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ON THE IDEAL OF VERONESEAN SURFACES

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ABSTRACT We consider the blowing up of $\mathbb{P}^2$ at s sufficiently general distinct points and its projective embedding by the linear system of the curves of a given degree through the points We study the ideal of the resulting (*Veronesean*) surface and find that it can be described by two matrices of linear forms in the sense that it is generated by the entries of the product matrix and the minors of complementary orders of the two matrices

By cutting the surface twice with general hyperplanes we also obtain some information about the generation (or even the resolution) of certain classes of points in projective space

Introduction. Let $Z = \{P_1, \quad , P_s\}$ be a set of s distinct points in $\mathbb{P}^2 = \mathbb{P}^2_{\mathfrak{k}}$ (where $\mathfrak{k}$ is an algebraically closed field) and let $J = \mathfrak{p}_1 \cap \quad \cap \mathfrak{p}_s \subset S = \mathfrak{k}[W_1, W_2, W_3]$ be the defining ideal of Z Let $\mathbb{P}^2(Z)$ be the surface obtained from $\mathbb{P}^2$ by blowing up the points of Z

The aim of this paper is to study the defining ideal of a projective embedding of $\mathbb{P}^2(Z)$ given by the linear system of curves associated to the vector space J_a, which is the degree a part of J The surface obtained in this way is called a *Veronesean surface*, as it can be obtained as a projection of a Veronese surface from points on the surface itself, equivalently, because the embedding which defines it is obtained by using the subsystem J_a of the complete linear system S_a on $\mathbb{P}^2$

We want to determine the elements of a minimal generating set for the ideal of this type of surface, and to do this by relating these generators to the ideal J of the points in $\mathbb{P}^2$

These kinds of questions have been considered by many authors, we mention the classical work by Castelnuovo ([C]), and the more recent work by Mumford ([M]), Green and Lazarsfeld ([Gr]), ([GL]), which more generally, relates properties of the ideal of a projective scheme with those of the linear system which embeds it

Our work is very much in the line of [G] and [GG], where the authors have given criteria to check when the embedded surface V is arithmetically Cohen-Macaulay (a C M , for short) or when the defining ideal of V, I_V, is generated by quadrics In particular (see §1 for definitions), they have shown that V is a C M when $a \geq \sigma(J)$ and I_V is generated by quadrics if $a \geq \sigma(J) + 1$

In the present paper we study the case of s sufficiently general points in $\mathbb{P}^2$, when we embed $\mathbb{P}^2(Z)$ with the linear system defined by $J_{\sigma(J)}$ From [G] and [GG] it is known that in the case $s = \binom{d+1}{2}$ the surface V (a "Room surface") is defined by quadrics (the 2×2

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minors of a $3 \times (d+1)$ matrix of linear forms), while in the case $s = \binom{d+2}{2}$ the surface V (a "White surface") is defined by cubics (the 3×3 minors of a $3 \times (d+1)$ matrix of linear forms), so we need only consider s such that $\binom{d+1}{2} < s < \binom{d+2}{2}$.

The aim of this paper is to generalize the construction of the two cases above: namely, we will show that if $s = \binom{d+1}{2} + k$, with $0 \leq k \leq d+1$, then I_V is generated in degrees 2 and 3, and it is not determinantal, but almost, in the sense that it can be viewed as given via two matrices of linear forms $\mathcal{X}$ and $\mathcal{B}$, in the following way (see also [P]): the generators of I_V are the entries of $\mathcal{B} \cdot \mathcal{X}$, the 2×2 minors of $\mathcal{X}$ and the 3×3 minors of $\mathcal{B}$, where $\mathcal{X}$ is a $3 \times (d-k+1)$ matrix and $\mathcal{B}$ a $k \times 3$ matrix:

$$\mathcal{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ \vdots & \vdots & \vdots \\ B_{k1} & B_{k2} & B_{k3} \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} X_{11} & \cdots & X_{1,d-k+1} \\ X_{21} & \cdots & X_{2,d-k+1} \\ X_{31} & \cdots & X_{3,d-k+1} \end{bmatrix}.$$

For instance, in Example 3.1 below, we work out the case of a set of 13 points in $\mathbb{P}^2$. We consider the map to $\mathbb{P}^7$ described by the linear system of plane quintics through the 13 points. The image of this map is a surface V of degree 12 in $\mathbb{P}^7$, whose defining ideal I_V can be described as follows:

$I_V =$ (minors of order 2 of $\mathcal{X}$, entries of $\mathcal{B} \cdot X$, det $\mathcal{B}$), where

$$\mathcal{B} = \begin{bmatrix} 0 & 2Y_1 + X_{12} - 5X_{22} & -4Y_2 4X_{31} + 5X_{32} \\ 4X_{12} & -Y_1 & -X_{32} \\ X_{11} & -X_{22} & -X_{31} + X_{32} \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{bmatrix}.$$

Note that in the cases $k = 0,\ d+1$, we get again the Room and the White surfaces, respectively; while in the cases $k = 1$ or $k = 2$, $\mathcal{B}$ has no minors of order 3, and so V is generated by quadrics only.

Another way to look at this presentation of I_V is the following: if $V \subset \mathbb{P}^N$, denote by R the coordinate ring of $\mathbb{P}^N$ and consider the sequence

$$R^{d-k+1} \xrightarrow{\mathcal{X}} R^3 \xrightarrow{\mathcal{B}} R^k;$$

then we can view V as the locus where the above sequence is an exact complex.

The layout of the paper is the following: after a section of preliminaries, in §2 we study the ideal, I_V, of the surface V; in §3 we define an ideal I constructed as above, and finally prove the main result ($I = I_V$) in §4. In §5 we apply this result to the case of points, after cutting V twice with general hyperplanes.

Most of the computations were done with the help of the symbolic computation system "CoCoA" by A. Giovini and G. Niesi, in the MS/DOS version due to E. Armando.

1. **Generalities.** It is known that, if A is the (homogeneous) coordinate ring of an a. C. M. variety of projective dimension $p-1$ (with defining ideal I), then its Hilbert function is non-decreasing, and the p-th difference of its Hilbert function, $\Delta^p H_A(m)$, is eventually 0 (see, for instance, [L1, end of §1]). Define:

$$\alpha(I) = \min\{m \mid I_m \neq (0)\} \text{ and } \sigma(I) = \min\{m \mid \Delta^p H_A(m) = 0\}.$$

Let $Z = \{P_1, \ldots, P_s\}$ be a set of s distinct points of $\mathbb{P}^2$ and let $J \subset S = \mathfrak{k}[W_1, W_2, W_3]$ be their ideal.

Let d be the least integer such that $s < \binom{d+2}{2}$, so that we can write

$$s = \binom{d+1}{2} + k = \binom{d+2}{2} - (d - k + 1),$$

with $0 \leq k < d + 1$. This also means that $\alpha(J) \leq d$.

For general points, we know that $\alpha(J) = d$ and $d \leq \sigma(J) \leq d + 1$, with $d = \sigma(J)$ only when $k = 0$.

Also, the ideal generation conjecture states that, for general points of $\mathbb{P}^2$, J should be minimally generated by $d - k + 1$ forms of degree d and h forms of degree $d + 1$, where h is either 0 or $2k - d$, according to whether $d \geq 2k$ or not.

Because in $\mathbb{P}^2$ the ideal generation conjecture holds (see, for instance, [GGR] or [GM]), and is equivalent to the minimal resolution conjecture, we say that we choose $P_1, \ldots, P_s$ to have "generic resolution".

In other words, if we denote by $F_1, \ldots, F_{d-k+1}$ the generators of J of degree d and by $G_1, \ldots, G_{2k-d}$ those of degree $d + 1$, then the $W_i F_J$'s are all linearly independent over $\mathfrak{k}$, when $d < 2k$; while if $d \geq 2k$, then there are no G_l's, and the $W_i F_j$ need not be linearly independent (certainly not, as soon as $d > 2k$).

Furthermore, because of the Hilbert-Burch Theorem (see, for instance, [CGO]), we can view the F_j's and the G_l's as the $\rho + 1$ minors of order ρ of a $\rho \times (\rho + 1)$ matrix $\mathcal{A}$, where

$$\rho = \begin{cases} k & \text{if } d \leq 2k \\ d - k & \text{if } d \geq 2k \end{cases}.$$

In the case when $d < 2k$, the matrix $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{bmatrix} L_{1,1} & \cdots & L_{1,2k-d} & Q_{1,1} & \cdots & Q_{1,d-k+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ L_{k,1} & \cdots & L_{k,2k-d} & Q_{k,1} & \cdots & Q_{k,d-k+1} \end{bmatrix},$$

where the $L_{u,j}$'s are linear forms and the $Q_{u,l}$'s are forms of degree 2. Then, for all $j = 1, \ldots, d - k + 1$, F_j is the minor obtained by deleting column $2k - d + j$, and for all $l = 1, \ldots, 2k - d$, G_l is the minor obtained by deleting column l.

