective action to quadratic order, which requires knowledge of the quadratic part of the generating functional only. From the definition

$$W_{in\text{-}out}[J] \sim \frac{i}{2\hbar}\int d^4x d^4y\, J(x)\langle\varphi(x)\varphi(y)\rangle_c J(y) \tag{6.45}$$

where $\langle\varphi(x)\varphi(y)\rangle_c$ is the sum of all connected Feynman graphs ending in two external legs as shown. If we further expand in powers of g, keeping the constraint equation (6.44), we may appeal to Wick's theorem to write

$$\langle\varphi(x)\varphi(y)\rangle_c = \Delta(x-y) - i\int d^4z d^4z' \;\; \Delta(x-z)\,\Sigma(z-z')\,\Delta(z'-y) \tag{6.46}$$

where

$$i\Sigma(z-z') = \frac{g^2}{2\hbar^2}\left[\Delta(z-z')\right]^2 \tag{6.47}$$

The important issue is which propagator is exactly $\Delta(x-y)$. It is given by

$$\Delta(x-y) = \frac{\int D\varphi\, e^{i\hbar^{-1}S_{\mathrm{f}}}\varphi(x)\varphi(y)}{\int D\varphi\, e^{i\hbar^{-1}S_{\mathrm{f}}}} \tag{6.48}$$

where S_{f} means the free action, that is, the action with $g = h = 0$. Since the path integral time orders whatever is inside, Δ must correspond to the Feynman propagator Δ_{F} for the free theory. We then have

$$\begin{aligned}\tilde{\phi}(x) &= \frac{\delta W_{in\text{-}out}[J]}{\delta J(x)} \\ &= i\hbar^{-1}\int d^4y\, K(x-y)\, J(y)\end{aligned} \tag{6.49}$$

where

$$K(x-y) = \Delta_{\mathrm{F}}(x-y) - i\int d^4z d^4z' \;\; \Delta_{\mathrm{F}}(x-z)\,\Sigma_{\mathrm{F}}(z-z')\,\Delta_{\mathrm{F}}(z'-y) \tag{6.50}$$

Since

$$\left[\nabla^2 - m^2\right]\Delta_{\mathrm{F}}(x,x') = i\hbar\delta(x,x') \tag{6.51}$$

we get the "equation of motion"

$$\left[\nabla^2 - m^2\right] \tilde{\phi}(x) - \hbar \int d^4z \;\; \Sigma_{\mathrm{F}}(x - z)\, \tilde{\phi}(z) = -J(x) + O\left(g^4\right) \tag{6.52}$$

To appreciate the content of this equation, let us compute the kernel Σ explicitly, using the results from Chapter 5. The infinite and constant terms may be absorbed into a redefinition of m^2 and do not concern us now. For the remainder, we may already be able to make a crucial observation: since $\Sigma_{\mathrm{F}}(p)$ is an even function of p^0, Σ_{F} will be an even function of $x^0 - z^0$. Therefore, equation (6.52) cannot possibly yield a causal dynamics: the behavior of $\tilde{\phi}$ at any given event will depend on the whole future, and not only on the past of that event. This is clear also if we seek the response of $\tilde{\phi}$ to an impulse by setting J to be a delta function in equation (6.49): far from turning on when the source does, $\tilde{\phi}$ is nonzero everywhere.

Compared to this, the fact that equation (6.52) will generally predict a complex $\tilde{\phi}$ even for real sources is a lesser sin. This follows from the fact that the argument of the logarithm in the fish may be negative if $-p^2 > 4m^2$. Thus $\Sigma_{\mathrm{F}}(p)$ develops an even imaginary part, which passes on to $\Sigma_{\mathrm{F}}(x - z)$, Chapter 5, equation (5.27). Since the field operator is Hermitian, its expectation value must be real; we conclude that $\tilde{\phi}$ cannot possibly be that expectation value.

In summary, we find that the generating functional $W_{in\text{-}out}[J]$ is not generating expectation values of observables – it is generating something else, see below. Neither is it useful as a way to derive equation (6.52) because this equation is not admissible as a true dynamical law, since it is not causal.

In order to proceed, we must understand why an approach which worked fine in equilibrium situations fares so badly off-equilibrium. The key is in the boundary conditions which are conspicuously absent from the Euclidean path integral (6.34). The reason why we do not need to introduce explicit boundary conditions in equation (6.34) is that only the vacuum-to-vacuum amplitude survives the limiting procedure of taking the time interval t_{E} to infinity. Anything else becomes negligible against the vacuum-to-vacuum contribution.

This is not true of the Lorentzian path integral, and in fact equation (6.39) is meaningless unless the boundary conditions on the path of integration are specified. We implicitly assumed that the Lorentzian path integral was defined as the analytic continuation of the Euclidean path integral. This is implemented by replacing m^2 by $m^2 - i\varepsilon$ in the classical action. Therefore the path integral came to represent a vacuum-to-vacuum transition amplitude. While in a truly Poincaré-invariant situation there is only one vacuum (up to a phase), off-equilibrium the vacuum $|0in\rangle$ in the distant past may be very different from the vacuum $|0out\rangle$ in the far future.

Let us describe the situation in canonical terms. We have a physical idea of what the vacuum is, both in the distant past and future (for example, we have a particle detector we trust, and we know it is in the vacuum if the detector does not click). There is a state $|0in\rangle$ which corresponds to the vacuum at time $t=-t_{\mathrm{L}}$ (subscript L under t denotes Lorentzian time). If we adopt the Schrödinger picture, at time $t=t_{\mathrm{L}}$ this state has evolved into $e^{-2it_{\mathrm{L}}H}|0in\rangle$, which does *not* correspond to the physical vacuum. On the other hand, there is a *different* state $|0out\rangle$ (which we may regard as a state either in the Heisenberg picture or a Schrödinger picture at time $t=-t_{\mathrm{L}}$) evolving into $|0out,t_{\mathrm{L}}\rangle \equiv e^{-2iHt_{\mathrm{L}}}|0out\rangle$, which corresponds to the vacuum in the far future. We obtain the *in–out* generating functional by forcing $|0in\rangle$ to evolve not only under its own dynamics, but also under an external time dependent source $J(\mathbf{x},t)$, and comparing the result to $|0out,t_{\mathrm{L}}\rangle$

$$\begin{aligned} e^{i\hbar^{-1}W_{in\text{-}out}[J]} &= \langle 0out,t_{\mathrm{L}}|\,T\left[e^{-i\int_{-t_{\mathrm{L}}}^{t_{\mathrm{L}}}\hbar^{-1}dt\left(H-\int d^3\mathbf{x}J\Phi_S\right)}\right]|0in\rangle \\ &= \langle 0out|\,e^{2i\hbar^{-1}t_{\mathrm{L}}H}T\left[e^{-i\int_{-t_{\mathrm{L}}}^{t_{\mathrm{L}}}\hbar^{-1}dt\left(H-\int d^3\mathbf{x}J\Phi_S\right)}\right]|0in\rangle \end{aligned} \tag{6.53}$$

where H, $\Phi_S(\mathbf{x})$ are Schrödinger picture operators. Therefore

$$\tilde{\phi}(x) = \left.\frac{\delta W_{in\text{-}out}[J]}{\delta J(x)}\right|_{J=0} = \frac{\langle 0out|\,\Phi(x)\,|0in\rangle}{\langle 0out\,|0in\rangle} \tag{6.54}$$

where $\Phi(\mathbf{x},t)$ is the Heisenberg picture operator. This is a very different object from the true expectation value

$$\phi(x) = \langle 0in|\,\Phi(x)\,|0in\rangle \tag{6.55}$$

In particular, being a nondiagonal matrix element, $\tilde{\phi}(x)$ will be generally complex, and since it carries the information that the state must evolve into the vacuum in the far future, it is not surprising that its dynamics is acausal.

The true expectation value $\phi(x)$ must be real and evolve causally. We should therefore forget about $\tilde{\phi}$ but concentrate on finding the right equations of motion for ϕ. We want to make the same overall strategy work: we shall find the correct generating functional, and obtain an effective action as a Legendre transform of it based on the demand that the variation of this effective action will yield real and causal equations of motion. The correct generating functional and effective action will have to be different from their counterparts above. These are the necessary requirements for a consistent nonequilibrium functional formalism.

6.3 The closed time path effective action

As mentioned earlier, we shall study nonequilibrium dynamics through the evolution of expectation values of field operators and their correlation functions. To study this evolution, we shall derive equations of motion which represent successive truncations of the Schwinger–Dyson hierarchy (this being the quantum equivalent of the BBGKY hierarchy in classical statistical mechanics). The reason why we concentrate on the equations of motion rather than the propagators themselves is because it is more efficient: an approximation to the equation may be equivalent to summing an infinite set of graphs in the solution. A dramatic example of this improved efficiency is the *hard thermal loop* resummation scheme which we will discuss at length in Chapter 10.

In the simplest approach, we choose a single indicator, namely, the expectation value of the field operator. Allowing for a mixed initial state rather than the vacuum initial conditions assumed so far, we have $\phi(x) = \mathrm{Tr}\,[\Phi(x)\,\rho]$. Introducing the Heisenberg dynamical law for the field operator, this expectation value admits a representation as a CTP path integral

$$\phi(x) = \int D\varphi^1 D\varphi^2\ \rho\left[\varphi^1(0,\mathbf{x}), \varphi^2(0,\mathbf{x})\right] \exp\left\{(i/\hbar)\, S\left[\varphi^1, \varphi^2\right]\right\} \varphi^1(x) \quad (6.56)$$

where

$$S\left[\varphi^1, \varphi^2\right] = S\left[\varphi^1\right] - S\left[\varphi^2\right]^* \quad (6.57)$$

This suggests considering two mean fields ϕ^a to be derived from the *closed time path* (CTP) *generating functional* $W\left[J^1, J^2\right]$

$$e^{(i/\hbar)W[J]} = \int D\varphi^1 D\varphi^2\ \rho\left[\varphi^1(0,\mathbf{x}), \varphi^2(0,\mathbf{x})\right]$$
$$\exp\left\{(i/\hbar)\left[S\left[\varphi^1, \varphi^2\right] + \int d^4x\ \left[J^1\varphi^1 - J^2\varphi^2\right]\right]\right\} \quad (6.58)$$

through the variational formula

$$\phi^a(x) = \frac{\delta}{\delta J_a(x)} W\left[J, J'\right] \quad (6.59)$$

If after the variation we set $J_a = 0$, then $\phi^1 = \phi^2 = \langle\Phi(x)\rangle$. Here $a = 1, 2$ denotes the branch within the time path. Also there is a "metric" $c_{ab} = \mathrm{diag}\ (1, -1)$, so that $J_1(x) = J^1(x)$ and $J_2(x) = -J^2(x)$.

To obtain the equation of motion for these mean fields, we introduce the *CTP effective action* (EA) as the Legendre transform of the generating functional $\Gamma[\phi] = W[J] - J_A\phi^A$. The dynamical equations for the mean fields read

$$\frac{\delta\Gamma}{\delta\phi^A} = -J_A \quad (6.60)$$

The index A is (x, a), where x is a spacetime event, and $a = 1, 2$ denotes the branch within the time path. We apply a generalized Einstein convention whereby repeated indices are summed if they are discrete, or integrated if continuous. For example

$$J\phi = J_A\phi^A = \int d^4x\ J_a\phi^a = \int d^4x\ \left[J_1\phi^1 + J_2\phi^2\right] = \int d^4x\ \left[J^1\phi^1 - J^2\phi^2\right] \quad (6.61)$$

To obtain the equation of motion for the physical expectation value, we set $J = 0$ in equation (6.60), in which case, as we shall see below, the two equations (6.60) are actually equivalent.

