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A LOWER BOUND FOR THE REAL GENUS OF A FINITE GROUP

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ABSTRACT. Let G be a finite group. The *real genus* $\rho(G)$ is the minimum algebraic genus of any compact bordered Klein surface on which G acts. Here we obtain a good general lower bound for the real genus of the group G. We use the standard representation of G as a quotient of a non-euclidean crystallographic group by a bordered surface group. The lower bound is used to determine the real genus of several infinite families of groups; the lower bound is attained for some of these families. Among the groups considered are the dicyclic groups and some abelian groups. We also obtain a formula for the real genus of the direct product of an elementary abelian 2-group and an "even" dicyclic group. In addition, we calculate the real genus of an abstract family of groups that includes some interesting 3-groups. Finally, we determine the real genus of the direct product of an elementary abelian 2-group and a dihedral group.

1. **Introduction.** Let G be a finite group. The *real genus* $\rho(G)$ [8] is the minimum algebraic genus of any compact bordered Klein surface on which G acts. This parameter is called the "real" genus because of the correspondence between Klein surfaces and real algebraic curves [1]; the bordered surfaces correspond to curves with real points. The real genus parameter was introduced in [8], and several basic results about the parameter were obtained there. In particular, the groups with real genus $\rho \leq 3$ were classified and the real genus of each dicyclic group was determined. The groups of real genus four were classified in [9]. In addition, McCullough calculated the real genus of each finite abelian group [11], and the actions of metacyclic groups on bordered surfaces were investigated in [10].

Here we obtain a good general lower bound for the real genus of a finite group G. We use the standard representation of G as a quotient of a non-euclidean crystallographic group Γ by a bordered surface group K; then G acts on the Klein surface U/K, where U is the open upper half-plane. We apply our lower bound to determine the real genus of several infinite families of groups. The lower bound is attained for some of these families. We give a quick proof of the genus formula for dicyclic groups; we also consider the direct product of an elementary abelian 2-group and an "even" dicyclic group. Next we calculate the real genus of an abstract family of groups that includes some interesting 3-groups. In addition, the lower bound and a slight improvement give the real genus $\rho(A)$ for an abelian group A in some cases. Finally, we calculate the real genus of the direct product of an elementary abelian 2-group and a dihedral group.

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2. **Preliminaries.** We shall assume that all surfaces are compact. Let X be a bordered surface; X is characterized topologically by orientability, the number k of components of the boundary ∂X and the topological genus p. The bordered surface X can carry a dianalytic structure [1, p. 46] and be considered a Klein surface or a non-singular real algebraic curve. Thus the surface X has an algebraic genus g, which is given by the following relation:

$$g = \begin{cases} 2p + k - 1 & \text{if } X \text{ is orientable} \\ p + k - 1 & \text{if } X \text{ is non-orientable.} \end{cases}$$

The algebraic genus is the rank of the fundamental group of X, and this number appears quite naturally in bounds for the order of an automorphism group of a Klein surface (the monograph [2] contains several examples). The real genus of a group is defined in terms of the algebraic genus.

There is a general upper bound for the real genus of a finite group in terms of the orders of the elements in a generating set [8, §3]. This bound will be quite helpful here.

THEOREM A [8]. *Let G be a finite group with generators $z_1, \ldots, z_c$, where $o(z_i) = m_i$. Then*

$$\rho(G) \leq 1 + o(G)\Big[c - 1 - \sum_{i=1}^{c} \frac{1}{m_i}\Big]. \tag{2.1}$$

Non-euclidean crystallographic (NEC) groups have been quite useful in investigating group actions on bordered Klein surfaces. Here see [2], an excellent general reference for the work on bordered surfaces. Let $\mathcal{L}$ denote the group of automorphisms of the open upper half-plane U, and let $\mathcal{L}^+$ denote the subgroup of index 2 consisting of the orientation-preserving automorphisms. An NEC group is a discrete subgroup Γ of $\mathcal{L}$ (with the quotient space U/Γ compact). If $\Gamma \subset \mathcal{L}^+$, then Γ is called a *Fuchsian* group. Otherwise Γ is called a *proper NEC group*; in this case Γ has a canonical Fuchsian subgroup $\Gamma^+ = \Gamma \cap \mathcal{L}^+$ of index 2.