In the other case ($d \geq 2k$), the matrix $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{bmatrix} Q_{1,1} & Q_{1,2} & \cdots & Q_{1,d-k+1} \\ \vdots & \vdots & & \vdots \\ Q_{k,1} & Q_{k,2} & \cdots & Q_{k,d-k+1} \\ L_{1,1} & L_{1,2} & \cdots & L_{1,d-k+1} \\ \vdots & \vdots & & \vdots \\ L_{d-2k,1} & L_{d-2k,2} & \cdots & L_{d-2k,d-k+1} \end{bmatrix},$$

and, for all $j = 1, \ldots, d - k + 1$, F_j is the minor obtained by deleting column j.

Now let $E_1, \ldots, E_s$ be the divisor classes on $\mathbb{P}^2(Z)$ which contain the exceptional lines corresponding to the blow-ups of the points $P_1, \ldots, P_s$, respectively. If E_0 is the divisor class on $\mathbb{P}^2(Z)$ which contains the proper transform of a line in $\mathbb{P}^2$ which misses all the points of Z, then it is well-known that $\mathrm{Pic}\big(\mathbb{P}^2(Z)\big) \cong \mathbb{Z}^{s+1} \cong \langle E_0, E_1, \ldots, E_s \rangle$.

If C is a curve in $\mathbb{P}^2$ of degree a which has a singularity at P_ι with multiplicity m_ι, then it is also well-known that the proper transform of C on $\mathbb{P}^2(Z)$ is an effective divisor in the class $aE_0 - \sum_{\iota=1}^s m_\iota E_\iota$. In fact, it is possible to show that if $J' = \mathfrak{p}_1^{m_1} \cap \cdots \cap \mathfrak{p}_s^{m_s}$ and if we let $\mathcal{J}$ denote the ideal sheaf in $\mathcal{O}_{\mathbb{P}^2}$ corresponding to J', then:

$$\dim_{\mathfrak{k}} J'_a = h^0\big(\mathbb{P}^2, \mathcal{J}(a)\big) = h^0\Big(\mathbb{P}^2(Z), aE_0 - \sum_{\iota=1}^s m_\iota E_\iota\Big);$$

$$\sum_{\iota=1}^s \frac{m_\iota(m_\iota+1)}{2} - H(S/J, a) = h^1\big(\mathbb{P}^2, \mathcal{J}(a)\big) = h^1\Big(\mathbb{P}^2(Z), aE_0 - \sum_{\iota=1}^s m_\iota E_\iota\Big).$$

It is well-known (and easy to see) that:

(a) $aE_0 - \sum_{\iota=1}^s E_\iota$ is base-point free for $a \geq \sigma$;

(b) $aE_0 - \sum_{\iota=1}^s E_\iota$ is very ample for $a \geq \sigma + 1$.

Moreover:

THEOREM A ([DG, THEOREM (3.1)]). *The following are equivalent:*

i) $\sigma E_0 - \sum_{\iota=1}^s E_\iota$ *is very ample on* $\mathbb{P}^2(Z)$*;*

ii) no σ *elements of* Z *lie on a line of* $\mathbb{P}^2$.

If ii) holds, then, for all $a \geq \sigma$, each divisor class $aE_0 - \sum_{\iota=1}^s E_\iota$ defines a morphism

$$\Phi_{a,Z} \colon \mathbb{P}^2(Z) \longrightarrow \mathbb{P}^N,$$

where $N = h^0(aE_0 - \sum_{\iota=1}^s E_\iota) - 1$. These morphisms embed $\mathbb{P}^2(Z)$ into $\mathbb{P}^N$ and we shall denote the image of $\Phi_{a,Z}$ by $V_{a,Z}$.

Since $a \geq \sigma$, it is easy to see that:

$$N = \left[\binom{a+2}{2} - s\right] - 1, \quad \deg V_{a,Z} = a^2 - s;$$

and that the general hyperplane section of $V_{a,Z}$ is a curve of genus

$$g = \frac{(a-1)(a-2)}{2}.$$

Let us go back to ideal $J \subset S = \mathfrak{k}[W_1, W_2, W_3]$ of the points $P_1, \ldots, P_s$, and notice that, from our hypotheses on the points it follows, in particular, that they have maximal Hilbert function (or, generic postulation), and so $\sigma(J) = d + 1$.

If we add the hypothesis that no $d+1$ of the points lie on a line, then, by Theorem A, the linear system

$$J_{d+1} = \langle W_\iota F_j, G_l \mid i = 1,2,3; j = 1, \ldots, d-k+1; l = 1, \ldots, 2k-d \rangle$$

(where $\langle * \rangle$ denotes the span of $*$) induces an embedding of $\mathbb{P}^2(Z)$ in $\mathbb{P}^N$, where $N = \dim_{\mathfrak{k}} J_{d+1} - 1 = 2d - k + 2$

For simplicity, we call $V = V_{d+1\,Z}$ the image of this embedding, I_V the defining ideal of V, and A_V its (homogeneous) coordinate ring

Then, clearly, V is an irreducible surface, of degree $t = (d+1)^2 - s = \binom{d+2}{2} - k$

Furthermore, the following is proved in [G], though not stated precisely in this form

THEOREM B (SEE [G, PROPOSITION 2 1)] *Let Z satisfy ii) of Theorem A Then $V_{a\,Z}$ is* a C M *for every $a \geq \sigma$*

Hence, by Theorem B, our surface V is a C M

More detailed information about the defining ideal of the surfaces $V_{a\,Z}$ can be found in [GG] For instance

THEOREM C ([GG, THEOREM 2 1]) *Let $a \geq \sigma + 1$ and let I_V be the ideal of $V_{a\,Z}$ in $\mathbb{P}^N$ Then I_V is generated by quadrics*

Our aim is to describe the ideal I_V in an almost determinantal way, and in relation with the ideal J

As the case $k = 0$ has been dealt with in [GG], we are actually interested in the range $1 \leq k \leq d$

2 **The ideal of the surface** V**.** A tool we shall need in what follows is the knowledge of the Hilbert function of V $H_V(\lambda) = \dim_{\mathfrak{k}}(A_V)_\lambda, \forall \lambda \in \mathbb{N}$

Let $\mathcal{O}_V$ be the structure sheaf of V Since V is a C M , we have

$$(A_V)_\lambda \cong H^0\big(\mathcal{O}_V(\lambda)\big)$$

REMARK 2 1 $H^1\big(\mathcal{O}_V(\lambda)\big) = 0, \forall \lambda \in \mathbb{N}$

PROOF Since V is a C M we have that $h^2\big(I_V(\lambda)\big) = 0$ for all λ Since $H^2\big(I_V(\lambda)\big) \cong H^1\big(\mathcal{O}_V(\lambda)\big)$ in any case, we are done ∎

PROPOSITION 2 2 *For every $\lambda \in \mathbb{N}$, the Hilbert function of A_V is given by*

$$H_V(\lambda) = \frac{\lambda^2}{2}\left[\binom{d+2}{2} - k\right] - \frac{\lambda}{4}(d^2 - 5d + 2k - 6) + 1$$

PROOF It follows easily from Serre's duality that $H^2\big(\mathcal{O}_V(\lambda)\big) = 0$, while $H^1\big(\mathcal{O}_V(\lambda)\big) = 0$ by Remark 2 1, therefore we can compute the dimension $h^0\big(\mathcal{O}_V(\lambda)\big)$ by using the Riemann-Roch Theorem on V

$$\begin{aligned} h^0\big(\mathcal{O}_V(\lambda)\big) &= \frac{1}{2}\big((\lambda\mathbf{H})^2 - \lambda\mathbf{H}\ K_V\big) + 1 \\ &= \frac{\lambda^2}{2}\left[\binom{d+2}{2} - k\right] - \frac{\lambda}{2}\left[3d + 3 - \binom{d+1}{2} - k\right] + 1 \\ &= \frac{\lambda^2}{2}\left[\binom{d+2}{2} - k\right] - \frac{\lambda}{4}(d^2 - 5d + 2k - 6) + 1 \end{aligned}$$ ∎

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COROLLARY 2.3. *The ideal I_V can always be generated by forms of degree less than or equal to 3.*

PROOF. Observe that, from Proposition 2.2 we get that $\alpha(I_V) = 2$, as $\dim_{\mathfrak{k}}(I)_\lambda = \binom{2d-k+2+\lambda}{\lambda} - H_V(\lambda) = 0$, for $\lambda < 2$. Since the Hilbert function of A_V is a polynomial of degree 2 in λ for every λ (*i.e.* it equals the Hilbert polynomial from degree 0 on), a simple computation shows that the third difference of the Hilbert function of A_V (equivalently, the Hilbert function of an Artinian reduction of A_V) becomes 0 from degree 3 on, *i.e.* that $\sigma(I_V) = 3$. Thus, as I_V is perfect, we get (see *e.g.* [L1, Theorem 2.2]) that I_V is generated (at most) in degrees 2 and 3. ∎

REMARK. In the proof above we noticed that the Hilbert function of A_V coincides with the Hilbert polynomial in each degree, *i.e.* I_V is what in [A] is called a *Hilbertian* ideal.