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Given the flourishing of applications of the CTPEA, it would be impossible to give a complete set of references. Some papers which were influential in the development of the subject are [Sch60, Sch61, BakMah63, Kel64, ChoSuHa80, CSHY85, SCYC88, DeW86, Jor86, CalHu87, CalHu88, CalHu89].

6.3.1 An example

Before we proceed further with the formalism it is useful to show an example. We continue with the $g\Phi^3$ theory we introduced in (6.43).

Let us assume that the initial conditions are set in the distant past, where the initial state is the *in* vacuum. This is implemented, as above, by shifting m^2 to $m^2 - i\varepsilon$. Therefore

$$e^{(i/\hbar)W[J]} = \int D\varphi^1 D\varphi^2 \exp\left\{(i/\hbar)\left[S\left[\varphi^1\right] - S\left[\varphi^2\right]^* + \int d^4x \,\left[J^1\varphi^1 - J^2\varphi^2\right]\right]\right\} \tag{6.62}$$

In the second branch the mass is shifted to $m^2 + i\varepsilon$. Observe that the two branch integrations are not independent, as they couple through the "CTP boundary condition" $\varphi^1(T,\mathbf{x}) = \varphi^2(T,\mathbf{x})$ for all $\mathbf{x}$ at some very large time T. In canonical terms, this expression is equivalent to

$$e^{(i/\hbar)W[J]} = \langle 0in|\, U_{J^2}(-T,T)\, U_{J^1}(T,-T)\,|0in\rangle|_{T\to\infty} \tag{6.63}$$

where U_J is the evolution operator for the field interacting with the external c-number source J

$$U_J(t,t') = T\left[\exp\left\{\left(-\frac{i}{\hbar}\right)\int_{t'}^{t} dt \int d^3\mathbf{x}\, \left(H - J\Phi\right)\right\}\right] \tag{6.64}$$

The CTP boundary condition arises from inserting a complete set of states at time T between the two evolution operators. It is interesting to compare the CTP generating functional (6.63) to its "open path" or *in-out* counterpart (6.53). Observe that the *out* vacuum plays no role in the CTP expression.

As in our earlier example, we shall compute only the quadratic part of the generating functional. Since we enforce

$$\left.\frac{\delta W}{\delta J^{1,2}(x)}\right|_{J=0} = 0 \tag{6.65}$$

by conveniently tuning the linear term h in equation (6.43), the quadratic part is

$$W[J] = \frac{i}{2\hbar} G^{AB} J_A J_B \tag{6.66}$$

From equation (6.62)

$$G^{AB} = \int D\varphi^1 D\varphi^2 \exp\left\{(i/\hbar)\left[S\left[\varphi^1\right] - S\left[\varphi^2\right]^*\right]\right\}\varphi^A\varphi^B \tag{6.67}$$

From equation (6.63)

$$G^{11}(x,y) = G_F(x,y) \tag{6.68}$$
$$G^{22}(x,y) = G_D(x,y) \tag{6.69}$$
$$G^{21}(x,y) = G^{+}(x,y) \tag{6.70}$$
$$G^{12}(x,y) = G^{-}(x,y) \tag{6.71}$$

So G^{AB} are the "path-ordered" propagators: Path-ordering is equivalent to time-ordering for points on the first time branch, anti-time ordering on the second time branch, and placing points on the second branch to the left of points on the first branch.

Formally, the mean fields are given by

$$\phi^A = i\hbar^{-1} G^{AB} J_B \tag{6.72}$$

which is inverted to

$$-J_A = i\hbar \left[G^{-1}\right]_{AB} \phi^B \tag{6.73}$$

Comparing with equation (6.60) we find

$$\Gamma = \frac{i\hbar}{2} \left[G^{-1}\right]_{AB} \phi^A \phi^B \tag{6.74}$$

The actual equation of motion is obtained when $\phi^1 = \phi^2$. Thus the equation of motion is

$$i\hbar \int d^4y \ \left[\left[G^{-1}\right]_{11}(x,y) + \left[G^{-1}\right]_{12}(x,y)\right] \phi(y) = -J(x) \tag{6.75}$$

This equation is real and causal as we shall soon prove.

For free fields, the Klein–Gordon equations for the fundamental propagators

$$\left[\nabla^2 - m^2\right] \Delta_F(x,x') = -\left[\nabla^2 - m^2\right] \Delta_D(x,x') = i\hbar\delta(x,x') \tag{6.76}$$
$$\left[\nabla^2 - m^2\right] \Delta^{+}(x,x') = \left[\nabla^2 - m^2\right] \Delta^{-}(x,x') = 0 \tag{6.77}$$

may be summarized as

$$c_{AB}\left[\nabla^2 - m^2\right] \Delta^{BC} = i\hbar\delta_A^C \tag{6.78}$$

So

$$\left[\Delta^{-1}\right]_{AB} = \left(-i\hbar^{-1}\right)\left[\nabla^2 - m^2\right] c_{AB} \tag{6.79}$$

where $c_{AB} = c_{ab}\delta(x,x')$. Never mind for now that we are claiming $\left[\Delta^{-1}\right]_{AB}$ is diagonal, while Δ^{AB} is conspicuously not. The nondiagonal elements of Δ^{AB} are retrieved by inverting $\left[\Delta^{-1}\right]_{AB}$ under the CTP constraints

$$\Delta^{11}(x,y) = \theta\left(x^0 - y^0\right)\Delta^{21}(x,y) + \theta\left(y^0 - x^0\right)\Delta^{12}(x,y) \tag{6.80}$$
$$\Delta^{22}(x,y) = \theta\left(x^0 - y^0\right)\Delta^{12}(x,y) + \theta\left(y^0 - x^0\right)\Delta^{21}(x,y) \tag{6.81}$$

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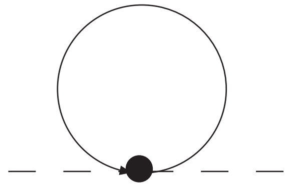

Figure 6.1 The tadpole graph; there are no propagators on the external lines.

With this form for the inverse propagators, the equation of motion is just the Klein–Gordon equation, which is correct but not very illuminating.

For interacting fields, define the self-energies Σ_{AB} from

$$\left[G^{-1}\right]_{AB} = \left[\Delta^{-1}\right]_{AB} + i\Sigma_{AB} \tag{6.82}$$

leading to the perturbative development

$$G^{AB} = \Delta^{AB} - i\Delta^{AC}\Sigma_{CD}\Delta^{DB} + \dots \tag{6.83}$$

Meanwhile the expansion of the propagators in powers of the coupling constant g reads

$$\begin{aligned} G^{ab}(x,y) = \Delta^{AB} - \frac{1}{2}\left(\frac{g}{6\hbar}\right)^2 \int d^4z d^4z' \; \Big\langle \varphi^a(x) \left[\left(\varphi^1\right)^3 - \left(\varphi^2\right)^3\right](z) \\ \times \left[\left(\varphi^1\right)^3 - \left(\varphi^2\right)^3\right](z')\,\varphi^b(y)\Big\rangle_{\mathrm{f}} + \dots \end{aligned} \tag{6.84}$$

where $\langle\,\rangle_{\mathrm{f}}$ denotes a path ordered expectation value computed for free fields, enforcing the constraint of vanishing "tadpoles." These expectation values are reduced to products of propagators applying Wick's theorem. Comparing both expansions, we conclude

$$i\Sigma_{11}(x,y) = \frac{g^2}{2\hbar^2}\left[\Delta_{\mathrm{F}}(x,y)\right]^2 \tag{6.85}$$

$$i\Sigma_{12}(x,y) = -\frac{g^2}{2\hbar^2}\left[\Delta^{-}(x,y)\right]^2 \tag{6.86}$$

$$i\Sigma_{21}(x,y) = -\frac{g^2}{2\hbar^2}\left[\Delta^{+}(x,y)\right]^2 \tag{6.87}$$

$$i\Sigma_{22}(x,y) = \frac{g^2}{2\hbar^2}\left[\Delta_{\mathrm{D}}(x,y)\right]^2 \tag{6.88}$$

We may now write the equation of motion to order g^2

$$\left[\nabla^2 - m^2\right]\phi(x) + i\frac{g^2}{2\hbar}\int d^4y \; \left[\left[\Delta_{\mathrm{F}}(x,y)\right]^2 - \left[\Delta^{-}(x,y)\right]^2\right]\phi(y) = -J(x) \tag{6.89}$$

Comparing with equation (6.52), we see that there is an extra contribution to the nonlocal part. By simple inspection, we see that this new term makes the equation causal, since $\Delta_{\mathrm{F}}(x,y) = \Delta^{-}(x,y)$ when $y^0 > x^0$. From the results in

Chapter 5, we see that the equation is also real, as required by the physical meaning of ϕ as the expectation value of a Hermitian operator.

6.3.2 The structure of the closed time path effective action

The example above already shows several generic features of the CTPEA. We wish now to highlight these features which are general and *exact* (i.e. not dependent on the model or the order of coupling).

In the above example, we assumed vacuum initial conditions set up in the distant past. In general, we deal with an arbitrary initial state set up at some definite time, which we may take as $t = 0$. Then the CTP generating functional admits the representation

$$e^{(i/\hbar)W[J]} = \mathrm{Tr}\left[U_{J^2}(0,T)\,U_{J^1}(T,0)\,\rho\right]\big|_{T\to\infty} \tag{6.90}$$

where, as before, U_J is the evolution operator for the field interacting with the external c-number source J as in equation (6.64). Variation yields, in the coincidence limit

$$\phi(x) = \frac{\delta}{\delta J(x)} W\left[J, J'\right]\Big|_{J'\to J} = \mathrm{Tr}\left[\Phi(x)\,U_J(t,0)\,\rho U_J(0,t)\right] \tag{6.91}$$

ϕ is the expectation value of the field operator with respect to the state which evolves from ρ under the influence of the source J.

The first property of the CTPEA we wish to discuss is its "Hermiticity," namely, for Hermitian field operators, $\Gamma\left[\phi^2, \phi^1\right] = -\Gamma\left[\phi^{1*}, \phi^{2*}\right]^*$ (since the field operators are Hermitian, we assume they couple to real c-number sources; however, we may be sure that the mean fields ϕ^a are real in the coincidence limit $J^1 = J^2$ only). To see this "Hermiticity," observe that, provided the density matrix ρ in equation (6.90) is itself Hermitian, then a similar property holds for the CTP generating functional, namely $W\left[J^2, J^1\right] = -W\left[J^1, J^2\right]^*$. Taking variations, we get

$$\phi_1\left[J^2, J^1\right] = -\phi_2\left[J^1, J^2\right]^*; \qquad \phi_2\left[J^2, J^1\right] = -\phi_1\left[J^1, J^2\right]^* \tag{6.92}$$

In other words, if the external sources necessary to sustain the given background fields $\left(\phi^1, \phi^2\right)$ are $\left(J^1, J^2\right)$, then the sources necessary to sustain the mean fields $\left(\phi^{2*}, \phi^{1*}\right)$ are $\left(J^2, J^1\right)$ (note the position of the indices). Thus

$$\Gamma\left[\phi^{2*}, \phi^{1*}\right] = W\left[J^2, J^1\right] - \left(J^2\phi^{2*} - J^1\phi^{1*}\right) = -\Gamma\left[\phi^1, \phi^2\right]^* \tag{6.93}$$

QED. An equivalent formulation is that, if the background fields $\left(\phi^1, \phi^2\right)$ are real, then

$$\mathrm{Re}\Gamma\left[\phi^1, \phi^2\right] = -\mathrm{Re}\Gamma\left[\phi^2, \phi^1\right]; \qquad \mathrm{Im}\Gamma\left[\phi^1, \phi^2\right] = \mathrm{Im}\Gamma\left[\phi^2, \phi^1\right] \tag{6.94}$$

If ρ is trace-class ($\mathrm{Tr}\left[\rho\right] = 1$) *and* the evolution operator is unitary, then the CTP generating functional vanishes on the diagonal ($W\left[J, J\right] = 0$). In the coincidence limit, therefore, there is a single mean field, since $\phi^1 = \phi^2 \equiv \phi$ (again, the position of the indices matters). Equation (6.92) shows that ϕ must be real (this can also be seen directly from equation (6.91)). We then find that the CTPEA is also trivial along the diagonal $\Gamma[\phi, \phi] \equiv 0$. This dispels the apparent mystery of having two equations for a single mean field ϕ: they are linearly dependent. The single equation reads

$$\left.\frac{\delta\Gamma}{\delta\phi^1}\right|_{\phi^2=\phi^1=\phi} = -\left.\frac{\delta\Gamma}{\delta\phi^2}\right|_{\phi^2=\phi^1=\phi} = -J \tag{6.95}$$

where J is the common value of J^1 and J^2. Observe that although Γ is generally complex, when ϕ is real the variation of the imaginary part must vanish in the coincidence limit (this follows from equations (6.94)), and so the physical equation (6.95) is explicitly a real equation.