Associated with the NEC group Γ is its *signature*, which has the form

$$\big(p;\ \pm;\ [\lambda_1, \ldots, \lambda_r];\ \{(\nu_{11}, \ldots, \nu_{1s_1}), \ldots, (\nu_{k1}, \ldots, \nu_{ks_k})\}\big). \tag{2.2}$$

The quotient space $X = U/\Gamma$ is a surface with topological genus p and k holes. The surface is orientable if the plus sign is used and non-orientable if the minus sign is used. The integers $\lambda_1, \ldots, \lambda_r$, called the *ordinary periods*, are the ramification indices of the natural quotient mapping from U to X in fibers above interior points of X. The integers $\nu_{i1}, \ldots, \nu_{is_i}$, called the *link periods*, are the ramification indices in fibers above points on the i-th boundary component of X. Associated with the signature (2.2) is a presentation for the NEC group Γ. For more information about signatures, see [6], [14], and [2].

Let G be a finitely presented group. If the generating set has the minimum size, then this number of generators is called the *rank* of G.

The canonical presentation for an NEC group almost always involves redundant generators. Let Γ be an NEC group with signature (2.2) and associated canonical presentation. Suppose $k \geq 1$ and exactly ℓ of the k period cycles are empty. Regardless of whether the plus or minus sign is present, the number of generators in the presentation [2, p. 14] is

$$N = r + B + k + \gamma + 1,$$

where γ is the algebraic genus of the quotient space U/Γ and $B = s_1 + \cdots + s_k$, the number of boundary points of U/Γ above which the quotient map is ramified. Of these, $Q = 1 + (k - \ell)$ are clearly redundant. Thus Γ has a simplified presentation with $N - Q$ generators, and $\operatorname{rank}(\Gamma) \leq N - Q$, that is,

$$\operatorname{rank}(\Gamma) \leq \gamma + r + B + \ell.$$

Let Γ be an NEC group with signature (2.2) and assume $k \geq 1$ so that the quotient space U/Γ is a bordered surface. The non-euclidean area $\mu(\Gamma)$ of a fundamental region for Γ can be calculated directly from its signature [14, p. 235]:

$$\mu(\Gamma)/2\pi = \gamma - 1 + \sum_{i=1}^{r}\Big(1 - \frac{1}{\lambda_i}\Big) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}\Big(1 - \frac{1}{\nu_{ij}}\Big), \tag{2.3}$$

where γ is the algebraic genus of the quotient space U/Γ. If Λ is a subgroup of finite index in Γ, then

$$[\Gamma : \Lambda] = \mu(\Lambda)/\mu(\Gamma). \tag{2.4}$$

An NEC group K is called a *surface group* if the quotient map from U to U/K is unramified. Fuchsian surface groups contain no elements of finite order. If the quotient space U/K has a non-empty boundary, then K is called a *bordered surface group*. Bordered surface groups contain reflections but no other elements of finite order.

Let X be a bordered Klein surface of algebraic genus $g \geq 2$. Then X can be represented as U/K where K is a bordered surface group with $\mu(K) = 2\pi(g - 1)$. Let G be a group of dianalytic automorphisms of the Klein surface X. Then there are an NEC group Γ and a homomorphism $\phi\colon \Gamma \longrightarrow G$ onto G such that kernel $\phi = K$. The group $G \cong \Gamma/K$, so that from (2.4) we obtain

$$g = 1 + o(G) \cdot \mu(\Gamma)/2\pi. \tag{2.5}$$

Minimizing g is therefore equivalent to minimizing $\mu(\Gamma)$. Among the NEC groups Γ for which G is a quotient of Γ by a bordered surface group, then, we want to identify one for which $\mu(\Gamma)$ is as small as possible.

3. **A lower bound.** Here we establish a useful lower bound for the real genus of a finite group. This bound is usually a significant improvement on the rough lower bound in [8, §4].