Now, in order to describe I_V we first give a slightly different description of V, and hence of I_V.

We set $N' = 2d - 2k + \rho + 2$ (where ρ is as defined in §1), and define $\phi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^{N'}$ by putting, for every $(a,b,c) \in \mathbb{P}^2 \setminus Z$,

$$\phi(a,b,c) = \big(aF_1(a,b,c), \ldots, cF_{d-k+1}(a,b,c), G_1(a,b,c), \ldots, G_{2k-d}(a,b,c)\big).$$

Let $R = \mathfrak{k}[X_{ij}, Y_l]$ $(i = 1,2,3; j = 1,\ldots,d-k+1; l = 1,\ldots,2k-d)$ and let S' be the (graded) subring of $S = \mathfrak{k}[W_1, W_2, W_3]$ defined by $S' = \oplus_k S'_k$, where $S'_k = S_{k(d+1)}$.

Now define $\theta: R \longrightarrow S' \subset S$ by$X_{ij} \longmapsto W_iF_j$ and $Y_l \longmapsto G_l$. Then θ is a graded ring homomorphism, whose kernel is a homogeneous ideal which will be related to V.

In fact, when $d < 2k$, we have that $N' = 2d-k+2 = N$ and that the set $\{W_iF_j, G_l \mid i = 1,2,3; j = 1,\ldots,d-k+1; l = 1,\ldots,2k-d\}$ is a basis of J_{d+1}. Therefore $V = \overline{\operatorname{Im}\phi} \subset \mathbb{P}^N$ (where "$\overline{}$" denotes the closure in the Zariski topology), and hence $I_V = \operatorname{Ker}\theta$.

When $d \geq 2k$, we have that $N' = 3d-3k+2$ and, except for $d = 2k$, that the W_iF_j's are not linearly independent (there are no G_l's in this case). However, consider the (square) matrix we obtain by repeating the l-th row of $\mathcal{A}$:

$$\mathcal{A}_l = \begin{bmatrix} & \mathcal{A} & \\ L_{l,1} & \cdots & L_{l,d-k+1} \end{bmatrix} = \begin{bmatrix} Q_{1,1} & Q_{1,2} & \cdots & Q_{1,d-k+1} \\ \vdots & \vdots & & \vdots \\ Q_{k,1} & Q_{k,2} & \cdots & Q_{k,d-k+1} \\ L_{1,1} & L_{1,2} & \cdots & L_{1,d-k+1} \\ \vdots & \vdots & & \vdots \\ L_{d-2k,1} & L_{d-2k,2} & \cdots & L_{d-2k,d-k+1} \\ L_{l,1} & L_{l,2} & \cdots & L_{l,d-k+1} \end{bmatrix}$$

and write $L_{l,j} = \sum_{i=1}^{3} \delta_i^{l,j} W_i$, for every $l = 1,\ldots,d-2k$. Then, by using cofactor expansion along the last row of $\mathcal{A}_l$, we obtain

$$\begin{aligned} 0 = \det \mathcal{A}_l &= \sum_{j=1}^{d-k+1} L_{l,j}F_j = \sum_{j=1}^{d-k+1} \Big(\sum_{i=1}^{3} \delta_i^{l,j} W_i\Big) F_j \\ &= \sum_{j=1}^{d-k+1} \Big(\sum_{i=1}^{3} \delta_i^{l,j} \theta(X_{ij})\Big) = \theta\Big(\sum_{j=1}^{d-k+1} \sum_{i=1}^{3} \delta_i^{l,j} X_{ij}\Big). \end{aligned}$$

In other words, $\ker\theta$ contains the $d-2k$ linear forms,

$$H_l = \sum_{j=1}^{d-k+1} \sum_{i=1}^{3} \delta_i^{l,j} X_{ij} \quad (l = 1, \ldots, d-2k).$$

LEMMA 2.4. *The linear forms described above,*

$$H_l = \sum_{j=1}^{d-k+1} \sum_{i=1}^{3} \delta_i^{l,j} X_{ij} \quad (l = 1, \ldots, d-2k),$$

are linearly independent.

PROOF. We view the H_l's as a system of linear equations, and, after giving the X_{ij}'s the lexicographic order,

$$X_{11}, \ldots, X_{1,d-k+1}, X_{21}, \ldots, X_{2,d-k+1}, \ldots, X_{31}, \ldots, X_{3,d-k+1},$$

write the matrix of its coefficients:

$$\Delta = \begin{bmatrix} \delta_1^{1,1} & \cdots & \delta_1^{1,d-k+1} & \delta_2^{1,1} & \cdots & \delta_2^{1,d-k+1} & \delta_3^{1,1} & \cdots & \delta_3^{1,d-k+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \delta_1^{d-2k,1} & \cdots & \delta_1^{d-2k,d-k+1} & \delta_2^{d-2k,1} & \cdots & \delta_2^{d-2k,d-k+1} & \delta_3^{d-2k,1} & \cdots & \delta_3^{d-2k,d-k+1} \end{bmatrix}$$

Then it sufficies to prove that $\operatorname{rk}\Delta = d-2k$.

To this end, we may assume none of the P_i's is the point $(0,0,1)$, and that F_1 does not vanish at $(0,0,1)$. Recall that

$$F_1 = \det \begin{bmatrix} Q_{1,2} & \cdots & Q_{1,d-k+1} \\ \vdots & & \vdots \\ Q_{k,2} & \cdots & Q_{k,d-k+1} \\ L_{1,2} & \cdots & L_{1,d-k+1} \\ \vdots & & \vdots \\ L_{d-2k,2} & \cdots & L_{d-2k,d-k+1} \end{bmatrix}.$$

and write $Q_{u,j} = \sum_{i,h=1}^{3} \beta_h^{u,i,j} W_h W_i$, for every $u = 1, \ldots, k$ and every $j = 2, \ldots, d-k+1$. Then, if we put:

$$M = \begin{bmatrix} \beta_3^{1,3,2} & \cdots & \beta_3^{1,3,d-k+1} \\ \vdots & & \vdots \\ \beta_3^{k,3,2} & \cdots & \beta_3^{k,3,d-k+1} \\ \delta_3^{1,2} & \cdots & \delta_3^{1,d-k+1} \\ \vdots & & \vdots \\ \delta_3^{d-2k,2} & \cdots & \delta_3^{d-2k,d-k+1} \end{bmatrix},$$

we have $\det M = F_1(0,0,1) \neq 0$.

Now, if all the minors of order $d-2k$ of M involving the δ's were 0, then the last $d-2k$ rows of M would be linearly dependent, contradicting the fact that $\det M \neq 0$.

Therefore there must be a minor of order $d-2k$, involving only the δ's, which is different from 0. But such a minor sits inside the matrix Δ defined above, and so $\operatorname{rk}\Delta = d-2k$, as we wanted. ∎

As $N' - (d-2k) = 2d-k+2 = N$, Lemma 2.4 enables us to say that $\overline{\operatorname{Im}\phi}$ is actually contained in $\mathbb{P}^N$, hence that we can identify I_V with $\frac{\operatorname{Ker}\theta}{(H_1, \ ,H_{d-2k})}$.

3. **The ideal** I. In this section we construct an ideal I, which we show is almost determinantally presented and which will turn out to be equal to the ideal of V.