The other fundamental property of equation (6.95) is that it is causal (we may say that doubling the degrees of freedom is the minimum price to pay to get a causal, real equation of motion for the mean field within a variational approach). Indeed, the solution to the physical equation (6.95) is given by the formal expression (6.91), which is obviously causal.

We may disclose further properties of the CTPEA by writing it as a function of new field variables $\phi_- = \phi^1 - \phi^2$ and $\phi_+ = \left(\phi^1 + \phi^2\right)/2$. Observe that $\Gamma[\phi, \phi] \equiv 0$ implies

$$\Gamma[\phi_- = 0, \phi_+] \equiv 0 \tag{6.96}$$

Therefore the Taylor development of Γ reads

$$\Gamma[\phi_-, \phi_+] = \int d^4x\ \phi_-(x)\,\mathbf{D}_x^{\text{full}}[\phi_+] + \frac{i}{2}\int d^4x d^4x'\ \phi_-(x)\,\mathbf{N}(x, x')\,\phi_-(x') + \ldots \tag{6.97}$$

To find the equations of motion, we first take its variation with respect to ϕ^1 and then set $\phi_- = 0$. Only the first term contributes, and the equations read

$$\mathbf{D}_x^{\text{full}}[\phi] = -J(x) \tag{6.98}$$

If the theory is set up so that $\mathbf{D}_x^{\text{full}}[0] = 0$, then $\mathbf{D}_x^{\text{full}}$ will have its own Taylor development

$$\mathbf{D}_x^{\text{full}}[\phi_+] = \int d^4x'\ \mathbf{D}^{\text{full}}(x, x')\,\phi_+(x') + \ldots \tag{6.99}$$

So the linearized equation of motion is

$$\int d^4x'\ \mathbf{D}^{\text{full}}(x, x')\,\phi(x') = -J(x) \tag{6.100}$$

The Hermiticity conditions (6.94) and the causality of the equations of motion (6.95) imply that the kernels $\mathbf{D}^{\text{full}}$ and $\mathbf{N}$ are real, and $\mathbf{D}^{\text{full}}$ is causal.

The appearance of the kernel $\mathbf{N}$ may seem redundant, since it does not contribute to the mean field equations of motion. However, equation (6.97) also suggests another way of looking at the CTPEA which discloses a surprising role for $\mathbf{N}$. Let us observe that the quadratic CTPEA has the same structure as an influence functional, and if we regard it this way, then $\mathbf{N}$ corresponds to the noise kernel. According to the theory of quantum open systems, we ought to replace the mean field equation by a Langevin equation $\mathbf{D}^{\text{full}}\phi = -J - \xi$, where J is the external source, if any, and ξ is a stochastic, c-number source with autocorrelation

$$\langle \xi (x) \, \xi (x') \rangle = \hbar \mathbf{N} (x, x') \tag{6.101}$$

In attention to this role for $\mathbf{N}$, we shall henceforth refer to it as the *noise kernel*. In general $\mathbf{N} (x, x')$ will be a functional of ϕ_+, leading to colored and multiplicative noise in the dynamical equations. A large part of the remainder of this book may be seen as the development of this theme.

6.4 Computing the closed time path effective action

6.4.1 The background field method

So far we have formally introduced the CTPEA and investigated some of its properties. Now we show how to actually compute it. As a start, let us observe that it is possible to give a definition of the CTPEA as the solution of a particular integral equation. To this end, we recall the definition of the generating functional (in condensed notation)

$$e^{(i/\hbar)W[J]} = \int D\Phi \, \exp \left\{ (i/\hbar) \left[S \left[\Phi \right] + J \Phi \right] \right\} \tag{6.102}$$

(observe that we do not write explicitly the initial density matrix; the mystery will be revealed below). The CTPEA is introduced as the Legendre transform $\Gamma[\phi] = W \left[J \right] - J_A \phi^A$. Note that, after all, $J = -\Gamma_{,\phi}$, and so we may write

$$e^{(i/\hbar)W[\phi]} = \int D\Phi \, \exp \left\{ (i/\hbar) \left[S \left[\Phi \right] - \frac{\delta \Gamma}{\delta \phi} \Phi \right] \right\} \tag{6.103}$$

which is the self-contained equation for the CTPEA.

Let us begin by rewriting it as

$$\Gamma \left[\phi^A \right] = (-i\hbar) \ln \int D\Phi^A \, \exp \left\{ (i/\hbar) \left[S \left[\Phi^A \right] - \frac{\delta \Gamma}{\delta \phi^A} \left(\Phi^A - \phi^A \right) \right] \right\} \tag{6.104}$$

Shift the integration variables by the mean fields $\Phi^A = \phi^A + \varphi^A$ and expand the classical action

$$S \left[\phi^A + \varphi^A \right] = S \left[\phi^A \right] + S_{,A} \varphi^A + S_r \left[\varphi^A \right] \tag{6.105}$$

For example, for a $\lambda \Phi^4$ theory with classical action

$$S \left[\phi^a \right] = \int d^4 x \, \left\{ \frac{c_{ab}}{2} \left[-\partial \phi^a \partial \phi^b - m^2 \phi^a \phi^b \right] - \frac{\lambda}{4} \, c_{abcd} \phi^a \phi^b \phi^c \phi^d \right\} \tag{6.106}$$

where c_{ab} is the CTP metric tensor and c_{abcd} is 1 if $a=b=c=d=1$, -1 if all indices are equal to 2, and vanishes otherwise, we obtain

$$S_r\left[\varphi^a\right]=\int d^4x\,\left\{\frac{c_{ab}}{2}\left[-\partial\varphi^a\partial\varphi^b-m^2\varphi^a\varphi^b\right]-\frac{\lambda}{4}\,c_{abcd}\phi^a\phi^b\varphi^c\varphi^d\right.$$
$$\left.-\frac{\lambda}{6}\,c_{abcd}\phi^a\varphi^b\varphi^c\varphi^d-\frac{\lambda}{24}\,c_{abcd}\varphi^a\varphi^b\varphi^c\varphi^d\right\} \tag{6.107}$$

Then

$$\Gamma\left[\phi^A\right]=S\left[\phi^A\right]-i\hbar\ln\int D\varphi^A\,\exp\left\{(i/\hbar)\left[S_r\left[\varphi^A\right]+\left(S_{,A}-\Gamma_{,A}\right)\varphi^A\right]\right\} \tag{6.108}$$

Next write

$$\Gamma\left[\phi^A\right]=S\left[\phi^A\right]+\Gamma_1\left[\phi^A\right] \tag{6.109}$$

$$\Gamma_1\left[\phi^A\right]=-i\hbar\ln\int D\varphi^A\,\exp\left\{(i/\hbar)\left[S_r\left[\varphi^A\right]-\Gamma_{1,A}\varphi^A\right]\right\} \tag{6.110}$$

The quantum correction Γ_1 has the form of a generating functional for a new theory, whose classical action is obtained from the original one by shifting the fields as in equation (6.105) and discarding constant and linear terms. This new generating functional must be evaluated at a particular value of the external source.

By performing the Legendre transform in reverse, we could write this generating functional in terms of the effective action for the φ field. We stress that the φ field represents a different field theory than the original one; for example, the action S_r for the φ field contains cubic interactions, which the action S for the ϕ field does not. In any case, to compute the generating functional, we must be able to compute the corresponding effective action at a mean field doublet $\bar{\varphi}$ equal to the expectation value $\langle\varphi\rangle_J$ of the Heisenberg operator $\varphi=\Phi-\phi$. Generally the new action S_r is not invariant under sign reversal (cf. equation (6.107)), and so there is no reason for this to vanish. However, the value $\Gamma_{1,A}$ is precisely the external force necessary to kill this expectation value. The conclusion is that we may ignore the external source, and compute the generating functional Γ_1 as the sum of all 1PI vacuum Feynman graphs: vacuum because we compute the effective action at $\bar{\varphi}=0$, and 1PI because, after all, it is an effective action.

To show that $\langle\varphi\rangle=0$, let us take the variational derivative of equation (6.110) with respect to the background field to obtain

$$\int D\varphi^A\,\left[S_{r,A}\left[\varphi\right]-\Gamma_{1,A}\left[\phi\right]-\Gamma_{1,AB}\varphi^B\right]\exp\left\{(i/\hbar)\left[S_r\left[\varphi\right]-\Gamma_{1,C}\varphi^C\right]\right\}\equiv 0 \tag{6.111}$$

and observe that

$$\frac{\delta S_r}{\delta\phi^A}=\frac{\delta S\left[\phi+\varphi\right]}{\delta\phi^A}-\frac{\delta S\left[\phi\right]}{\delta\phi^A}-\frac{\delta S\left[\phi\right]}{\delta\phi^A\delta\phi^B}\varphi^B=\frac{\delta S_r}{\delta\varphi^A}-\frac{\delta S\left[\phi\right]}{\delta\phi^A\delta\phi^B}\varphi^B \tag{6.112}$$

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The product

$$\left[\frac{\delta S_r}{\delta\varphi^A} - \Gamma_{1,A}\right] e^{\left\{(i/\hbar)\left[S_r[\psi]-\Gamma_{1,B}\varphi^B\right]\right\}} = -i\hbar\frac{\delta}{\delta\varphi^A}e^{\left\{(i/\hbar)\left[S_r[\varphi]-\Gamma_{1,A}\varphi^A\right]\right\}} \quad (6.113)$$

integrates to zero, and so we are left with the identity $\Gamma_{,AB}\left\langle\varphi^B\right\rangle = 0$. But the Hessian operator $\Gamma_{,AB}$ must be nonsingular, since it follows from the properties of Legendre transformation that

$$\Gamma_{,AB}\frac{\delta^2 W}{\delta J_B \delta J_C} = -\delta_A^C \quad (6.114)$$

and this establishes the vanishing of $\left\langle\varphi^b\right\rangle$, as desired. To summarize, Γ, the (vacuum) CTPEA consists of the classical action S plus a quantum correction Γ_1 (thereby the label *effective action*). This quantum correction is the sum of all *one-particle irreducible* (1PI) (that is, containing no one-particle insertions) *vacuum bubbles* (that is, containing no external vertices). This recipe will be the start of all computations based on the CTPEA.

For an exposition of the background field method, read, e.g. the classic papers by Jackiw and Iliopoulos, Itzykson and Martin [Jac74, IlItMa75].

6.4.2 The loop expansion

Having reduced the problem of computing the CTPEA in the theory with classical action S to the calculation of vacuum bubbles in the theory with classical action S_r, we proceed, as in the general case, to split the new action into its free and interacting components

$$S_r = \frac{1}{2}\frac{\delta^2 S}{\delta\phi^2}\varphi^2 + S_Q \quad (6.115)$$

For example, for a $\lambda\Phi^4$ theory the free and interacting parts correspond to the first and second lines in equation (6.107), respectively.