Let G be a finitely presented group and S a generating set for G. For $p = 2$ or $p = 3$, let $t_p(S)$ denote the number of generators in S of order p. Also let $t_h(S)$ be the number of generators of order larger than 3 (using "h" for high order). We will write simply t_p and t_h if the generating set is obvious. Then $|S| = t_2 + t_3 + t_h$. We define $\psi(G) =$ minimum $\{9t_h(S) + 8t_3(S) + 3t_2(S) \mid S$ a generating set for $G\}$. A generating set for which $\psi(G)$ is attained is said to be *ψ-minimal.* The parameter $\psi(G)$ appears in our lower bound for the real genus of a finite group. A similar parameter is used to study the graph-theoretical genus of a group in [12]. The following result is basic.

LEMMA 1. *Let G' be a quotient group of the finitely presented group G. Then*

$$\psi(G) \geq \psi(G').$$

PROOF. Let $\pi\colon G \longrightarrow G'$ denote the quotient map. Then let S be a ψ-minimal generating set for G, and let S' be the induced generating set for G'. Write $t_2 = t_2(S)$ and $t_2' = t_2(S')$ and so forth. Clearly $|S| \geq |S'|$ so that

$$t_h + t_3 + t_2 \geq t_h' + t_3' + t_2'.$$

Let p be 2 or 3. If $y \in S'$ with $o(y) > p$, then there is at least one generator x in S such that $\pi(x) = y$. But $o(x) > p$ also. Hence for $p = 2$ we have

$$t_h + t_3 \geq t_h' + t_3'.$$

With $p = 3$ we obtain

$$t_h \geq t_h'.$$

Now since S is ψ-minimal,

$$\begin{aligned}
\psi(G) &= 9t_h + 8t_3 + 3t_2 \\
&= 3(t_h + t_3 + t_2) + 5(t_h + t_3) + t_h \\
&\geq 3(t_h' + t_3' + t_2') + 5(t_h' + t_3') + t_h' \\
&\geq \psi(G'),
\end{aligned}$$

whether or not the generating set S' is ψ-minimal.

Next we obtain an upper bound for $\psi(G)$ for a group G that is a quotient of an NEC group Γ with the kernel a bordered surface group.

LEMMA 2. *Let G be a finite group. Suppose there exist an* NEC *group Γ with signature (2.2) and a homomorphism $\phi\colon \Gamma \longrightarrow G$ onto G such that $K =$* kernel ϕ *is a bordered surface group. Then $k \geq 1$; suppose exactly ℓ of the k period cycles are empty. For $p = 2$*

or $p = 3$, *let* r_p *denote the number of ordinary periods equal to* p; *let* r_h *be the number greater than* 3. *Then*

$$\psi(G) \leq 9(\gamma + r_h) + 8r_3 + 3(r_2 + B + \ell - 1),$$

where $B = s_1 + \cdots + s_k$ *and* γ *is the algebraic genus of the quotient space* U/Γ.

PROOF. Simplify the canonical presentation for Γ as in §2, and let S be the induced generating set for G. The bordered surface group $K = \text{kernel}\,\phi$ contains reflections. Each reflection in K is conjugate to some generating reflection c_{ij} [6, p. 1198]. Thus some c_{ij} is in K (the redundant reflections thrown out in the simplification are also conjugates of others). Therefore the generating set S has at most $\gamma + r + B + \ell - 1$ elements. Of the elements in S, clearly at most $\gamma + r_h + r_3$ can have order larger than two. Now apply the definition of $\psi(G)$.

Now we establish our general lower bound.

THEOREM 1. *Let G be a finite group. Then*

$$\rho(G) \geq 1 + o(G)[\psi(G) - 12]/12. \tag{3.1}$$

PROOF. It is a simple matter to check that the inequality holds for the groups with $\rho \leq 1$. Assume then that $\rho(G) \geq 2$, and let G act on the bordered surface X of algebraic genus $g \geq 2$. Then represent X as U/K where K is a bordered surface group, and obtain an NEC group Γ and a homomorphism $\phi\colon \Gamma \longrightarrow G$ onto G such that $\text{kernel}\,\phi = K$. We shall use the notation of Lemma 2. From (2.3)

$$\mu(\Gamma)/2\pi \geq \gamma - 1 + r_2 \cdot \frac{1}{2} + r_3 \cdot \frac{2}{3} + r_h \cdot \frac{3}{4} + B \cdot \frac{1}{4}.$$