First of all, notice that, for every $h,k = 1,2,3$ with $h \neq k$ and every $l,m = 1,\ldots,d-k+1$, with $l \neq m$, we have that

$$\theta(X_{hl}X_{km} - X_{kl}X_{hm}) = W_hF_lW_kF_m - W_kF_lW_hF_m = 0,$$

i.e. that the differences $X_{hl}X_{km} - X_{kl}X_{hm} \in \operatorname{Ker}\theta$. We view these differences, which are $\binom{3}{2}\binom{d-k+1}{2}$ forms of degree 2, as the 2×2 minors of the matrix

$$\mathcal{X} = \begin{bmatrix} X_{11} & \cdots & X_{1,d-k+1} \\ X_{21} & \cdots & X_{2,d-k+1} \\ X_{31} & \cdots & X_{3,d-k+1} \end{bmatrix}.$$

THE CASE $d < 2k$. In this case, for every $u = 1,\ldots,k$, consider the matrix obtained by repeating the u-th row of $\mathcal{A}$:

$$\mathcal{A}_u = \begin{bmatrix} & \mathcal{A} & \\ L_{u,1}\cdots L_{u,2k-d} & & Q_{u,1}\cdots Q_{u,d-k+1} \end{bmatrix}$$

Expanding $\det \mathcal{A}_u$ by cofactors of the last row we obtain:

$$0 = \det(\mathcal{A}_u) = \sum_{l=1}^{2k-d} L_{u,l}G_l + \sum_{j=1}^{d-k+1} Q_{u,j}F_j.$$

Keeping the notation already introduced in §2, we put $L_{u,l} = \sum_{\iota=1}^{3}\delta_\iota^{u,l}W_\iota$ and $Q_{u,j} = \sum_{\iota,h=1}^{3}\beta_h^{u,\iota,j}W_hW_\iota$, with the $\delta_\iota^{u,l}$'s and the $\beta_h^{u,\iota,j}$'s in the ground field $\mathfrak{k}$. Thus,

$$0 = \det(\mathcal{A}_u) = \sum_{l=1}^{2k-d}\Big(\sum_{\iota=1}^{3}\delta_\iota^{u,l}W_\iota\Big)G_l + \sum_{j=1}^{d-k+1}\Big(\sum_{\iota,h=1}^{3}\beta_h^{u,\iota,j}W_hW_\iota\Big)F_j;$$

and so, after multiplying by F_v (for every $v = 1,\ldots,d-k+1$),

$$\begin{aligned} 0 = F_v\det(\mathcal{A}_u) &= \sum_{\iota=1}^{3}\Big(\sum_{l=1}^{2k-d}\delta_\iota^{u,l}G_l\Big)W_\iota F_v + \sum_{\iota=1}^{3}\Big(\sum_{j=1}^{d-k+1}\sum_{h=1}^{3}\beta_\iota^{u,h,j}W_hF_j\Big)W_\iota F_v \\ &= \sum_{\iota=1}^{3}\Big(\sum_{l=1}^{2k-d}\delta_\iota^{u,l}\theta(Y_l)\Big)\theta(X_{\iota v}) + \sum_{\iota=1}^{3}\Big(\sum_{j=1}^{d-k+1}\sum_{h=1}^{3}\beta_\iota^{u,h,j}\theta(X_{hj})\Big)\theta(X_{\iota v}) \\ &= \theta\Big(\sum_{\iota=1}^{3}X_{\iota v}B_{u,\iota}\Big); \end{aligned}$$

where

$$B_{u,\iota} = \sum_{l=1}^{2k-d} \delta_\iota^{u,l} Y_l + \sum_{j=1}^{d-k+1} \sum_{h=1}^{3} \beta_\iota^{u,h,j} X_{hj}.$$

This tells us that the $M_{u,v} = \sum_{\iota=1}^{3} X_{\iota v} B_{u,\iota}$ ($u = 1, \ldots, k$ and $v = 1, \ldots, d-k+1$), describe $k(d-k+1)$ forms of degree 2 in $\operatorname{Ker}\theta$.

We view the $M_{u,v}$'s as the entries of the product matrix $\mathcal{B} \cdot \mathcal{X}$, where $\mathcal{X}$ is the $3 \times (d-k+1)$ matrix defined above and $\mathcal{B}$ is the $k \times 3$ matrix of linear forms given by $\mathcal{B} = (B_{u,\iota})_{u,\iota}$.

Notice that all these quadratic forms in $\operatorname{Ker}\theta$ (if different) number $3\binom{d-k+1}{2} + k(d-k+1) = \frac{3d^2+3d-4kd-k+k^2}{2}$, which is exactly the dimension of $(I_V)_2$, by Proposition 2.2.

Now we look for forms of degree 3 in $\operatorname{Ker}\theta$. To this end, let us call $\mathcal{C}$ the image under θ of $\mathcal{B} = (B_{u,\iota})_{u,\iota}$; *i.e.* $\mathcal{C} = (C_{u,\iota})_{u,\iota}$, where $C_{u,\iota} = \theta(B_{u,\iota})$. Also, for every $u = 1, \ldots, k$ and every $i = 1, 2, 3$, put

$$\mathcal{D}_{u,\iota} = \begin{bmatrix} & & \mathcal{A} & \\ \delta_\iota^{u,1} \cdots \delta_\iota^{u,2k-d} & \sum_{h=1}^{3} \beta_\iota^{u,h,1} W_h \cdots \sum_{h=1}^{3} \beta_\iota^{u,h,d-k+1} W_h \end{bmatrix}.$$

Then:

$$\det(\mathcal{D}_{u,\iota}) = \sum_{l=1}^{2k-d} \delta_\iota^{u,l} G_l + \sum_{j=1}^{d-k+1} \Big(\sum_{h=1}^{3} \beta_\iota^{u,h,j} W_h \Big) F_j = \theta(B_{u,\iota}) = C_{u,\iota}.$$

Call M_p ($p = 1, \ldots, \binom{k}{3}$) the minors of order 3 of $\mathcal{B}$. Then the $\theta(M_p)$'s are the minors of order 3 of $\mathcal{C}$. Now observe that, $\forall (a_1 : a_2 : a_3) \in \mathbb{P}^2$, we have

$$0 = \det\big(\mathcal{A}_u(a_1, a_2, a_3)\big) = \sum_{\iota=1}^{3} a_\iota \det \mathcal{D}_{u,\iota}(a_1, a_2, a_3) = \sum_{\iota=1}^{3} a_\iota C_{u,\iota}(a_1, a_2, a_3)$$

thus the rank of $\mathcal{C}(a_1, a_2, a_3)$ is less than 3, for every $(a_1 : a_2 : a_3) \in \mathbb{P}^2$, whence so is the rank of $\mathcal{C}$. Therefore the M_p's all belong to $\operatorname{Ker}\theta$ (and they all have degree 3).

Define I as the ideal generated by the 2×2 minors of $\mathcal{X} = (X_{\iota v})_{\iota v}$, the entries of the product matrix $\mathcal{B} \cdot \mathcal{X}$, and the 3×3 minors of $\mathcal{B}$. Thus we have $I \subset \operatorname{Ker}\theta$.

THE CASE $d \geq 2k$. As in the previous case, for each $u = 1, \ldots, k$, consider the matrix obtained by repeating the u-th row of $\mathcal{A}$,

$$\mathcal{A}_u = \begin{bmatrix} \mathcal{A} \\ Q_{u,1} \cdots Q_{u,d-k+1} \end{bmatrix},$$

so that $\det \mathcal{A}_u = 0$, and put $Q_{u,j} = \sum_{\iota,h=1}^{3} \beta_h^{u,\iota,j} W_h W_\iota$. Then, for every $u = 1, \ldots, k$ and every $v = 1, \ldots, d-k+1$, we have, after multiplying $\det \mathcal{A}_u$ by F_v:

$$0 = F_v \det \mathcal{A}_u = F_v \sum_{j=1}^{d-k+1} Q_{u,j} F_j = F_v \sum_{j=1}^{d-k+1} \Big(\sum_{\iota=1}^{3} \sum_{h=1}^{3} \beta_h^{u,\iota,j} W_h W_\iota \Big) F_j$$

$$= \sum_{\iota=1}^{3} \sum_{j=1}^{d-k+1} \sum_{h=1}^{3} \beta_\iota^{u,h,j} \theta(X_{hj}) \theta(X_{\iota v}) = \theta\Big(\sum_{\iota=1}^{3} X_{\iota v} B_{u,\iota} \Big),$$

where

$$B_{u,i} = \sum_{j=1}^{d-k+1} \sum_{h=1}^{3} \beta_i^{u,h,j} X_{hj}.$$

Therefore, by setting $M_{u,v} = \sum_{i=1}^{3} X_{iv} B_{u,i}$, we obtain $k(d-k+1)$ quadratic forms $M_{u,v}$ which belong to $\operatorname{Ker}\theta$.