We generate the Feynman graphs by expanding the exponential of S_Q. The different vertices shall be connected through lines, and associated with each line there is a propagator

$$\left\langle\varphi^A\varphi^B\right\rangle = \int D\varphi^C\ \left[\varphi^A\varphi^B\right]\exp\left\{\frac{i}{2\hbar}\frac{\delta^2 S}{\delta\phi^2}\varphi^2\right\} \equiv i\hbar\left[\frac{\delta^2 S}{\delta\phi^A\delta\phi^B}\right]^{-1} \quad (6.116)$$

To make this formula well defined, we assume the usual Gell-Mann–Low boundary condition which states that interactions are adiabatically switched off in the distant past, so the *in* vacuum for the φ field is the same as for the Φ field. Moreover, we assume this state to be properly normalized, so it is not necessary to normalize explicitly the expectation value (6.116).

The neat split of Γ into a classical and a quantum part can be continued by analyzing further its development in powers of $\hbar$. The idea is that each vertex contributes one inverse power of $\hbar$ to the amplitude of the graph, while each line contributes $\hbar$. So the overall power of $\hbar$, including the one in the beginning

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of equation (6.110), is $L = I - V + 1$, where I is the number of lines, and V of vertices. This is also the number of independent loops in the graph, and so the expansion of Γ in powers of $\hbar$ is equivalent to a topological classification of graphs according to the number of loops.

Note that I and V also satisfy the constraint $2I - 3V_3 - 4V_4 = 0$, where V_3 (V_4) is the number of cubic (quartic) vertices in the graph. Solving for the number of vertices of each type, we find $V_3 = 2I - 4(L-1)$ and $V_4 = 3(L-1) - I$. Since each of the numbers I, V_3, V_4 must be nonnegative, we conclude that for each value of L only a finite number of graphs are allowed. For example, for $L = 2$ we must have either $I = 2$, $V_3 = 0$, $V_4 = 1$ or $I = 3$, $V_3 = 2$, $V_4 = 0$, etc.

If $L = 1$, then we must have $V_4 = V_3 = 0$. In this limit, the integral in equation (6.110) is Gaussian and we may write

$$\Gamma_1\left[\phi^A\right] = -i\hbar \ln \mathrm{Det}\left[S''\right]^{-1/2} + O\left(\hbar^2\right) \tag{6.117}$$

Of course, since the propagators themselves depend on the background fields, we do not mean that individual graphs are easy to compute. The loop expansion, however, provides us with a classification scheme to consider the different processes contributing to a given amplitude in order of increasing complexity.

6.4.3 The one-loop closed time path effective action for the $g\Phi^3$ theory

As an example, let us compute the one-loop approximation to the CTPEA in the familiar scalar field theory with cubic self-interaction.

We assume the simplest case of vacuum initial conditions specified in the distant past. The classical potential is given in equation (6.43), where m^2 is shifted to $m^2 - i\varepsilon$. The CTP action is given by equation (6.57) and the CTPEA by equation (6.109). To construct the new action S_r which appears in this equation, we write the old action in terms of a displaced field variable $\phi + \varphi$, and then discard constant and linear terms in φ. Therefore, splitting S_r into a free part and an interaction part as in equation (6.115),

$$\frac{1}{2}\frac{\delta^2 S}{\delta\phi^2}\varphi^2 = \int d^4x\,\left\{\frac{c_{ab}}{2}\left[-\partial\varphi^a\partial\varphi^b - m^2\varphi^a\varphi^b\right] - \frac{g}{2}\,c_{abc}\phi^a\varphi^b\varphi^c\right\} \tag{6.118}$$

$$S_Q\left[\varphi\right] = \int d^4x\left\{-\frac{g}{6}\,c_{abc}\varphi^a\varphi^b\varphi^c\right\} \tag{6.119}$$

where $c_{111} = -c_{222} = 1$, all other components being zero. In principle, $\Gamma_1\left[\phi^A\right]$ is the sum of all 1PI vacuum graphs for this new theory. The one-loop approximation consists of discarding S_Q, so that $\Gamma_1\left[\phi^A\right]$ reduces to

$$\Gamma_1\left[\phi^A\right] = -i\hbar \ln \int D\varphi^A \exp\left\{(i/\hbar)\int d^4x\;\varphi^a\left\{\frac{c_{ab}}{2}\left[\nabla^2 - m^2\right] - \frac{g}{2}\,c_{abc}\phi^c\right\}\varphi^b\right\} \tag{6.120}$$

with the formal solution (6.117).

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We shall have more to say about the full one-loop approximation later in the book, but for now let us simply use equation (6.120) to recover the quadratic part of the CTPEA.

Expanding equation (6.120) to quadratic order we get

$$\Gamma_1\left[\phi^A\right] \sim \int d^4x\,\left(-\frac{g}{2}\right)\,c_{abc}\phi^c\left\langle\varphi^a\varphi^b\right\rangle_{\mathrm{c}} + \frac{i}{2\hbar}\left(\frac{g}{2}\right)^2\int d^4xd^4x'\,c_{abc}c_{def}\phi^c\left(x\right)\phi^f\left(x'\right)\left\langle\left(\varphi^a\varphi^b\right)\left(x\right)\left(\varphi^d\varphi^e\right)\left(x'\right)\right\rangle_{\mathrm{c}} \tag{6.121}$$

where $\langle\,\rangle_{\mathrm{c}}$ denotes the expectation value of path-ordered products of free field operators, keeping only the connected contributions. The first-order term will be canceled by the h term in the classical action, so we only need to worry about the second, which reads

$$\begin{aligned}\Gamma_1\left[\phi^A\right] \sim \frac{i}{2\hbar}\left(\frac{g}{2}\right)^2\int d^4xd^4x'\,\Big\{&\phi^1\left(x\right)\phi^1\left(x'\right)\left\langle T\left[\varphi^2\left(x\right)\varphi^2\left(x'\right)\right]\right\rangle_{\mathrm{c}}\\ &-\phi^2\left(x\right)\phi^1\left(x'\right)\left\langle\varphi^2\left(x\right)\varphi^2\left(x'\right)\right\rangle_{\mathrm{c}}-\phi^1\left(x\right)\phi^2\left(x'\right)\left\langle\varphi^2\left(x'\right)\varphi^2\left(x\right)\right\rangle_{\mathrm{c}}\\ &+\phi^2\left(x\right)\phi^2\left(x'\right)\left\langle\tilde{T}\left[\varphi^2\left(x\right)\varphi^2\left(x'\right)\right]\right\rangle_{\mathrm{c}}\Big\}\end{aligned} \tag{6.122}$$

Now write

$$\phi^{1,2} = \phi_+ \pm \frac{1}{2}\phi_- \tag{6.123}$$

to get

$$\begin{aligned}\Gamma_1\left[\phi^A\right] \sim \frac{i}{2\hbar}\left(\frac{g}{2}\right)^2\int d^4xd^4x'\,\Big\{&\phi_-\left(x\right)\phi_+\left(x'\right)\Big[\left\langle T\left[\varphi^2\left(x\right)\varphi^2\left(x'\right)\right]\right\rangle_{\mathrm{c}}\\ &+\left\langle\varphi^2\left(x\right)\varphi^2\left(x'\right)\right\rangle_{\mathrm{c}}-\left\langle\varphi^2\left(x'\right)\varphi^2\left(x\right)\right\rangle_{\mathrm{c}}-\left\langle\tilde{T}\left[\varphi^2\left(x\right)\varphi^2\left(x'\right)\right]\right\rangle_{\mathrm{c}}\Big]\\ &+\frac{1}{2}\phi_-\left(x\right)\phi_-\left(x'\right)\left\langle\left\{\varphi^2\left(x\right),\varphi^2\left(x'\right)\right\}\right\rangle_{\mathrm{c}}\Big\}\end{aligned} \tag{6.124}$$

which, under the definitions (6.5) for temporal and anti-temporal order, yields

$$\begin{aligned}\Gamma_1\left[\phi^A\right] \sim \frac{i}{4\hbar}\left(\frac{g}{2}\right)^2\int d^4xd^4x'\,\,\Big\{&4\phi_-\left(x\right)\phi_+\left(x'\right)\left\langle\left[\varphi^2\left(x\right),\varphi^2\left(x'\right)\right]\right\rangle_{\mathrm{c}}\theta\left(x^0-x'^0\right)\\ &+\phi_-\left(x\right)\phi_-\left(x'\right)\left\langle\left\{\varphi^2\left(x\right),\varphi^2\left(x'\right)\right\}\right\rangle_{\mathrm{c}}\Big\}\end{aligned} \tag{6.125}$$

Comparing with equations (6.97) and (6.99), we identify

$$\mathbf{D}^{\mathrm{full}}\left(x,x'\right) = \left[\nabla^2-m^2\right]\delta\left(x,x'\right)-\Sigma_{\mathrm{ret}}\left(x,x'\right) \tag{6.126}$$

$$i\Sigma_{\mathrm{ret}}\left(x,x'\right) = \frac{1}{\hbar}\left(\frac{g}{2}\right)^2\left\langle\left[\varphi^2\left(x\right),\varphi^2\left(x'\right)\right]\right\rangle_{\mathrm{c}}\theta\left(x^0-x'^0\right) \tag{6.127}$$

$$\mathbf{N}\left(x,x'\right) = \frac{1}{2\hbar}\left(\frac{g}{2}\right)^2\left\langle\left\{\varphi^2\left(x\right),\varphi^2\left(x'\right)\right\}\right\rangle_{\mathrm{c}} \tag{6.128}$$

so the CTPEA takes the influence functional structure, as expected. Observe that $\mathbf{D}^{\text{full}}$ is real, and obviously causal. Expanding the expectation value using Wick's theorem, the dynamical equation (6.100) gives back equation (6.89).

Although we still found no use for the noise kernel (see Chapter 8), it is undeniably nonzero. To lowest order, we find

$$i\Sigma_{\text{ret}}\left(x,x'\right)=\frac{g^2}{2\hbar}\left[\left[\Delta_F\left(x,x'\right)\right]^2-\left[\Delta^-\left(x,x'\right)\right]^2\right] \tag{6.129}$$

$$\mathbf{N}\left(x,x'\right)=\frac{g^2}{4\hbar}\left[\left(\Delta^+\left(x,x'\right)\right)^2+\left(\Delta^-\left(x,x'\right)\right)^2\right] \tag{6.130}$$

They can be expressed in terms of the U, ν and μ kernels introduced in Chapter 5

$$-\Sigma_{ret}=\frac{g^2\hbar}{2}\left[\frac{U}{2}+\mu\right] \tag{6.131}$$

$$\mathbf{N}=\frac{g^2\hbar}{4}\nu \tag{6.132}$$

It is interesting to observe a relationship between the Fourier transforms of these kernels. Since Σ_{ret} is causal it satisfies the Kramers–Kronig relations (6.23),

$$\Sigma_{\text{ret}}\left(p\right)=\frac{1}{\pi}\int\frac{d\omega}{\omega-p^0-i\varepsilon}\text{Im }\Sigma_{\text{ret}}\left(\omega,\mathbf{p}\right)+\text{ local terms} \tag{6.133}$$

The imaginary part comes from the Fourier transform of the μ kernel

$$\text{Im }\Sigma_{\text{ret}}\left(\omega,\mathbf{p}\right)=\text{sign }\left(\omega\right)\ \Pi\left(\omega^2-\mathbf{p}^2\right)\ \theta\left(\omega^2-\mathbf{p}^2-4m^2\right) \tag{6.134}$$

where

$$\Pi\left(\sigma^2\right)=\frac{g^2\hbar}{32\pi}\sqrt{1-\frac{4m^2}{\sigma^2}} \tag{6.135}$$

Also

$$\frac{1}{\omega-p^0-i\varepsilon}+\frac{1}{\omega+p^0+i\varepsilon}=\frac{2\omega}{\omega^2-\left(p^0+i\varepsilon\right)^2} \tag{6.136}$$

Writing $\omega^2-\mathbf{p}^2=\sigma^2$ we have

$$\Sigma_{\text{ret}}\left(p\right)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{d\sigma^2}{\left(p+i\varepsilon\right)^2+\sigma^2}\Pi\left(\sigma^2\right)+\text{ local terms} \tag{6.137}$$

The Fourier transform of $\mathbf{N}$ comes from the ν kernel

$$\mathbf{N}\left(p\right)=\frac{g^2\hbar}{32\pi}\sqrt{1-\frac{4m^2}{\left(-p^2\right)}}\theta\left(-p^2-4m^2\right) \tag{6.138}$$

Comparing with equation (6.135), we see that the noise kernel coincides up to a sign with the imaginary part (in frequency domain) of the dissipation kernel. We shall see in Chapter 8 that the imaginary part of the dissipation kernel describes the dissipation of the mean field by its interaction with quantum fluctuations. The relationship between the noise and dissipation kernels in this simple example is just a basic manifestation of the fluctuation–dissipation theorem at zero temperature.