Therefore

$$12[\mu(\Gamma)/2\pi] \geq 12\gamma + 9r_h + 8r_3 + 6r_2 + 3B - 12. \tag{3.2}$$

We always have $\ell \leq k \leq \gamma + 1$, and thus $12\gamma \geq 9\gamma + 3(\ell - 1)$. Now

$$12[\mu(\Gamma)/2\pi] \geq 9(\gamma + r_h) + 8r_3 + 3(r_2 + B + \ell - 1) - 12.$$

Applying Lemma 2 yields

$$12[\mu(\Gamma)/2\pi] \geq \psi(G) - 12.$$

Now from (2.5) we have $g \geq 1 + o(G)[\psi(G) - 12]/12$. Thus $\rho(G) \geq 1 + o(G)[\psi(G) - 12]/12$.

This result should be compared with the corresponding result for the graph-theoretical genus [12, §2]. In addition, it would be nice to see the companion result for the symmetric genus.

We believe the lower bound (3.1) is quite useful, in general. We shall see examples of infinite families of groups for which the lower bound gives the real genus.

4. **Dicyclic groups.** First we consider the family of dicyclic groups. For $n \geq 2$, let H_n be the group with generators x, y and defining relations

$$x^{2n} = 1, \quad x^n = y^2, \quad y^{-1}xy = x^{-1}. \tag{4.1}$$

Then H_n is called the *dicyclic* group of order $4n$ [5, p. 7]. Each element outside the big cyclic subgroup $\langle x \rangle$ has order 4, and there is a unique element (x^n) of order 2. The group H_n is also generated by the two elements $w = xy$ and y of order 4 [5, p. 8] with defining relations

$$w^2 = y^2 = (w^{-1}y)^n. \tag{4.2}$$

The smallest dicyclic group H_2 is isomorphic to the quaternion group Q, and the group H_3 is isomorphic to the nonabelian group T of order 12 that is not A_4 and not D_6. Also, the group T is a semidirect product of Z_3 by Z_4 [13, p. 138].

LEMMA 3. *If $n \neq 3$, $\psi(H_n) = 18$. Furthermore $\psi(H_3) = 17$.*

PROOF. From either presentation (4.1) or (4.2), we immediately have $\psi(H_n) \leq 18$. Also, $\psi(H_3) \leq 17$. Now let H_n have the presentation (4.1). Regardless of the value of n, the subgroup $J = \langle x^n \rangle$ contains the only element of order two and is normal in H_n. Since $H_n/J \cong D_n$, there must be at least two generators of order larger than two in any generating set for H_n.

Suppose H_n has elements of order 3. Then 3 divides n, of course. So write $n = 3\ell$. In this case, the two elements of H_n of order 3 and the element of order 2 are contained in the normal subgroup $N = \langle x^\ell \rangle$, and it is not hard to see that the quotient group H_n/N is the dihedral group D_ℓ if $\ell > 1$. Also, H_3 is not generated by two elements of order 3, so that $\psi(H_3) = 17$.

Assume $n \neq 3$. Whether H_n has elements of order 3 or not, it follows that there are at least two generators of order larger than three in any generating set for H_n, and now clearly $\psi(H_n) = 18$.

THEOREM B [8]. *If $n \neq 3$, $\rho(H_n) = 1 + 2n$. Furthermore $\rho(H_3) = 6$.*

PROOF. Since H_n is generated by two elements of order 4, (2.1) gives $\rho(H_n) \leq 1+2n$. We also have $\rho(H_3) \leq 6$. The lower bound is supplied by Lemma 3 and (3.1).

The dicyclic groups are an infinite family of non-abelian groups for which the general lower bound (3.1) is attained. It is interesting that (3.1) also gives the genus in the exceptional case $n = 3$. The ideas of §3 allow a nice, short proof of Theorem B; contrast with the proof [8, §7].

Next we consider the direct product of an elementary abelian 2-group and an "even" dicyclic group.