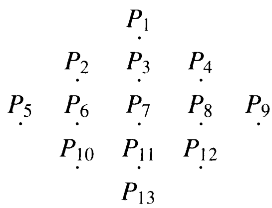
Let $\mathcal{B}$ be the $k \times 3$ matrix of linear forms: $\mathcal{B} = (B_{u,i})_{u,i}$. Consider the quadratic forms given by the 2×2 minors of $\mathcal{X}$ and by the entries of the product matrix $\mathcal{B} \cdot \mathcal{X}$, and the cubic forms given by the 3×3 minors of $\mathcal{B}$. Let I be the ideal generated by the residue classes of all those forms in $\frac{R}{(H_1,\ldots,H_{d-2k})}$.

As before, we need to prove that $I = I_V$, after identifying I_V with $\frac{\operatorname{Ker}\theta}{(H_1,\ldots,H_{d-2k})}$.

EXAMPLE 3.1. Let $s = 13$ (here $d = 4$ and $k = 3$) and let $P_1, \ldots, P_{13}$ be 13 points in $\mathbb{P}^2$ with generic resolution,

$$0 \longrightarrow S(-6)^3 \longrightarrow S(-4)^2 \oplus S(-5)^2 \longrightarrow J \longrightarrow 0,$$

like, for instance, the 13 points in the configuration below:



We may assume the 13 point have the following coordinates:

$$P_1 = (1,0,2),\ P_2 = (1,-1,1),\ P_3 = (1,0,1),\ P_4 = (1,1,1),$$
$$P_5 = (1,-2,0),\ P_6 = (1,-1,0),\ P_7 = (1,0,0),\ P_8 = (1,1,0),\ P_9 = (1,2,0),$$
$$P_{10} = (1,-1,-1),\ P_{11} = (1,0,-1),\ P_{12} = (1,1,-1),\ P_{13} = (1,0,-2).$$

Then the ideal of these points is generated by the maximal minors of the matrix:

$$\mathcal{A} = \begin{pmatrix} 2W_2 & -4W_3 & W_2^2 - 4W_3^2 & -5W_2^2 + 5W_3^2 \\ -W_2 & 0 & 0 & 4W_1^2 - W_3^2 \\ 0 & 0 & W_1^2 - W_3^2 & -W_2^2 + W_3^2 \end{pmatrix}.$$

In this case $N = 2d - k + 2 = 7$ and $R = \mathfrak{k}[X_{ij}, Y_l]$, with $i = 1,2,3$, $j = 1,2$ and $l = 1,2$. Thus:

$$\mathcal{X} = \begin{pmatrix} X_{11}, & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{pmatrix}.$$

Moreover, from $\mathcal{A}$ we compute the δ's and the β's and obtain:

$$\mathcal{B} = \begin{pmatrix} 0 & 2Y_1 + X_{12} - 5X_{22} & -4Y_2 - 4X_{31} + 5X_{32} \\ 4X_{12} & -Y_1 & -X_{32} \\ X_{11} & -X_{22} & -X_{31} + X_{32} \end{pmatrix}.$$

Therefore

$$\begin{aligned} I &= (\text{minors of order 2 of } \mathcal{X}, \text{entries of } \mathcal{B}\cdot\mathcal{X}, \det\mathcal{B}) \\ &= (X_{12}X_{21}-X_{11}X_{22},\ X_{12}X_{31}-X_{11}X_{32},\ X_{22}X_{31}-X_{21}X_{32}, \\ &\quad X_{12}X_{21}-5X_{21}X_{22}-4X_{31}^2+5X_{31}X_{32}+2X_{21}Y_1-4X_{31}Y_2, \\ &\quad X_{12}X_{22}-5X_{22}^2-4X_{31}X_{32}+5X_{32}^2+2X_{22}Y_1-4X_{32}Y_2, \\ &\quad 4X_{11}X_{12}-X_{31}X_{32}-X_{21}Y_1, 4X_{12}^2-X_{32}^2-X_{22}Y_1, \\ &\quad X_{11}^2-X_{21}X_{22}-X_{31}^2+X_{31}X_{32},\ X_{11}X_{12}-X_{22}^2-X_{31}X_{32}+X_{32}^2). \end{aligned}$$

The computation for this example was done partially by hand, partially with the help of "CoCoA" by Giovini-Niesi (in the MS/DOS version by Armando).

4. ***I* is the ideal of *V*.** We shall prove the equality we are after by showing that the set of zeros of I, $W = Z(I)$, coincides with V and that I is prime.

The main tool used to prove that I is prime is a theorem by Huneke (see [H, Theorem 60]), which we rephrase as follows:

THEOREM 4.1 (HUNEKE). *Let* $\mathbf{X} = (X_{ij})$ *be an* $r \times s$ *matrix of indeterminates and* $\mathbf{Y} = (Y_{jk})$ *an* $s \times t$ *matrix of indeterminates. Let* $\mathfrak{k}$ *be a field and* J *be the ideal in* $\mathfrak{k}[x_{ij}, y_{jk}]$ *generated by the entries of the product matrix* $\mathbf{X}\cdot\mathbf{Y}$, *all* $(a+1)\times(a+1)$ *minors of* $\mathbf{X}$ *and all* $(b+1)\times(b+1)$ *minors of* $\mathbf{Y}$. *If* $a+b \leq s$, *then* J *is prime and perfect.*

THEOREM 4.2. *For a generic choice of the points* $P_1, \ldots, P_s$ *in* $\mathbb{P}^2$, *the ideal* I *is prime (and perfect).*

PROOF. Let us consider the case $d < 2k$ first. In this case the ideal I is not given by matrices made of indeterminates, but we can consider the ring $R' = \mathfrak{k}[X_{hj}, Y_l, Z_{ui}]$ (where $h, i = 1,2,3$; $l = 1,\ldots,2k-d$; $j = 1,\ldots,d-k+1$ and $u = 1,\ldots,k$) and the ideal $I' \subseteq R'$, defined as in Huneke's theorem, with the matrix $B' = (Z_{ui})$ as $\mathbf{X}$, the matrix $\mathcal{X}$ as $\mathbf{Y}$, and with $a = 2$, $b = 1$, and $s = 3$. Then I' is prime and perfect, and we can easily see that we can view I as the quotient ideal of I' in the ring $\mathfrak{k}[X_{ij}, Y_l, Z_{ui}]/(H_{ui})$, where:

$$H_{ui} = Z_{ui} - \sum_{l=1}^{2k-d} \delta_i^{u,l} Y_l - \sum_{j=1}^{d-k+1}\sum_{h=1}^{3} \beta_h^{u,i,j} X_{hj}.$$

By Huneke's theorem the scheme W' defined by I' is a. C. M.

Moreover, W' is an integral (*i.e.* a reduced, irreducible) scheme in $\mathbb{P}^M$, where $M = 2d+2k+2$, and the ideal I defines the section W obtained by cutting W' with the $3k$ hyperplanes H_{ui}'s.

Observe also that, since $W \supset V$, we have $2 \leq \dim W \leq \dim W'$.

Now, it is well known that if we cut an integral projective scheme of dimension greater than 1 with a generic hyperplane, we obtain an integral scheme. So we have to check that the H_{ui}'s are "generic enough". This will also imply that every H_{ui} is not a zero divisor modulo I', hence that the ideal (I', H_{ui}), which is the homogeneous ideal associated to

$W' \cap Z(H_{ui})$, is again a perfect ideal, *i.e.* that $W' \cap Z(H_{ui})$ is a. C. M. By iteration, we shall get that $(I', H_{11}, \ldots, H_{k3})$ is perfect and that W is a. C. M.

In other words, we have to show that a generic choice of $\mathcal{A}$ (hence a generic choice of $P_1, \ldots, P_s$) corresponds to a generic choice of the $3k$ hyperplanes H_{ui}, hence to a generic $3k$-codimensional linear space $\Pi_{\mathbf{A}}$ in $\mathbb{P}^M$ and to a generic section $W = \Pi_{\mathbf{A}} \cap W'$. Thus, by Bertini's Theorem (see, for instance, [J, Theorem 6.3]), W will be integral, *i.e.* I will be prime.

More precisely: consider any $\rho \times (\rho + 1)$ matrix with entries given by linear and quadratic forms of $S = \mathfrak{k}[W_1, W_2, W_3]$ (*i.e.* of the same type as the matrix $\mathcal{A}$ described in §1 and §3). The coefficients of its entries will be of type $(\delta_i^{u,l}, \beta_i^{u,j,h})$, with $h, i = 1, 2, 3$; $u = 1, \ldots, k$; $l = 1, \ldots, 2k - d$; $j = 1, \ldots, d - k + 1$.