6.4.4 The large N expansion

Computing Feynman graphs is easy. The harder question is how many Feynman graphs must be computed to achieve a prescribed accuracy.

The number N of replicas of essentially identical fields (like the N scalar fields in an $O(N)$ invariant theory, or the N^2-1 gauge fields in a $SU(N)$ invariant nonabelian gauge theory) suggests using $1/N$ as a natural small parameter, with a well-defined physical meaning. Unlike coupling constants, this is not subjected to renormalization or radiative corrections. By ordering the perturbative expansion in powers of this small parameter, several nonperturbative effects (in terms of coupling constants) may be systematically investigated.

The ability of the $1/N$ framework to address the nonperturbative aspects of quantum field dynamics has motivated a detailed study of the properties of these systems [CoJaPo74, Roo74]. In nonequilibrium situations, this formalism has been applied to the dynamics of symmetry breaking [HKMP96, CHKMPA94, CKMP95, BBHKP98, BVHS99a, LoMaRi03] and self-consistent semiclassical cosmological models (see Chapter 15).

In the case of the $O(N)$ invariant theory, in the presence of a nonzero background field (or an external gravitational or electromagnetic field interacting with the scalar field) we may distinguish the longitudinal quantum fluctuations in the direction of the background field, in field space, from the $N-1$ transverse (Goldstone or pion) fluctuations perpendicular to it. To first order in $1/N$, the longitudinal fluctuations drop out of the formalism, so we effectively are treating the background field as classical. Likewise, quantum fluctuations of the external field are overpowered by the fluctuations of the N scalar fields. In this way, the $1/N$ framework provides a systematic and quantitative measure of the semiclassical approximation [HarHor81].

To leading order (LO), the theory reduces to $N-1$ linear fields with a time-dependent mass, which depends on the background field and on the linear fields themselves through a gap equation local in time. This depiction of the dynamics agrees both with the Gaussian approximation for the density matrix [EbJaPi88, MazPaz89] and with the Hartree approximation [HKMP96].

For example, let us consider an $O(N)$ invariant scalar field theory, in the limit $N\to\infty$. The action is

$$S=\int d^4x\,\left\{\frac{-1}{2}\partial_\mu\Psi^\alpha\partial^\mu\Psi^\alpha-\frac{1}{2}M^2\Psi^\alpha\Psi^\alpha-\frac{\lambda}{8N}\left(\Psi^\alpha\Psi^\alpha\right)^2\right\} \tag{6.139}$$

or by a rescaling $\Psi^\alpha=\sqrt{N}\Phi^\alpha$,

$$S=N\int d^4x\,\left\{\frac{-1}{2}\partial_\mu\Phi^\alpha\partial^\mu\Phi^\alpha-\frac{1}{2}M^2\Phi^\alpha\Phi^\alpha-\frac{\lambda}{8}\left(\Phi^\alpha\Phi^\alpha\right)^2\right\} \tag{6.140}$$

whereby the classical equations are

$$\nabla^2 \phi^\alpha - \left[M^2 + \frac{\lambda}{2} \left(\phi^\beta \phi^\beta \right) \right] \phi^\alpha = 0 \tag{6.141}$$

To compute the 1PIEA we shift the field $\Phi \to \phi + \varphi$ and discard linear terms to get

$$S_r [\varphi] = N \int d^4x \left\{ \frac{-1}{2} \partial_\mu \varphi^\alpha \partial^\mu \varphi^\alpha - \frac{1}{2} M^2_{\alpha\beta} \varphi^\alpha \varphi^\beta - \frac{\lambda}{2} \phi^\beta \varphi^\beta \varphi^\alpha \varphi^\alpha - \frac{\lambda}{8} \left(\varphi^\alpha \varphi^\alpha \right)^2 \right\} \tag{6.142}$$

where

$$M^2_{\alpha\beta} = \left[M^2 + \frac{\lambda}{2} \phi^\gamma \phi^\gamma \right] \delta_{\alpha\beta} + \lambda \phi^\alpha \phi^\beta \tag{6.143}$$

We see that the fluctuation field in the direction of ϕ^α has a different propagator than the "pions," namely the fluctuations orthogonal to the mean field. However, since there are $N - 1$ pions, they dominate the perturbative expansion, and we may think only of them (or even simpler, let us consider the loop expansion at $\phi = 0$).

In this theory, propagators carry a weight of N^{-1} and vertices a weight N. Therefore, an individual vacuum Feynman graph carries a weight N^C, where $C = 1 - L$ is the number of vertices minus the number of internal lines (L is the number of loops in the graph). However, when we sum over internal indices, it acquires an additional power of N for each independent choice of the $O(N)$ index α. For this reason, adding a tadpole to an internal line does not affect the overall power of N in the graph: although there are two more lines after the insertion, there is also one more vertex and one more $O(N)$ index to sum over. In particular, the "double-bubble" graph has an overall power of N (that is, the same scaling as the classical action itself) coming from two internal lines, one vertex, and two possible choices of the internal indices. By adding tadpoles to the double-bubble in all possible ways, we obtain an infinite family of graphs of weight N, the so-called "daisy" graphs.

If we wish to say something about the large N limit of the theory, we must be able to add up the daisy graphs. This will be achieved by a more powerful formalism, the so-called two-particle irreducible (2PI) or Cornwall–Jackiw–Tomboulis (CJT) effective action, to which we now turn.

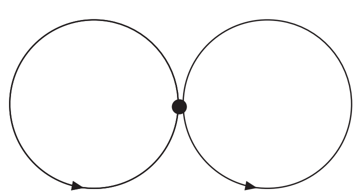

Figure 6.2 The "double-bubble" graph.

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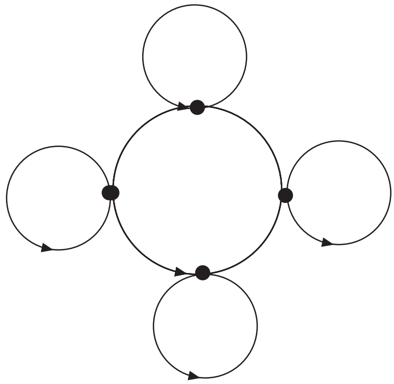

Figure 6.3 A “daisy” graph.

6.5 The two-particle irreducible (2PI) effective action

One clear advantage of working with the CTPEA rather than with the CTP generating functional is that the perturbative expansion is simpler: all connected graphs contribute to the latter, while only one-particle irreducible graphs contribute to the former. It is possible to simplify the perturbative expansion even more by writing Feynman graphs where internal lines represent the full propagators G^{ab}, rather than free propagators Δ^{ab} or some intermediate object. This means that graphs which just dress some internal line of some simpler graph must be disregarded, since all possible corrections are already taken into account in G^{ab}; the remaining graphs are those where no nontrivial subgraph can be isolated by cutting two internal lines, the so-called *two-particle irreducible* (2PI) graphs.

The basic idea is that, when computing the CTP generating functional, we want to constrain the deviations φ^A from the mean field ϕ^A so that not only their expectation value vanishes but also their fluctuations are known. We achieve this by adding suitable Lagrange multipliers: our already familiar source J^A, associated with the first constraint, and four new two-point sources $K^{AB} = K^{ab}(x, x')$ to enforce the second. The 2PI generating functional then reads, written as a path integral over full field Φ configurations

$$e^{(i/\hbar)W[J,K]} = \int D\Phi^A \exp\left\{(i/\hbar)\left[S\left[\Phi^A\right] + J_A\Phi^A + \frac{1}{2}K_{AB}\Phi^A\Phi^B\right]\right\} \quad (6.144)$$

The sources are connected to the mean fields and propagators through

$$\frac{\delta W}{\delta J_A} = \phi^A; \qquad \frac{\delta W}{\delta K_{AB}} = \frac{1}{2}\left[\phi^A\phi^B + G^{AB}\right] \quad (6.145)$$

The 2PI CTP EA is the full Legendre transform

$$\Gamma_2[\phi, G] = W[J, K] - J_A\phi^A - \frac{1}{2}K_{AB}\left[\phi^A\phi^B + G^{AB}\right] \quad (6.146)$$

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The equations of motion for mean fields and propagators are then

$$\frac{\delta\Gamma_2}{\delta\phi^A} = -J_A - K_{AB}\phi^B; \qquad \frac{\delta\Gamma_2}{\delta G^{AB}} = -\frac{1}{2}K_{AB} \tag{6.147}$$

To implement the background field method, write

$$e^{i\Gamma_2/\hbar} = \int D\Phi^A \; e^{(i/\hbar)\left[S\left[\Phi^A\right]+J_A\left(\Phi^A-\phi^A\right)+\frac{1}{2}K_{AB}\left(\Phi^A\Phi^B-\phi^A\phi^B-G^{AB}\right)\right]} \tag{6.148}$$

The exponent becomes

$$S\left[\Phi^A\right] - \frac{\delta\Gamma_2}{\delta\phi^A}\left(\Phi^A-\phi^A\right) - \frac{\delta\Gamma_2}{\delta G^{AB}}\left[\left(\Phi^A-\phi^A\right)\left(\Phi^B-\phi^B\right)-G^{AB}\right] \tag{6.149}$$

Write $\Phi = \phi + \varphi$ and expand the classical action as before,

$$S\left[\phi^A + \varphi^A\right] = S\left[\phi^A\right] + S_{,A}\varphi^A + \frac{1}{2}S_{,AB}\varphi^A\varphi^B + S_Q \tag{6.150}$$

From our previous experience with the 1PI CTPEA, we know that the effective action will be equal to the classical action plus $O(\hbar)$ corrections, which will be given in terms of a Feynman path integral over the φ field. At the lowest order, this integral will be Gaussian, and will yield a term like $-\left(i\hbar/2\right)\ln\mathrm{Det}\left\langle\varphi^A\varphi^B\right\rangle$. On the other hand, the formalism is set up so that $\left\langle\varphi^A\varphi^B\right\rangle \equiv G^{AB}$, thus we expect $\Gamma_2 = S[\phi] - (i\hbar/2)\ln\mathrm{Det}G^{AB} + \ldots$. This effective action should generate the equations of motion for both the mean field and the propagators. To lowest order in $\hbar$, the Schwinger–Dyson equation for the propagators may be written as $(i/\hbar)S_{,AB} = -G^{-1}_{AB}$ (this is the statement that the Hessian of the effective action is the inverse of the propagators, specialized to lowest order). Since the variation of $-i\hbar\ln\mathrm{Det}G^{AB}$ with respect to G^{AB} yields $-i\hbar G^{-1}_{AB}/2$, we get the right equation by adding a term whose variation is $S_{,AB}/2$. With these considerations in mind, we make the ansatz

$$\Gamma_2\left[\phi^A, G^{AB}\right] = S\left[\phi^A\right] + \frac{1}{2}S_{,AB}G^{AB} - \frac{1}{2}i\hbar\mathrm{Tr}\;\ln G + \Gamma_Q - \frac{1}{2}i\hbar\delta^A_A \tag{6.151}$$

(the final term does not affect the equations of motion and may be disregarded in practice) to get

$$e^{i\Gamma_Q/\hbar} = \left[\mathrm{Det}\; G\right]^{-1/2}\int D\varphi^A \\ \times\exp\left\{\frac{-1}{2}G^{-1}_{AB}\varphi^A\varphi^B + (i/\hbar)\left[S_Q - \tilde{J}_A\varphi^A - \tilde{K}_{AB}\left(\varphi^A\varphi^B - G^{AB}\right)\right]\right\} \tag{6.152}$$

where

$$\tilde{J}_A = \frac{1}{2}S_{,ABC}G^{BC} + \frac{\delta\Gamma_Q}{\delta\phi^A}; \qquad \tilde{K}_{AB} = \frac{\delta\Gamma_Q}{\delta G^{AB}} \tag{6.153}$$

We see that the 2PI effective action is given, besides the terms already explicit in equation (6.151), by the sum of all *two-particle irreducible vacuum* graphs in a theory with action $(i/2)\,G^{-1}_{AB}\varphi^A\varphi^B + S_Q$. They are vacuum because there is

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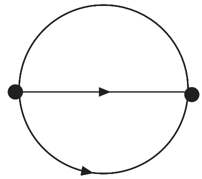

Figure 6.4 The setting sun graph.