LEMMA 4. *Let $G = (Z_2)^a \times H_n$, where $a \geq 1$ and n is even. Then* $\operatorname{rank}(G) = a + 2$ *and $\psi(G) = 3a + 18$.*

PROOF. Obviously, $\operatorname{rank}(G) \leq a + 2$. Let H_n have the presentation (4.1), and let $M = \langle x^2 \rangle$. Then it is easy to see that $H_n/M \cong D_2$ (since n is even), so that the elementary abelian 2-group of rank $a + 2$ is a quotient of G. Hence $\operatorname{rank}(G) \geq a + 2$.

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Since G is generated by a elements of order 2 and two of order 4, $\psi(G) \leq 9 \cdot 2 + 3 \cdot a$. But there are at least two generators of order larger than three in any generating set for H_n. Since $\operatorname{rank}(G) = a + 2$, it follows that $\psi(G) \geq 9 \cdot 2 + 3 \cdot a$.

For these groups it is usually possible to improve the lower bound slightly.

LEMMA 5. *Let $G = (Z_2)^a \times H_n$, with n even. If $a \geq 3$, then*

$$\rho(G) \geq 1 + o(G)[\psi(G) - 9]/12.$$

PROOF. First note that $\rho(G)$ is not 0 or 1 [8, §6]. We re-examine the proof of Theorem 1 and use the same notation. Write $M = 12[\mu(\Gamma)/2\pi]$. Then (3.2) becomes

$$M \geq 12\gamma + 9r_h + 8r_3 + 6r_2 + 3B - 12. \tag{4.3}$$

Always $\ell \leq k \leq \gamma + 1$. Here we consider two cases.

CASE I. $\ell \leq \gamma$ or $r_2 \geq 1$. Then easily, $2r_2 + \gamma \geq r_2 + \ell$. From (4.3)

$$\begin{aligned} M &\geq 9(\gamma + r_h) + 8r_3 + 3(2r_2 + \gamma + B) - 12 \\ &\geq 9(\gamma + r_h) + 8r_3 + 3(r_2 + \ell + B) - 12 \\ &= 9(\gamma + r_h) + 8r_3 + 3(r_2 + \ell + B - 1) - 9 \\ &\geq \psi(G) - 9, \end{aligned}$$

using Lemma 2.

CASE II. $\ell = k = \gamma + 1$ and $r_2 = 0$. In this case all of the period cycles are empty so that $B = 0$. With $r_2 = B = 0$, from (4.3) we have

$$M \geq 12\gamma + 9r_h + 8r_3 - 12. \tag{4.4}$$

As in Lemma 2, we must have $2\gamma + r_h + r_3 \geq a + 2 = \operatorname{rank}(G)$. Thus $M \geq 6(2\gamma + r_h + r_3) - 12 \geq 6(a + 2) - 12 = 6a \geq 3a + 9 = \psi(G) - 9$, using Lemma 4 and the condition that $a \geq 3$.

Then, just as in the proof of Theorem 1, we find $\rho(G) \geq 1 + o(G)[\psi(G) - 9]/12$.

Note that the proof in Case I does not depend on the structure of the group G.

THEOREM 2. *Let $G = (Z_2)^a \times H_n$, with n even. Then*

$$\rho(G) = \begin{cases} 1 + 2^a(a+2)n & \text{if } a = 1 \text{ or } a = 2 \\ 1 + 2^a(a+3)n & \text{if } a \geq 3. \end{cases}$$

PROOF. First suppose $a \geq 3$. Then Lemmas 4 and 5 yield $\rho(G) \geq 1 + 2^a(a+3)n$. Let Γ be an NEC group with signature

$$\left(0;\ +;\ [4,4];\ \{(2^{a+1})\}\right).$$

(There is one period cycle with $a+1$ periods equal to two). From (2.3) we calculate $\mu(\Gamma)/2\pi = (a+3)/4$. It is easy to construct a homomorphism $\phi\colon \Gamma \longrightarrow G$ of Γ onto G such that $K = \operatorname{kernel}\phi$ is a bordered surface group. Then G acts on the surface $X = U/K$. If g is the algebraic genus of X, then from (2.5) we find $g = 1 + 2^a(a+3)n$. Hence $\rho(G) \leq 1 + 2^a(a+3)n$.