For any choice of u (hence of a row in the given matrix) and any choice of $i = 1, 2, 3$, the coefficients $\delta_i^{u,l}$, $\beta_h^{u,i,j}$ will determine a hyperplane H_{ui} in $\mathbb{P}^M$. Hence we have a map μ which associates to the given matrix a $3k$-tuple of hyperplanes in $\mathbb{P}^M$. For a generic matrix the $3k$ hyperplanes will be independent, so we can see the image of the map μ as a $3k$-codimensional subspace of $\mathbb{P}^M$.

We have to check that the image of this map covers an open set of $\mathrm{Gr}(M - 3k, M)$, the grassmannian which parameterizes the linear spaces of codimension $3k$ in $\mathbb{P}^M$.

In the space $[(\mathbb{P}^M)^v]^{3k}$, which parameterizes the $3k$-tuples of hyperplanes in $\mathbb{P}^M$, consider the open set U which consists of $3k$-tuples of independent hyperplanes. There is an obvious map, $\lambda\colon U \longrightarrow \mathfrak{G} = \mathrm{Gr}(M - 3k, M)$, which associates to each $a \in U$ the $3k$-codimensional space which is the intersection of the $3k$ hyperplanes corresponding to a. Such a map λ is obviously surjective.

For any $a \in U$, we can consider the matrix, $\mathcal{M}_a$, of the coefficients of the $3k$ equations defining the hyperplanes (up to multiplication of every row by a constant). In other words, if $a = (H_1, \ldots, H_{3k})$, with

$$H_m = \Big\{ \sum_{i=1}^{3} \sum_{u=1}^{k} \alpha_{ui}^m Z_{ui} - \sum_{l=1}^{2k-d} \delta_l^m Y_l - \sum_{j=1}^{d-k+1} \sum_{h=1}^{3} \beta_{h,j}^m X_{hj} = 0 \Big\},$$

for every $m = 1, \ldots, 3k$, then

$$\mathcal{M}_a = \big[(\alpha_{ui}^m) \,\vdots\, (\delta_l^m) \,\vdots\, (\beta_{h,j}^m) \big] =$$
$$\left[\begin{array}{ccccc|cccc|ccccc} \alpha_{1,1}^1 & \alpha_{1,2}^1 & \alpha_{1,3}^1 & \cdots & \alpha_{k,3}^1 & \delta_2^1 & \delta_2^1 & \cdots & \delta_{2k-d}^1 & \beta_{1,1}^1 & \beta_{2,1}^1 & \beta_{3,1}^1 & \cdots & \beta_{3,d-k+1}^1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_{1,1}^{3k} & \alpha_{1,2}^{3k} & \alpha_{1,3}^{3k} & \cdots & \alpha_{k,3}^{3k} & \delta_1^{3k} & \delta_2^{3k} & \cdots & \delta_{2k-d}^{3k} & \beta_{1,1}^{3k} & \beta_{2,1}^{3k} & \beta_{3,1}^{3k} & \cdots & \beta_{3,d-k+1}^{3k} \end{array}\right]$$

Let A_a be the $3k \times 3k$ matrix (a_{ui}^m) and B_a the $3k \times (2d - k + 3)$ matrix $(\delta_i^m, \beta_{h,j}^m)$, so that $\mathcal{M}_a = (A_a, B_a)$.

Now consider the subet U' of U given by those a's such that A_a has maximal rank. Then $\lambda|_{U'}$ is no longer surjective, but $\lambda(U')$ is a (Zariski) open subset of $\mathfrak{G}$, since $\mathfrak{G} \setminus \lambda(U')$ is given by those $3k$-codimensional suspaces of $\mathbb{P}^M$ contained in some hyperplane.

Now, let $a \in U'$, and consider the matrix $\mathcal{M}_a$. Because $\operatorname{rk} A_a = 3k$, we can find an invertible $3k \times 3k$ matrix E_a, such that $E_a \cdot \mathcal{M}_a = (I, E_a \cdot B_a)$, where I is the $3k \times 3k$ identity matrix.

Note that $\forall a \in U'$, $\exists v \in U'$ such that $\mathcal{M}_v = E_a \cdot \mathcal{M}_a = (I, E_a \cdot B_a)$ and $\lambda(v) = \lambda(a)$; hence if U'' is the subset of U' of the v's such that $\mathcal{M}_v = (I, B_v)$, then $\lambda(U'') = \lambda(U')$.

Now consider $\mathfrak{M} = M(S; k, k+1)$, the space parameterizing the matrices of size $k \times (k+1)$ with entries in S_1, for the first $2k - d$ columns and in S_2 for the remaining ones. Also recall the map $\mu \colon \mathfrak{M} \longrightarrow U''$, which we have considered above, defined by $\mu(\mathcal{M}) = (H_{1,1}, \ldots, H_{k,3})$. Then we get that

$$\mathcal{M}_{\mu(\mathcal{A})} = \left[\begin{array}{ccc|cccc|cccc} 1 & \cdots & 0 & \delta_1^{1,1} & \delta_1^{1,2} & \cdots & \delta_1^{1,2k-d} & \beta_1^{1,1,1} & \beta_2^{1,1,1} & \cdots & \beta_3^{1,1,d-k+1} \\ 0 & \cdots & 0 & & & & & & & & \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & \delta_3^{k,1} & \delta_3^{k,2} & \cdots & \delta_3^{k,2k-d} & \beta_1^{k,3,1} & \beta_2^{k,3,1} & \cdots & \beta_3^{k,3,d-k+1} \end{array}\right]$$

The map μ is surjective and $\lambda(U'')$ is dense in $\mathfrak{G}$, so $\lambda \circ \mu \colon \mathfrak{M} \longrightarrow \mathfrak{G}$ has an open (dense) image too. It is not hard to check that both λ and μ are continuous (with respect to the Zariski topology). Now let $\mathfrak{U}$ be the open subset $\lambda \circ \mu(\mathfrak{M}) \subset \mathfrak{G}$ corresponding to $3k$-codimensional spaces "generic enough" to intersect W' (scheme-theoretically) in an integral scheme.

Finally, let $\mathcal{U}$ be the open subset of $\mathfrak{M}$ parameterizing those matrices which correspond to ideals of distinct points $P_1, \ldots, P_s$ in $\mathbb{P}^2$ with generic resolution. Then the set $(\lambda \circ \mu)^{-1}(\mathfrak{U}) \cap \mathcal{U} \subseteq \mathfrak{M}$ will give us a (non-empty) open set where we can choose our matrix $\mathcal{A}$ (hence our points) in such a way that the ideal I will be prime. This completes the proof in the case $d < 2k$.

When $d \geq 2k$, we have to consider the ring $R' = \mathfrak{k}[X_{ij}, Z_{ui}]$, with $h, i = 1, 2, 3$; $j = 1, \ldots, d-k+1$; $u = 1, \ldots, k$, and we define an ideal I' (and the associated variety W'), defined as above. Then we have to cut W', with the hyperplanes

$$H_{ui} = Z_{ui} - \sum_{j=1}^{d-k+1} \sum_{h=1}^{3} \beta_h^{u,i,j} X_{hj} \text{ and } H_l = \sum_{j=1}^{d-k+1} \sum_{i=1}^{3} \delta_i^{l,j} X_{ij}$$

(see §2).

We can do this in two steps. First we cut with the H_{ui}'s, and we work as in the previous case. Namely: we set

$$\mathfrak{M} = M(S; d-k, d-k+1)$$

to be the space parameterizing matrices of size $(d-k) \times (d-k+1)$ with entries in S_2 for the first k rows, and in S_1 for the last $d - 2k$ rows (as the matrix $\mathcal{A}$). In this case we will have a map μ which associates the given matrix $\mathcal{M} \in \mathfrak{M}$ a $3k$-tuple of hyperplanes $\mu(\mathcal{M}) = (H_{1,1}, \ldots, H_{k,3})$, whose coefficients are given by the first k rows of $\mathcal{M}$. If λ is as before, we will get

$$\mathcal{M}_{\mu(\mathcal{A})} = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & \beta_1^{1,1,1} & \beta_2^{1,1,1} & \cdots & \beta_3^{1,1,d-k+1} \\ 0 & 1 & \cdots & 0 & & & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \beta_1^{k,3,1} & \beta_2^{k,3,1} & \cdots & \beta_3^{k,3,d-k+1} \end{array}\right]$$

Working as in the previous case we get, for a generic choice of $\mathcal{A}$, a (perfect) prime ideal I'' in

$$R'' = \mathfrak{k}[X_{ij}, Z_{ul}]/(H_{1,1}, \ldots, H_{k,3}).$$

We can view I as the quotient of I'' by the ideal $(\bar{H}_1, \ldots, \bar{H}_{d-2k})$, where $\bar{H}_l$ is the image of H_l in R''. Finally, by using again the genericity of $\mathcal{A}$ to check that the H_l's are "generic enough" to preserve primeness, we conclude the argument. ■

Recall that we denoted by W the scheme associated to the ideal I defined in §3.