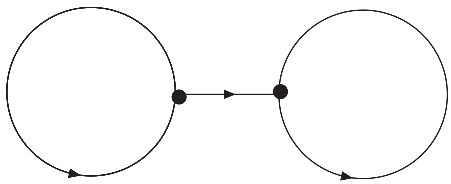

Figure 6.5 Two tadpoles joined by one line.

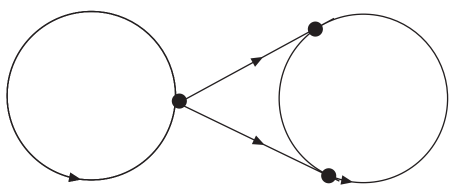

Figure 6.6 The "horn" graph.

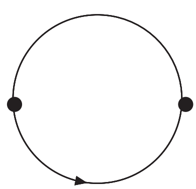

Figure 6.7 The fish graph.

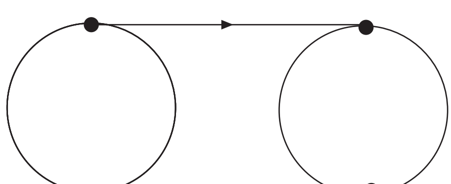

Figure 6.8 Two fishes joined by two lines.

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no φ mean field, and 2PI because the nonlocal source $\tilde{K}_{AB}$ ensures that G^{AB} is the full propagator (we shall show this explicitly right away). The reduction from connected to 1PI to 2PI graphs entails a substantial increase in efficiency.

For example, for a $\lambda\Phi^4$ theory, S_Q contains both cubic and quartic vertices. The simplest Feynman graphs are the "double-bubble" and the "setting sun," both of which are of order $\hbar^2$. We may discard a graph consisting of two tadpoles joined by an internal line, since this is not 1PI. At the following order we may also discard graphs like the "horn," and two fishes joined by two internal lines, which are 1PI but not 2PI.

To conclude, let us verify that the sources enforce the proper constraints. As in our earlier discussion of the loop expansion in the 1PI CTPEA, this will follow from the invertibility of the relationship of sources to fields, namely, that the operator

$$\begin{pmatrix} \dfrac{\delta J_A}{\delta\phi^C} & \dfrac{\delta J_A}{\delta G^{CD}} \\ \dfrac{\delta K_{AB}}{\delta\phi^C} & \dfrac{\delta K_{AB}}{\delta G^{CD}} \end{pmatrix} \tag{6.154}$$

is nonsingular. In terms of derivatives of the 2PIEA, this becomes (minus)

$$\begin{pmatrix} \dfrac{\delta^2\Gamma_2}{\delta\phi^A\delta\phi^C} - 2\dfrac{\delta\Gamma_2}{\delta G^{AC}} - 2\dfrac{\delta^2\Gamma_2}{\delta G^{AB}\delta\phi^C}\phi^C & \dfrac{\delta^2\Gamma_2}{\delta\phi^A\delta G^{CD}} - 2\dfrac{\delta^2\Gamma_2}{\delta G^{AB}\delta G^{CD}}\phi^B \\ 2\dfrac{\delta^2\Gamma_2}{\delta G^{AB}\delta\phi^C} & 2\dfrac{\delta^2\Gamma_2}{\delta G^{AB}\delta G^{CD}} \end{pmatrix} \tag{6.155}$$

It is clear that this operator will be nonsingular if and only if the simpler matrix

$$\begin{pmatrix} \dfrac{\delta^2\Gamma_2}{\delta\phi^A\delta\phi^C} - 2\dfrac{\delta\Gamma_2}{\delta G^{AC}} & \dfrac{\delta^2\Gamma_2}{\delta\phi^A\delta G^{CD}} \\ 2\dfrac{\delta^2\Gamma_2}{\delta G^{AB}\delta\phi^C} & 2\dfrac{\delta^2\Gamma_2}{\delta G^{AB}\delta G^{CD}} \end{pmatrix} \tag{6.156}$$

also is. By taking variations of equation (6.152) with respect to ϕ^A and G^{AB}, and after some algebra, we obtain the set of equations

$$\begin{pmatrix} \dfrac{\delta^2\Gamma_2}{\delta\phi^A\delta\phi^C} - 2\dfrac{\delta\Gamma_2}{\delta G^{AC}} & \dfrac{\delta^2\Gamma_2}{\delta\phi^A\delta G^{CD}} \\ 2\dfrac{\delta^2\Gamma_2}{\delta G^{AB}\delta\phi^C} & 2\dfrac{\delta^2\Gamma_2}{\delta G^{AB}\delta G^{CD}} \end{pmatrix} \begin{pmatrix} \langle\varphi^C\rangle \\ \langle\varphi^C\varphi^D\rangle - G^{CD} \end{pmatrix} = 0 \tag{6.157}$$

and so the constraints are enforced. In practice, this means that we can forget about $\tilde{J}$ and $\tilde{K}$ when computing the nonlinear correction Γ_Q to the 2PIEA, provided that we omit all one- and two-particle reducible graphs, and use the full propagator G^{AB} in internal lines. The vertices, of course, are those contained in S_Q, and will generally depend on the mean fields.

As for the CTP method more generally, it is impossible to give a complete list of references for the 2PIEA. For some of the pioneering papers, see

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[LutWar60, DomMar64a, DomMar64b, DahLas67, CoJaTo74]. This method has been generalized and applied to the establishment of a quantum kinetic field theory (Chapter 11). It has been applied to problems in gravitation and cosmology (Chapter 15), particles and fields (Chapter 14), Bose–Einstein condensates and condensed matter systems (Chapter 13) as well as to address the issues of thermalization and quantum phase transitions (Chapters 9 and 12). More generally, we may regard the 2PIEA as an implementation of the Φ-derivable approach to be discussed in Chapter 13.

6.5.1 The 2PI effective action in the $g\Phi^3$ theory

Let us test our understanding of this new object by applying it to the $g\Phi^3$ field theory. The 2PI CTPEA is given by equation (6.151), where, as before, the classical potential is given in equation (6.43), and m^2 is shifted to $m^2 - i\varepsilon$ on the first branch, $m^2 + i\varepsilon$ on the second. The second derivatives of the classical action may be read off from equation (6.118). We do not need an explicit knowledge of $\ln G$, other than the formal property

$$\frac{\delta \ln G}{\delta G^{AB}} = \left[G^{-1}\right]_{AB} \tag{6.158}$$

Γ_Q is the sum of all 2PI vacuum bubbles with vertices from equation (6.119) and full propagators G^{AB} in internal lines. Observe that in this model, Γ_Q is independent of the background fields, which is rather exceptional. The lowest order contribution to Γ_Q has two loops

$$\Gamma_Q \sim \frac{i}{2\hbar}\left(\frac{g^2}{6}\right) c_{abc}c_{def}\int d^4x d^4x'\; G^{ad}\left(x,x'\right) G^{be}\left(x,x'\right) G^{cf}\left(x,x'\right) \tag{6.159}$$

The equations of motion are derived from the variations with respect to ϕ^A and G^{AB}. In the physical case where there are no external sources, we get

$$S_{,A} + \frac{1}{2}S_{,ABC}G^{BC} = 0 \tag{6.160}$$

$$\frac{1}{2}S_{,AB} - \frac{1}{2}i\hbar\left[G^{-1}\right]_{AB} + \frac{\delta\Gamma_Q}{\delta G^{AB}} = 0 \tag{6.161}$$

The equation for the propagators reduces to equation (6.82) after we identify

$$\frac{\delta\Gamma_Q}{\delta G^{AB}} = -\frac{\hbar}{2}\Sigma_{AB} \tag{6.162}$$

It is customary to rewrite it as

$$S_{,AB}G^{BC} - \hbar\Sigma_{AB}G^{BC} = i\hbar\delta_A^C \tag{6.163}$$

More explicitly (we write the equation for ϕ^1, after setting $\phi^1 = \phi^2 = \phi$)

$$\nabla^2\phi\left(x\right) - m^2\phi\left(x\right) + \frac{1}{2}g\left[\phi^2\left(x\right) + G^{11}\left(x,x\right)\right] = -h \tag{6.164}$$

$$\begin{aligned}&\left[\nabla^2 - m^2\right] G^{ac}\left(x,y\right) + gc^{ag}c_{gdb}\phi^d\left(x\right) G^{bc}\left(x,y\right)\\ &\quad - \hbar c^{ag}\int d^4z\; \Sigma_{gb}\left(x,z\right) G^{bc}\left(z,y\right) = i\hbar c^{ac}\delta\left(x,y\right)\end{aligned} \tag{6.165}$$

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where

$$i\Sigma_{gb}(x,z) = \frac{g^2}{2\hbar^2} c_{gde} c_{fhb}\, G^{df}(x,z)\, G^{eh}(x,z) \tag{6.166}$$

We may begin to see the power of the 2PIEA. Our linearized one-loop equation derived above from the 1PIEA is equivalent to neglecting the Σ kernels. In this case, the propagators decouple, and solving for the Feynman propagator in powers of ϕ, we recover the known results. Using the 2PIEA, although we are also doing a one-loop approximation to the Schwinger–Dyson equations, we have a much more complete description of the physics, including some of the nonlinear interactions between fluctuations (we shall return to this in Chapter 11).