If $a = 1$, then (3.1) provides the lower bound, and G is a quotient of an NEC group with signature $\big(0; +; [4]; \{(\ )^2\}\big)$ such that the kernel is a bordered surface group. If $a = 2$, then use (3.1) and an NEC group with signature $\big(0; +; [\ ]; \{(\ )^3\}\big)$.

An interesting special case involves the quaternion group $Q = H_2$.

COROLLARY. *If $a \geq 3$, then $\rho\big((Z_2)^a \times Q\big) = 1 + 2^{a+1}(a+3)$.*

5. **A family of abstract groups.** Let $(p, q|r, m)$ denote the group with generators R and S and defining relations

$$R^p = S^q = (RS)^r = (R^{-1}S)^m = 1. \tag{5.1}$$

This family of groups and two related families were studied in [4]; also see [5]. These families contain many interesting groups.

Here we consider each group $M_t = (3, 3|3, t)$. The group M_t has order $3t^2$ [4, p. 83] and contains a normal subgroup (denoted $(3, 3, 3; 1)$ in [4]) of order t^2; this subgroup is isomorphic to $Z_t \times Z_t$ [4, p. 96].

The smallest group M_2 is isomorphic to A_4 and thus has real genus 3 [8, Theorem 6]. The group M_3 is the nonabelian group of order 27 with no element of order 9.

LEMMA 6. *Let $t \geq 3$. Then* $\operatorname{rank}(M_t) = 2$ *and* $\psi(M_t) = 16$.

PROOF. Write $G = M_t$. Clearly $\operatorname{rank}(G) = 2$, and $\psi(G) \leq 16$ since G is generated by two elements of order three. If t is odd, then G has no elements of order 2 and obviously $\psi(G) \geq 16$. Suppose t is even. The group G contains a normal abelian subgroup $A \cong Z_t \times Z_t$. The Sylow 2-subgroup S of A is characteristic in A and normal in G. Since $[G : A] = 3$, it follows that S is the Sylow 2-subgroup of G. Let H be the subgroup of G generated by all involutions of G. Then $H \subset S \subset A$, and H is a characteristic subgroup of A. Therefore H is normal in G, and obviously G/H is not cyclic (since A/H is not). Hence, in any generating set for G, there must be at least two elements with order larger than 2. Thus $\psi(G) \geq 16$.

Using the presentation (5.1) and the bounds (2.1) and (3.1), we obtain the following.

THEOREM 3. *If $t \geq 3$, $\rho(M_t) = 1 + t^2$.*

Thus we have another infinite family of groups for which the lower bound (3.1) is attained. If t is a power of 3, then each group M_t is a 3-group, of course. These 3-groups are especially interesting, because each is a 3-group of the maximum possible order for the value of the genus; here see [3] and [8, §4].

6. **Abelian and dihedral groups.** The structure of finite abelian groups is well understood, of course. A finite abelian group A of rank c has a unique *canonical form*

$$A = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_c}$$

such that m_i divides m_{i+1} for $i = 1, \ldots, c-1$ and $m_1 > 1$ [7, p. 387]. A *canonical generating set* consists of generators for these cyclic groups. This canonical form is especially useful for studying genus parameters; see [12] and [15].

We can calculate $\psi(A)$ for an abelian group A directly from its canonical form. We need the following preliminary result.

LEMMA 7. *Let A be a finite abelian group, and let p be 2 or 3. Let A_p be the subgroup of A generated by the elements of order p. If S is a generating set for A, then*

$$|S| - t_p(S) \geq \operatorname{rank}(A/A_p).$$

PROOF. The induced generating set for the quotient group A/A_p contains at most $|S| - t_p(S)$ elements.

LEMMA 8. *Let the abelian group A have the canonical form $Z_{m_1} \times \cdots \times Z_{m_c}$, where $m = m_1 = m_2 = \cdots = m_a$ and $m_{a+1} \neq m$. Then*

$$\psi(A) = \begin{cases} 9c - 6a & \text{if } m = 2 \\ 9c - a & \text{if } m = 3 \\ 9c & \text{if } m \geq 4 \end{cases}.$$

PROOF. Let $m = 2$. By considering a canonical generating set for A, we obtain

$$\psi(A) \leq 9(c-a) + 3a = 9c - 6a.$$

Now let S be a ψ-minimal generating set for A. Applying Lemma 7 with $p = 2$ yields $t_h + t_3 \geq \operatorname{rank}(A/A_2) = c - a$. For $p = 3$ we obtain $t_h + t_2 \geq \operatorname{rank}(A/A_3) = c$. We have

$$\begin{aligned} \psi(A) &= 9t_h + 8t_3 + 3t_2 \\ &= 6(t_h + t_3) + 3(t_h + t_2) + 2t_3 \\ &\geq 6(c-a) + 3c \\ &= 9c - 6a. \end{aligned}$$

The arguments for $m = 3$ and $m \geq 4$ are quite similar and no more difficult, and we omit them.