LEMMA 4.3. *If $d < 2k$, then the points $P = (x_{ij}, y_l) \in W$ (where $i = 1,2,3$; $j = 1,\ldots,d-k+1$; $l = 1,\ldots,2k-d$) such that $x_{ij} = 0$, for all i and j, form a closed subset of W of dimension less than or equal to* 1.

PROOF. First assume $k \geq 3$ and consider the subspace $\mathbb{P}^{2k-d-1}$ of $\mathbb{P}^N$, defined by the equations $\{X_{ij} = 0, \forall i,j\}$ and its coordinate ring $S' = \mathfrak{k}[Y_1, \ldots, Y_{2k-d}]$. Consider also the matrix $\mathcal{B}' = (B'_{ul})$, where $B'_{ul} = \sum_{l=1}^{2k-d} \delta_l^{u,l} Y_l$. Its maximal minors define an ideal $Y \subseteq S'$ which is the image of I in the quotient ring $S' = S/(X_{ij})$. The zero set $Z(Y)$ can be viewed as $W \cap \mathbb{P}^{2k-d-1}$, which is exactly the set that we want to determine. If we prove that $\dim Z(Y) \leq 1$, we will be done (recall that $\dim W \geq 2$, since $V \subset W$).

Let Y' be the ideal generated by the 3×3 minors of the generic matrix $\mathcal{B}'' = (Z_{ul})$, in $\mathfrak{k}[Z_{ul}, Y_l]$. We can view Y as the quotient ideal of Y' (which is a prime, perfect ideal of height $k - 3 + 1 = k - 2$) modulo the linear forms $H_{ul} = Z_{ul} - \sum_{l=1}^{k-d} \delta_l^{u,l} Y_l$. By using a "genericity argument", as in the proof of Theorem 4.2, one gets that either Y is an irrelevant ideal, or it has the same height as Y', *i.e.* $\mathrm{ht}(Y) = k - 2$.

In the first case, $Z(Y) = \emptyset$, and we are done; while in the second we have that $\dim Z(Y) = 2k - d - 1 - (k-2) = k - d + 1$, and so (since $k \leq d$), $\dim Z(Y)$ equals 1 or 0.

Now assume $k < 3$. First of all, $k = 1$, combined with $d < 2k$, would give $d = 1$, while we assume $d \geq 2$, to avoid trivial cases. Thus, we may assume $k = 2$, hence we have that d is 2 or 3, and so $l = 2k - d = 1, 2$. Therefore the subspace of $\mathbb{P}^N$ defined by the equations $\{X_{ij} = 0, \forall i,j\}$ has already dimension less than or equal to 1, hence, *a fortiori*, so does $Z(Y)$. ■

PROPOSITION 4.4. *Let I be the ideal defined in §3 and V the surface introduced in §1; then $W = Z(I) = V$, as sets.*

PROOF. We have, by construction, that $W \supset V$, so we have to prove the reverse inclusion.

Assume first that $d \geq 2k$ and let $P \in Z(I) \subset \mathbb{P}^N \subset \mathbb{P}^{N'}$. Recall that, in this case, $\rho = d - k$, hence $N' = 3d - 3k + 2$; and think of P as a point of $\mathbb{P}^{N'}$:

$$P = (x_{11}, x_{21}, x_{31}, \ldots, x_{1,d-k+1} x_{2,d-k+1}, x_{3,d-k+1}),$$

recalling that

$$(\dagger) \qquad H_l(P) = \sum_{j=1}^{d-k+1} \sum_{i=1}^{3} \delta_l^{i,j} x_{ij} = 0, \quad \forall l = 1, \ldots, d-2k.$$

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Because $P \in Z(I)$, the matrix

$$\mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{1,d-k+1} \\ x_{21} & \cdots & x_{2,d-k+1} \\ x_{31} & \cdots & x_{3,d-k+1} \end{bmatrix}$$

has rank 1.

Now, the x_{ij}'s are obviously not all 0, therefore we can assume there is a j such that $x_{3j} \neq 0$.

Then, $\mathrm{rk}(\mathbf{X}) = 1$ tells us that the first two rows of $\mathbf{X}$ are multiples of the third one, *i.e.* there are non-zero $\eta, \zeta \in \mathfrak{k}$ such that $\begin{cases} x_{1j} = \eta x_{3j} \\ x_{2j} = \zeta x_{3j} \end{cases}$, $\forall j = 1, \ldots, d-k+1$. Now, write $\eta = w_1/w_3$ and $\zeta = w_2/w_3$, for suitable $w_1, w_2, w_3 \in \mathfrak{k}$ with $w_3 \neq 0$, and put $x_{3_j} = c_j$. Then we can write:

$$P = (w_1c_1, w_2c_1, w_3c_1, \ldots, w_1c_{d-k+1}, w_2c_{d-k+1}, w_3c_{d-k+1}),$$

with the c_j's not all 0.

We also have:

$$(\ddagger) \qquad M_{u,v}(P) = \sum_{i=1}^{3} x_{iv}\Big(\sum_{j=1}^{d-k+1}\sum_{h=1}^{3} \beta_i^{u,h,j} x_{hj}\Big) = 0,$$

for all $u = 1, \ldots, k$ and all $v = 1, \ldots, d-k+1$.

Now, from (†) we get

$$\sum_{j=1}^{d-k+1}\sum_{i=1}^{3} \delta_i^{l,j} w_i c_j = \sum_{j=1}^{d-k+1} L_{l,j}(w_1, w_2, w_3)c_j = 0,$$

for all $l = 1, \ldots, d-2k$; while (‡) yields

$$\sum_{i=1}^{3} w_i c_v \Big(\sum_{j=1}^{d-k+1}\sum_{h=1}^{3} \beta_i^{u,h,j} w_h c_j\Big) = 0,$$

for all $u = 1, \ldots, k$ and all $v = 1, \ldots, d-k+1$; with some $c_v \neq 0$. Thus:

$$\sum_{j=1}^{d-k+1}\Big(\sum_{i,h=1}^{3} \beta_i^{u,h,j} w_i w_h\Big)c_j = \sum_{j=1}^{d-k+1} Q_{u,j}(w_1, w_2, w_3)c_j = 0,$$

for all $u = 1, \ldots, k$.

In other words, $Q = (w_1, w_2, w_3)$ is a point of $\mathbb{P}^2$ such that

$$\mathcal{A}(w_1, w_2, w_3)\begin{bmatrix} c_1 \\ \vdots \\ c_{d-k+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If Q is not one of the initial points $P_1, \ldots, P_s \in \mathbb{P}^2$, then $\mathcal{A}(w_1, w_2, w_3)$ has maximal rank (which is $\rho = d-k$), and so, by elementary linear algebra, after recalling that the F_j's are the maximal minors of $\mathcal{A}$, $(c_1, \ldots, c_{d-k+1})$ must be a multiple of

$$\big(F_1(w_1, w_2, w_3), \ldots, F_{d-k+1}(w_1, w_2, w_3)\big);$$

namely:

$$c_j = \gamma F_j(w_1, w_2, w_3), \quad \forall j = 1, \ldots, d-k+1,$$

for some $\gamma \in \mathfrak{k}$. In this case $P = \phi(Q)$, where ϕ is the map defined in §2.

If Q is one of the initial points $P_1, \ldots, P_2 \in \mathbb{P}^2$, then P belongs to the image in $\mathbb{P}^{N'}$ of one of the exceptional lines of $\mathbb{P}^2(Z)$, hence to the closure of $\operatorname{Im}\phi$ in $\mathbb{P}^{N'}$.

In both cases P belongs to $\overline{\operatorname{Im}\phi}$, and hence to V (after cutting with the hyperplanes $H_1, \ldots, H_{d-2k}$), as we wished.

Now assume $d < 2k$. In this case we work directly in $\mathbb{P}^N$, so we let

$$P = (x_{11}, x_{21}, x_{31}, \ldots, x_{1,d-k+1}, x_{2,d-k+1}, x_{3,d-k+1}, y_1, \ldots, y_l)$$

be a point of $Z(I)$.