The propagator equations are more transparent if we choose G_{ret} and G_1 as independent variables, rather than the four fundamental propagators. Recalling equation (6.9), we get

$$\left[\nabla^2 - m^2 + g\phi(x)\right] G_{\text{ret}}(x,y) - \int d^4z\, \Sigma_{\text{ret}}(x,z)\, G_{\text{ret}}(z,y) = -\delta(x,y) \tag{6.167}$$

where

$$\Sigma_{\text{ret}}(x,z) = \hbar\left[\Sigma_{11}(x,z) + \Sigma_{12}(x,z)\right] \tag{6.168}$$

and we have used that $G^{11} + G^{22} = G^{12} + G^{21}$. Observe that the kernel Σ_{ret} is causal. For the Hadamard propagator, we get

$$\begin{aligned}&\left[\nabla^2 - m^2 + g\phi(x)\right] G_1(x,y)\\ &- \int d^4z\, \Sigma_{\text{ret}}(x,z)\, G_1(z,y) = -i\hbar \int d^4z\, \Sigma_1(x,z)\, G_{\text{adv}}(z,y)\end{aligned} \tag{6.169}$$

$$\begin{aligned}\Sigma_1(x,z) &= \hbar\left[\Sigma_{11}(x,z) + \Sigma_{22}(x,z)\right]\\ &= -\hbar\left[\Sigma_{12}(x,z) + \Sigma_{21}(x,z)\right]\end{aligned} \tag{6.170}$$

We shall discuss further this equation in Chapter 8. For the time being, we remark that to lowest order in perturbation theory, $i\Sigma_1$ is just (twice) the noise kernel from the 1PIEA (cf. equation (6.130)).

6.5.2 Large N expansion (suite)

Let us return to the $O(N)$ invariant scalar field theory from the last section. After rescaling, the action is given by equation (6.140). We recall there was an infinite family of Feynman graphs all scaling as N in the large N limit, for which reason the 1PIEA was not easy to compute. As we shall see presently, the 2PI approach cures this problem. The reason is that all but one of the offending graphs are two-particle *reducible*, and therefore drop out of the effective action.

It follows that if we are satisfied with the leading order (LO) theory, we may simply write down the 2PIEA for the theory as given, and obtain a closed form

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expression. For simplicity, let us consider the *unbroken symmetry* case, $\phi^A = 0$. Therefore

$$\Gamma^{\mathrm{LO}} = \frac{N}{2}\left\{\left[\nabla^2 - M^2\right] G^{A\alpha,A\alpha}\right\} - \frac{i\hbar}{2}\mathrm{Tr}\ \ln G + \Gamma_Q^{\mathrm{LO}} + \mathrm{const.} \tag{6.171}$$

$$\Gamma_Q^{\mathrm{LO}} = -N\frac{\lambda_{abcd}}{8}\int d^4x\left\{G^{a\alpha,b\alpha}(x,x)\, G^{c\beta,d\beta}(x,x) + 2G^{a\alpha,b\beta}(x,x)\, G^{c\alpha,d\beta}(x,x)\right\} \tag{6.172}$$

Of course, only the first term is truly LO. Discarding the terms which are not strictly LO we get

$$c_{ac}\left[\nabla^2 - M^2 - \frac{\lambda}{2}G_F^{\gamma\gamma}(x,x)\right] G^{c\alpha,b\beta}(x,x') = \frac{i\hbar}{N}\delta_{\alpha\beta}\delta_a^b\delta(x-x') \tag{6.173}$$

These equations are all there is to leading order. Observe that the only difference with the equations for a free theory is the mass shift: the real mass of the theory is not M^2 but rather $M^2_{\mathrm{phys}} = M^2 + (\lambda/2)\, G_F^{\gamma\gamma}(x,x)$. Since we may also solve the equation to obtain

$$G_F^{\alpha\beta}(x,x') = \frac{-i\hbar}{N}\delta_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-x')}}{k^2 + M^2_{\mathrm{phys}} - i\varepsilon} \tag{6.174}$$

this results in a nonlinear (gap) equation for the physical mass

$$M^2_{\mathrm{phys}} = M^2 - \frac{i\hbar\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2 + M^2_{\mathrm{phys}} - i\varepsilon} \tag{6.175}$$

The name "gap equation" is adopted from condensed matter physics, to the fact that M here is the energy of an excitation with zero momentum.

To the next order, we find that strings of *fish* graphs are all of order N^0, since there are $l-1$ fishes in the graph, and each may carry an independent index. So to get a closed expression to NLO, we must use the Coleman–Jackiw–Politzer trick [CoJaPo74] of including an auxiliary field χ, by adding a term to the action, which now reads

$$S = N\int d^4x\left\{\frac{-1}{2}\partial_\mu\Phi^\alpha\partial^\mu\Phi^\alpha - \frac{1}{2}M^2\Phi^\alpha\Phi^\alpha - \frac{\lambda}{8}\left(\Phi^\alpha\Phi^\alpha\right)^2 + \frac{1}{2}\left(\frac{\chi}{\sqrt{\lambda}} - \frac{\sqrt{\lambda}}{2}\left(\Phi^\alpha\Phi^\alpha\right)\right)^2\right\} \tag{6.176}$$

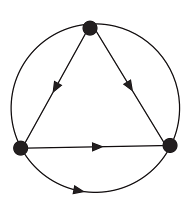

Figure 6.9 A string of three fishes.

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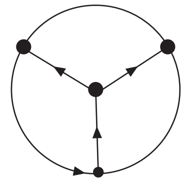

Figure 6.10 The three-pointed star graph.

Expanding this, we get

$$S = N \int d^4x \left\{ \frac{-1}{2} \partial_\mu \Phi^\alpha \partial^\mu \Phi^\alpha - \frac{1}{2} M^2 \Phi^\alpha \Phi^\alpha + \frac{\chi^2}{2\lambda} - \frac{\chi}{2} \left(\Phi^\alpha \Phi^\alpha \right) \right\} \quad (6.177)$$

The new classical equations of motion are

$$\nabla^2 \phi^\alpha - \left[M^2 + \chi \right] \phi^\alpha = 0 \quad (6.178)$$

$$\chi = \frac{\lambda}{2} \phi^\beta \phi^\beta \quad (6.179)$$

which are seen to be identical to the old ones.

In this new action, strings of fish graphs beyond two loops are no longer 2PI. The next nontrivial graph is the *three-pointed star*, which scales as N^{-1}. Thus, once again, we obtain a closed form for NLO large N.

To obtain this explicit expression, we begin by shifting the field $\Phi \to \phi + \varphi$, $\chi \to \bar{\chi} + \delta\kappa$. As usual, we discard linear terms, so

$$\delta S = N \int d^4x \left\{ \frac{-1}{2} \partial_\mu \varphi^\alpha \partial^\mu \varphi^\alpha - \frac{1}{2} \left(M^2 + \bar{\chi} \right) \varphi^\alpha \varphi^\alpha \right.$$
$$\left. + \frac{\delta\kappa^2}{2\lambda} - \delta\kappa \left(\phi^\alpha \varphi^\alpha \right) - \frac{\delta\kappa}{2} \left(\varphi^\alpha \varphi^\alpha \right) \right\} \quad (6.180)$$

It is convenient to eliminate the quadratic cross-term, shifting $\delta\kappa = \delta\chi + \lambda \phi^\alpha \varphi^\alpha$. We get

$$\delta S = \frac{N}{2} \int d^4x \left\{ - \left(\partial\varphi \right)^2 - M^2_{\alpha\beta} \varphi^\alpha \varphi^\beta + \frac{\delta\chi^2}{\lambda} - \delta\chi \left(\varphi^\alpha \varphi^\alpha \right) - \lambda \phi^\alpha \varphi^\alpha \varphi^\beta \varphi^\beta \right\}$$
$$(6.181)$$

$M^2_{\alpha\beta} = \left(M^2 + \bar{\chi} \right) \delta_{\alpha\beta} + \lambda \phi_\alpha \phi_\beta$, whereby the 2PIEA,

$$\Gamma^{\mathrm{NLO}} = S \left[\phi, \bar{\chi} \right] + \frac{N}{2} \left\{ \left[\nabla^2 \delta_{\alpha\beta} - M^2_{\alpha\beta} \right] G^{\alpha\beta} + \frac{H}{\lambda} \right\}$$
$$- \frac{i\hbar}{2} \left\{ \mathrm{Tr}\, \ln H + \mathrm{Tr}\, \ln G \right\} + \Gamma_Q^{NLO} + \mathrm{const.} \quad (6.182)$$

$$\Gamma_Q^{\mathrm{NLO}} = \frac{iN^2}{4\hbar} \int d^4x d^4x' \left\{ H \left(x, x' \right) G^{\gamma\delta} \left(x, x' \right)^2 + \lambda^2 \phi^\alpha \left(x \right) \phi^\beta \left(x' \right) \Delta^{\alpha\beta} \left(x, x' \right) \right\}$$
$$(6.183)$$

$$\Delta^{\alpha\beta} \left(x, x' \right) = G^{\alpha\beta} \left(x, x' \right) G^{\gamma\delta} \left(x, x' \right)^2 + 2 G^{\alpha\delta} \left(x, x' \right) G^{\gamma\delta} \left(x, x' \right) G^{\gamma\beta} \left(x, x' \right)$$
$$(6.184)$$

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Let us write the equations of motion for unbroken symmetry, leaving the CTP indices implicit

$$\left[\nabla^2 - M^2 - \chi\right]\delta_{\alpha\beta} - \frac{i\hbar}{N}G^{-1}_{\alpha\beta} + \frac{iN}{\hbar}H\left(x,x'\right)G^{\alpha\beta}\left(x,x'\right) = 0 \tag{6.185}$$

$$\chi - \frac{\lambda}{2}G^{\alpha\alpha}\left(x,x\right) = 0 \tag{6.186}$$

$$\mathbf{1} - \frac{i\lambda\hbar}{N}H^{-1} + \frac{i\lambda N}{2\hbar}G^{\gamma\delta}\left(x,x'\right)^2 = 0 \tag{6.187}$$

6.6 Handling divergences

As is well-known, the field theory of point particles is riddled with divergences. One needs to identify and remove them before one can begin to deal with physical applications. In this section we shall briefly summarize the most common types of divergences to be expected. By no means is this a complete treatment. As an example, we continue to use the $g\varphi^3$ theory to discuss its divergences.

6.6.1 Ultraviolet divergences

In field theories defined on flat spacetime and where the *in* and *out* vacuum agree to zeroth order in perturbation theory, to any finite order the 1PIEA is rendered free of ultraviolet divergences by renormalizing the parameters in the bare action in the same way one does for the *in–out* EA. This follows from the observation that a primitively divergent graph must have all its vertices on the same branch of the closed time path, and therefore, if we use free propagators in the internal lines, it is either equivalent to an *in–out* graph (all propagators are Feynman) or to its conjugate (all propagators are Dyson). By the assumed equivalence of the vacuum states, these are the same graphs appearing in the *in–out* EA.

If a graph does not have all vertices in the same branch, it cannot be primitively divergent. We say that a graph is primitively divergent when it diverges, and every subgraph is also divergent. Take a one-particle irreducible graph with vertices on both branches. Take one vertex, say, on the first branch, and consider the maximal set of vertices on the same branch which are connected to it. This set is not all the graph, because the graph has also second branch vertices. The maximal set is connected to the rest of the graph by at least two lines, because the graph is one-particle irreducible. These lines have mixed vertices at their ends, since otherwise they would be internal to the maximal set. When writing the corresponding amplitude, these lines will go on-shell, because both Δ^{21} and Δ^{12} are proportional to $\delta\left(p^2+m^2\right)$. Now consider a loop including these two lines: it has to be finite, because there are two on-shell lines. Therefore the graph is not primitively divergent.