It is now a simple matter to check that for the following two families, the lower bound (3.1) and the upper bound (2.1) agree.

COROLLARY 1. $\rho\big((Z_3)^c\big) = 1 + 3^{c-1}(2c-3)$ *for* $c \geq 1$.

COROLLARY 2. $\rho\big((Z_4)^c\big) = 1 + 4^{c-1}(3c-4)$ *for* $c \geq 1$.

Next we find a genus formula for an abelian group with most of the factors in the canonical form isomorphic to Z_2. Again it is possible to improve the lower bound slightly, as in Lemma 5.

LEMMA 9. *Let the abelian group A have the canonical form* $(Z_2)^a \times Z_{2m_1} \times \cdots \times Z_{2m_b}$, *where* $a > b \geq 0$ *and* $m_1 \neq 1$. *Then*

$$\rho(A) \geq 1 + o(A)[\psi(A) - 9]/12.$$

PROOF. First the inequality holds if $\rho(A) \leq 1$ (so that $b = 0$ and $a \leq 3$). Assume then that $\rho(A) \geq 2$. We proceed as in the proof of Lemma 5, using the notation of Theorem 1. Again write $M = 12[\mu(\Gamma)/2\pi]$, and note that (4.3) holds. We have $\ell \leq k \leq \gamma + 1$. If $\ell \leq \gamma$ or $r_2 \geq 1$, then exactly as in Lemma 5, $M \geq \psi(G) - 9$.

Suppose $\ell = k = \gamma + 1$ and $r_2 = 0$. Let $c = \operatorname{rank} A = a + b$. We also have $B = 0$ and (4.4) holds. Now by Lemma 7, $2\gamma + r_h \geq c = \operatorname{rank}(A/A_3)$. From (4.4) we obtain $M \geq 6(2\gamma + r_h) - 12 \geq 6c - 12$. But $a \geq b+1$ so that $c = a+b \geq 2b+1$. Then $6c \geq 3c+6b+3$, and again $M \geq 3c + 6b - 9 = \psi(A) - 9$. It follows that $\rho(A) \geq 1 + o(A)[\psi(A) - 9]/12$.

THEOREM 4. *Let the abelian group A have the canonical form* $(Z_2)^a \times Z_{2m_1} \times \cdots \times Z_{2m_b}$, *where* $a > b \geq 0$ *and* $m_1 \neq 1$. *Then*

$$\rho(A) = 1 + o(A)(3b + a - 3)/4.$$

PROOF. The formula holds for the groups with $\rho \leq 1$. Assume $\rho(A) \geq 2$ so that either $b > 0$ or $a \geq 4$. Let Δ be the NEC group with signature

$$(0; +; [\,]; \{(\,)^b, (2^{a-b+1})\}).$$

(There are $b + 1$ period cycles; b of these are empty.) It is not hard to construct a homomorphism $\phi\colon \Delta \longrightarrow A$ of Δ onto A such that $L = \operatorname{kernel} \phi$ is a bordered surface group. Then A acts on the surface $Y = U/L$. If g denotes the algebraic genus of Y, then from (2.3) and (2.5) we calculate $g = 1 + o(A)(3b + a - 3)/4$. Hence $\rho(A) \leq 1 + o(A)(3b + a - 3)/4$. But this agrees with the lower bound of Lemma 9, and the genus formula holds.

As a special case, we obtain the formula for the genus of an elementary abelian 2-group [8, Theorem 7].

COROLLARY 3 [8]. $\rho\big((Z_2)^a\big) = 1 + 2^{a-2}(a-3)$.