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When working with the 2PIEA, one does not aim to make the effective action finite, but rather to show that the equations of motion admit finite solutions. To be ready for renormalization, we build the 2PIEA on the bare action

$$S_{\text{bare}}\left[\Phi\right]=\int d^dx\,\left[-\frac{1}{2}Z_\varphi\left(\partial\Phi\right)^2-\frac{1}{2}m_b^2\Phi^2+\frac{1}{6}Z_gg\Phi^3+h_B\Phi\right]\tag{6.188}$$

leading to the equations of motion

$$Z_\varphi\nabla^2\phi\left(x\right)-m_b^2\phi\left(x\right)+\frac{1}{2}Z_gg\left[\phi^2\left(x\right)+\frac{1}{2}G^{11}\left(x,x\right)\right]=-h_B\tag{6.189}$$

$$\left[Z_\varphi\nabla^2-m_b^2+Z_gg\phi\left(x\right)\right]G_{\text{ret}}\left(x,y\right)-Z_g^2\int d^4z\,\Sigma_{\text{ret}}\left(x,z\right)G_{\text{ret}}\left(z,y\right)=-\delta\left(x,y\right)\tag{6.190}$$

$$\left[Z_\varphi\nabla^2-m_b^2+Z_gg\phi\left(x\right)\right]G_1\left(x,y\right)-Z_g^2\int d^4z\,\Sigma_{\text{ret}}\left(x,z\right)G_1\left(z,y\right)=-\mathcal{K}\left(x,y\right)\tag{6.191}$$

$$\mathcal{K}\left(x,y\right)=iZ_g^2\hbar\int d^4z\,\Sigma_1\left(x,z\right)G_{\text{adv}}\left(z,y\right)\tag{6.192}$$

(The Σ's are defined by equations (6.166), (6.168) and (6.170).) In order to analyze the possible divergences in these equations, we adopt some kind of perturbative approach. Let us assume that ϕ is constant, and that in the Σ kernels we may approximate the propagators by free propagators, corresponding to a yet unknown mass M^2. The propagators will then be translation invariant, and we may Fourier transform all equations to get

$$m_b^2\phi\left(x\right)-\frac{1}{2}Z_gg\left[\phi^2+\frac{1}{2}\int\frac{d^dp}{\left(2\pi\right)^d}G^{11}\left(p\right)\right]=-h_B\tag{6.193}$$

$$\left[-Z_\varphi p^2-m_b^2+Z_gg\phi-Z_g^2\Sigma_{\text{ret}}\left(p\right)\right]G_{\text{ret}}\left(p\right)=-1\tag{6.194}$$

$$\left[-Z_\varphi p^2-m_b^2+Z_gg\phi-Z_g^2\Sigma_{\text{ret}}\left(p\right)\right]G_1\left(p\right)=-iZ_g^2\Sigma_1\left(p\right)G_{\text{adv}}\left(p\right)\tag{6.195}$$

Since $\Sigma_1\left(p\right)$ is finite and $Z_g^2=1$ to lowest order, the third equation will be well defined if we can control the second.

At this point we need to relate the effective mass M^2 to the propagators. Two common choices are to define M^2 as the value of the inverse retarded propagator at $p=0$, or else as the position of the pole of the retarded propagator as a function of $-p^2$. This second choice has a greater physical appeal, but it is harder to implement in practice.

Let us therefore define M^2 as the value of the inverse retarded propagator at $p=0$. Recall that $\Sigma_{\text{ret}}\left(p\right)$ can be obtained from the results in Chapter 5, provided the Feynman prescription $p^2\to p^2-i\varepsilon=-p^{02}+\mathbf{p}^2-i\varepsilon$ is replaced by the causal prescription $p^2\to\left(p+i\varepsilon\right)^2=-p^{02}+\mathbf{p}^2-i\varepsilon\text{ sign }p^0$. It is convenient

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to parametrize m_b^2 in terms of the value M_0^2 of M^2 at $\phi = 0$ (which is always a solution of the equations of motion, by construction)

$$m_b^2 = M_0^2 + \frac{Z_g^2 g^2 \hbar}{16\pi^2}\left[\frac{1}{\varepsilon} + \text{ constant } - \frac{1}{2}\ln\left(\frac{M_0^2}{4\pi\mu^2}\right)\right] \tag{6.196}$$

The gap equation reads

$$M^2 - M_0^2 + Z_g g\phi + \frac{Z_g^2 g^2 \hbar}{32\pi^2}\ln\left(\frac{M^2}{M_0^2}\right) = 0 \tag{6.197}$$

In this model, the gap equation is explicitly finite, so we may simply set $Z_g = 1$. Otherwise, we may use this further degree of freedom to control any remaining divergence.

The wavefunction renormalization Z_φ may be determined, for example, by requiring that

$$\left.\frac{\partial G_{\text{ret}}^{-1}}{\partial\left(-p^2\right)}\right|_{p^2=0} = -1 \tag{6.198}$$

We get

$$1 = Z_\varphi + \frac{g^2\hbar}{192\pi^2 M^2}$$

which is finite.

After these choices, we have exhausted our freedom to redefine the parameters in the classical action, so the mean field equation ought to be explicitly finite. The mean field equation reads (recall the tadpole from Chapter 5, equation (5.24))

$$\begin{aligned} h_B = &\left\{M_0^2 + \frac{g^2\hbar}{16\pi^2}\left[\frac{1}{\varepsilon} + \text{ constant } - \frac{1}{2}\ln\left(\frac{M_0^2}{4\pi\mu^2}\right)\right]\right\}\phi\left(x\right) \\ &- \frac{1}{2}g\left\{\phi^2 - \frac{\hbar M^2}{8\pi^2}\left[\frac{1}{\varepsilon} + \text{ constant}' - \frac{1}{2}\ln\left(\frac{M^2}{4\pi\mu^2}\right)\right]\right\} \end{aligned} \tag{6.199}$$

Setting $\phi = 0$ we get

$$h_B = \frac{g\hbar M_0^2}{16\pi^2}\left[\frac{1}{\varepsilon} + \text{ constant}' - \frac{1}{2}\ln\left(\frac{M_0^2}{4\pi\mu^2}\right)\right] \tag{6.200}$$

so the coefficient of ε^{-1} is

$$\frac{g\hbar}{16\pi^2}\left[g\phi + M^2 - M_0^2\right] \tag{6.201}$$

which vanishes to lowest order by virtue of the gap equation.

For further discussion, we refer the reader to the literature [HeeKno02a, HeeKno02b, HeeKno02c, BlIaRe03].

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6.6.2 Initial time singularities

As we have seen in the last subsection, the handling of ultraviolet singularities in the nonequilibrium formalism is not really different from the usual field theory methods. We shall now discuss a new class of singularities which are specific to nonequilibrium problems [Lin87, CooMot87, Baa98, Baa00a, Baa00b, HaMoMo99, BaBoVe01].

These singularities arise when one attempts to solve the mean field equations of motion with Cauchy data at some initial time (which we may choose as $t = 0$ without any loss of generality). In a perturbative scheme, it seems "natural," to lowest order, to use free propagators to compute the Feynman graphs in the effective action, and to assume an initial state uncorrelated with the initial conditions for the mean fields. But actually this is wrong: the switching on of the mean field (or equivalently, of the coupling constant) in an arbitrarily short time-scale always has an impact on the initial state of the quantum fluctuations. Neglect of this effect introduces an inconsistency in the theory, thereby the divergences.

Let us consider the mean field equations for the $g\varphi^3$ model, as derived above from the one-loop 1PIEA, equation (6.89). We are interested in finding the free evolution of the mean field, from given initial conditions at $t = 0$. We shall assume the local terms (including the ultraviolet singularities) in the quantum correction have been absorbed in the parameters of the equation. We also assume the mean field is spatially homogeneous, so we may write

$$\left[\nabla^2 - m^2\right]\phi(t) + \frac{g^2\hbar}{32\pi^2}\int\frac{d\omega}{2\pi}\int_0^t du\; e^{-i\omega(t-u)}\int_{4m^2}^{\infty}\frac{d\sigma^2\sqrt{1-\frac{4m^2}{\sigma^2}}}{-(\omega+i\varepsilon)^2+\sigma^2}\phi(u) = 0 \tag{6.202}$$

Perform the integral over ω

$$\int\frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-u)}}{-(\omega+i\varepsilon)^2+\sigma^2} = \frac{\sin\left[\sigma(t-u)\right]}{\sigma} \tag{6.203}$$

$$\left[\nabla^2 - m^2\right]\phi(t) + \frac{g^2\hbar}{16\pi^2}\int_{4m^2}^{\infty} d\sigma\sqrt{1-\frac{4m^2}{\sigma^2}}\int_0^t du\;\sin\left[\sigma(t-u)\right]\;\phi(u) = 0 \tag{6.204}$$

We improve the convergence of the σ integral with an integration by parts

$$\begin{aligned}0 &= \left[\nabla^2 - m^2\right]\phi(t) + \frac{g^2\hbar}{16\pi^2}\int_{4m^2}^{\infty}\frac{d\sigma}{\sigma}\sqrt{1-\frac{4m^2}{\sigma^2}}\int_0^t du\;\left(\frac{d}{du}\cos\left[\sigma(t-u)\right]\right)\phi(u)\\ &= \left[\nabla^2 - m^2\right]\phi(t) - \frac{g^2\hbar}{16\pi^2}\int_{4m^2}^{\infty}\frac{d\sigma}{\sigma}\sqrt{1-\frac{4m^2}{\sigma^2}}\int_0^t du\;\cos\left[\sigma(t-u)\right]\frac{d\phi}{du}\\ &\quad + \delta m^2\phi(t) - \chi(t)\,\phi(0)\end{aligned} \tag{6.205}$$

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where

$$\delta m^2 = \frac{g^2\hbar}{16\pi^2}\left[\int_{4m^2}^{\infty}\frac{d\sigma}{\sigma}\sqrt{1-\frac{4m^2}{\sigma^2}}\right] \tag{6.206}$$

$$\chi(t) = \frac{g^2\hbar}{16\pi^2}\left[\int_{4m^2}^{\infty}\frac{d\sigma}{\sigma}\sqrt{1-\frac{4m^2}{\sigma^2}}\,\cos[\sigma t]\right] \tag{6.207}$$

The logarithmically divergent term δm^2 may be absorbed in m^2. At issue is the "source term" $\chi(t)\,\phi(0)$. At finite times, we may expect that the oscillatory behavior of the cosine will be enough to make the integral convergent. However, at $t = 0$ this improved convergence is lost, and $\chi(0)$ is ill defined. This is the initial time singularity.

In physical terms, it is as though we set $g = 0$ for $t < 0$, thereby allowing the quantum fluctuations to reach equilibrium as a free field (in this case, at zero temperature, but allowing for equilibrium at a finite temperature only makes the problem worse), and then suddenly we switch the interaction and the mean field on. This sudden transition will necessarily create particles, so it is inconsistent to assume that the state of the quantum fluctuations is the vacuum at any positive time, no matter how short.

The problem may be cured by adopting a more physical initial condition; we refer the reader to the literature for details [Lin87, CooMot87, Baa98, Baa00a, Baa00b, HaMoMo99, BaBoVe01].

6.6.3 Other divergences

Unfortunately, ultraviolet and initial time singularities are not the only problems to watch out for [CarKob98, CaKoPe98, Bed, Dad99, BoVeWa00, GeScSe01]. Among other common complications, we may mention infrared singularities, which appear when some quantum fluctuations are massless. Massless fields are rather common: they appear in problems related to unbroken gauge symmetries, at critical points in models of phase transitions, and as Goldstone bosons when a global symmetry is broken. For example, an $O(N)$ model in the broken symmetry phase has $N-1$ massless fields in its spectrum. The spectrum of excitations above a homogeneous Bose–Einstein condensate also generally contains one massless mode, which arises from the breaking of global $U(1)$ invariance.

Although we shall connect with finite temperature field theory in a later chapter it is timely to mention that real time perturbation theory at finite temperature also has its peculiar kind of divergences. The free thermal propagators contain terms proportional to mass-shell delta functions $\delta\left(-p^2-m^2\right)$, and so they produce singularities whenever two propagators are evaluated at collinear momenta in the same graph.

It is important to beware of singularities arising from a nonjudicious application of perturbation theory. Regardless of the formal order in g, $\hbar$ or N^{-1}, large corrections must be included to get consistent results. For example, weak damping of fluctuations due to higher order processes modifies the behavior of the propagators near the mass shell, and may cure some singularities. Judging the situation by the right physics is often the best way to handle the unfamiliar pathologies.

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