The formulas of Theorem 4 and the three corollaries can be obtained from the general results in [11], although they do not appear there explicitly. These formulas are nice applications of the ideas in §3, and we have tried to be brief. Also, the approach in [11] utilizes graphs of groups and is quite different.

Finally we consider the product of an elementary abelian 2-group and a dihedral group. Since $Z_2 \times D_n \cong D_{2n}$ for n odd, we only need to consider even n. The dihedral group D_n is generated by two involutions, of course, and if n is even, D_n has D_2 as a quotient group. Thus we have the following.

LEMMA 10. *Let $G = (Z_2)^a \times D_n$, where $a \geq 1$ and n is even. Then* $\operatorname{rank}(G) = a + 2$ *and $\psi(G) = 3a + 6$.*

LEMMA 11. *Let $G = (Z_2)^a \times D_n$, where $a \geq 1$ and n is even. Then*

$$\rho(G) \geq 1 + o(G)[\psi(G) - 9]/12.$$

PROOF. If $a = 1$, then $\rho(G) = 1$ [8, Theorem 4] and the inequality holds. Assume that $a \geq 2$ so that $\rho(G) \geq 2$ and proceed again as in Lemma 5, with $M = 12[\mu(\Gamma)/2\pi]$. If $\ell \leq \gamma$ or $r_2 \geq 1$, then, as before, $M \geq \psi(G) - 9$.

Suppose $\ell = k = \gamma + 1$ and $r_2 = 0$ so that $B = 0$ and (4.4) holds. As in Lemma 2, we have $2\gamma + r_h + r_3 \geq a + 2 = \operatorname{rank}(G)$. Thus $M \geq 6(2\gamma + r_h + r_3) - 12 \geq 6(a + 2) - 12 = 6a \geq 3a + 6 = \psi(G)$. Thus in either case, $M \geq \psi(G) - 9$, and again we find $\rho(G) \geq 1 + o(G)[\psi(G) - 9]/12$.

THEOREM 5. *Let $G = (Z_2)^a \times D_n$, where $a \geq 1$ and n is even. Then*

$$\rho(G) = 1 + 2^{a-1}(a - 1)n.$$

PROOF. First the formula holds for $a = 1$. Let $a \geq 2$ and let Γ be an NEC group with signature $\big(0; +; [\ ]; \{(2^{a+3})\}\big)$. It is a simple matter to construct a homomorphism $\phi: \Gamma \longrightarrow G$ of Γ onto G such that $K = \operatorname{kernel} \phi$ is a bordered surface group. Then G acts on the surface $X = U/K$. If g is the algebraic genus of X, then from (2.3) and (2.5) we calculate $g = 1 + 2^{a-1}(a - 1)n$. Hence $\rho(G) \leq 1 + 2^{a-1}(a - 1)n$. The lower bound is provided by Lemma 11.

7. **Open problems.**

There are many unsolved problems about the real genus parameter. We mention three of the more natural ones related to our work here. Some additional problems are in [8, §8] and [10, §6].

PROBLEM 1. Determine $\rho\big((Z_2)^a \times H_n\big)$, where n is odd.

A *hamiltonian* group is a non-abelian group in which every subgroup is normal. The finite hamiltonian groups have the form

$$Q \times A \times B,$$

where A is an elementary abelian 2-group and B is an abelian group of odd order [5, p. 8]. The groups in the corollary to Theorem 2 are the hamiltonian groups with no odd order part.

PROBLEM 2. Finish the calculation of $\rho(G)$ for each finite hamiltonian group G.

We expect that techniques similar to those applied here may be used to attack the more general hamiltonian groups. This is also suggested by the work of Pisanski and White [12].

The real genus of a group is naturally related to the symmetric genus. The *symmetric genus* $\sigma(G)$ of a finite group G is the minimum genus of any Riemann surface on which G acts (possibly reversing orientation). Some basic relationships between the symmetric genus and the real genus are in [8, §5]. We always have $\sigma(G) \leq \rho(G)$. Further, if $\rho(G) > 0$, then $\sigma(G) < \rho(G)$ [8, p. 716].

PROBLEM 3. Obtain a general lower bound (similar to (3.1)) for the symmetric genus of a finite group.

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