In this case, we are not sure that the x_{ij}'s are not all 0. Nevertheless, it is enough to prove the statement for a point $P \in U$, where U is the open set where the x_{ij}'s are not all 0, which is not empty by Lemma 4.3. In fact, if we prove that $U \subset V$, then we also have $\bar{U} \subset V$. On the other hand, by Theorem 4.2, W is irreducible, hence $\bar{U} = W$, and so we obtain $W \subset V$. Thus we can assume the x_{ij}'s are not all 0.

Then, as in the other case, we write

$$P = (w_1c_1, w_2c_1, w_3c_1, \ldots, w_1c_{d-k+1}, w_2c_{d-k+1}, w_3c_{d-k+1}, y_1, \ldots, y_{2k-d})$$

with the c_j's not all 0.

As $P \in Z(I)$, we also have:

$$M_{u,v}(P) = \sum_{i=1}^{3} w_i c_v \Big(\sum_{l=1}^{2k-d} \delta_i^{u,l} y_l + \sum_{j=1}^{d-k+1} \sum_{h=1}^{3} \beta_i^{u,h,j} w_h c_j \Big) = 0,$$

for all $u = 1, \ldots, k$ and all $v = 1, \ldots, d-k+1$. From this, for $c_v \neq 0$, we obtain:

$$\begin{aligned} 0 &= \sum_{l=1}^{2k-d} \Big(\sum_{i=1}^{3} \delta_i^{u,l} w_i \Big) y_l + \sum_{j=1}^{d-k+1} \Big(\sum_{i,h=1}^{3} \beta_i^{u,h,j} w_i w_h \Big) c_j \\ &= \sum_{l=1}^{2k-d} L_{u,l}(w_1, w_2, w_3) y_l + \sum_{j=1}^{d-k+1} Q_{u,j}(w_1, w_2, w_3) c_j, \end{aligned}$$

for all $u = 1, \ldots, k$. In other words, $Q = (w_1, w_2, w_3)$ is a point of $\mathbb{P}^2$ such that

$$\mathcal{A}(w_1, w_2, w_3) \begin{bmatrix} Y_1 \\ \vdots \\ Y_{2k-d} \\ c_1 \\ \vdots \\ c_{d-k+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Now, as before, if Q is note one of the initial points $P_1, \ldots, P_s \in \mathbb{P}^2$, then $\mathcal{A}(w_1, w_2, w_3)$ has maximal rank (which is $\rho = k$), and so, as the F_j's and the G_l's are the maximal minors of $\mathcal{A}$, $(y_1, \ldots, y_{2d-k}, c_1, \ldots, c_{d-k+1})$ must be a multiple of

$$\big(G_1(Q), \ldots, G_{2k-d}(Q), F(Q), \ldots, F_{d-k+1}(Q)\big),$$

i.e.

$$c_j = \gamma F_j(w_1, w_2, w_3), \quad \forall j = 1, \ldots, d-k+1,$$
$$y_l = \gamma G_l(w_1, w_2, w_3), \quad \forall l = 1, \ldots, 2d-k,$$

for some $\gamma \in \mathfrak{k}$.

This allows us to conclude the argument, as in the previous case. ∎

By combining Theorem 4.2 with Proposition 4.4, we obtain the main result:

COROLLARY 4.5. *Let I be the ideal defined in §3 and I_V the ideal of the Veronesean surface V. Then $I = I_V$.*

REMARK 4.6. While proving Proposition 4.4, we did not make use of the cubics of I, in the case $d \geq 2k$. In other words, in that range, V is set-theoretically generated by quadrics. However, this is not true ideal-theoretically, *i.e.* the cubics are not redundant in a minimal set of generators of $I = I_V$. This follows from the fact that they are obviously needed to generate the ideal I' defined in the proof of Theorem 4.2, and that I is obtained from I' by cutting with general hyperplanes.

5. **Applications.** By cutting the surface V twice with general hyperplanes, we obtain $t = \deg V = \binom{d+2}{2} - k$ points in $\mathbb{P}^n$, where $n = N - 2 = 2d - k \geq d$.

Clearly, $n + 1 < t < \binom{n+2}{2}$: in fact, $t = (n+1) + \binom{d}{2}$.

The Hilbert function of the t points so obtained, is the second difference of the Hilbert function of V, $\Delta^2 H_V$, which is inductively defined by:

$$\Delta^2 H_V(m) = \Delta H_V(m) - \Delta H_V(m-1),$$

where

$$\Delta H_V(m) = H_V(m) - H_V(m-1).$$

Therefore,

$$\Delta^2 H_V = H_V(m) - 2H_V(m-1) + H_V(m-2);$$

and so, by using Proposition 2.2, we obtain:

$$\Delta^2 H_V(0) = H_V(0) = 1, \quad \Delta^2 H_V(1) = H_V(1) - 2 = 2d - k + 1 = n + 1,$$

and

$$\Delta^2 H_V(m) = \binom{d+2}{2} - k = t, \text{ for } m \geq 2.$$

In other words, the t points have maximal (or, *generic*) Hilbert function.

Since our points also satisfy the Uniform Position Property in the sense of Harris (all subsets of the same cardinality have the same Hilbert function—see [Ha] and [DiG]), every subset still has generic Hilbert function (*i.e.* the points are in *uniform position*—see [DiG, Lemma 15]).

The genericity of the Hilbert function implies, in particular, that the ideal of our points, like that of V, can be generated in degrees 2 and 3 (see also [GO, Corollary 1.6]) and needs the same number of generators as I_V in each degree.

Obviously, one needs all the quadrics through the points, which are $\binom{n+2}{2} - t = \dim_{\mathbf{f}}(I_V)_2$.

The *ideal generation conjecture*, first stated in [GO], predicts that, for a "general" set of t points (with generic Hilbert function) the minimum number of cubics needed should depend only on t and should equal

$$\min\left\{0, nt - 2\binom{n+2}{3}\right\}.$$

As for the ideal of our points, we know there are no cubics when $k = 1$ or $k = 2$; and so the corresponding t points do satisfy the ideal generation conjecture. Note that, since $n = 2d - k$, n and k have the same parity, and $d = \frac{n+k}{2}$, so that $k = 1$ forces n odd while $k = 2$ forces n even.

With this in mind, start from any integer n and put

$$d = \begin{cases} \frac{n+2}{2} & \text{for } n \text{ even} \\ \frac{n+1}{2} & \text{for } n \text{ odd} \end{cases}, \quad \text{and} \quad s = \begin{cases} \binom{d+2}{2} + 2 & \text{for } n \text{ even} \\ \binom{d+1}{2} + 2 & \text{for } n \text{ odd} \end{cases}.$$

Now, let Z be a set of s points in $\mathbb{P}^2$ with generic resolution and let $V = V_{d+1,Z}$ be the surface of $\mathbb{P}^{n+2}$ constructed as in §1. Finally, cut V twice with generic hyperplanes to obtain t points in $\mathbb{P}^n$, where

$$t = \begin{cases} \frac{(n+1)(n+7)}{8} & \text{for } n \text{ odd} \\ \frac{n^2+10n+8}{8} & \text{for } n \text{ even} \end{cases},$$

which satisfy the ideal generation conjecture.

Because generation in the lowest degree can be extended to general subsets (see, for instance, Proposition 3.1 of [L3]), we can say that some—hence every, by Theorem 11 of [DiG]—subset of the t points satisfies the ideal generation conjecture. We just proved:

COROLLARY 5.1. *Let n be any integer and let*

$$t = \begin{cases} \frac{(n+1)(n+7)}{8} & \textit{for } n \textit{ odd} \\ \frac{n^2+10n+8}{8} & \textit{for } n \textit{ even} \end{cases},$$

then any set of ρ points in $\mathbb{P}^n$, with $n+1 \leq \rho \leq t$, cut on V by general hyperplane sections, satisfies the ideal generation conjecture.

It was known (Corollary 2.2 of [MV]) that $t \leq 2n$ points in uniform position satisfy the ideal generation conjecture; thus we obtain new cases as soon as $n > 7$, when n is odd, and $n > 4$, for n even.

The ideal generation conjecture can be extended to predict what a graded minimal free resolution of t "general" points should look like (see [L3]): linear almost everywhere, except in one place where two degrees (or, at best, a jump in degree) will show up:

$$0 \to \cdots \to T\big(-(i+4)\big)^{\beta_{i+1}} \to T\big(-(i+2)\big)^{\alpha_i} \oplus T\big(-(i+3)\big)^{\beta_i} \to T\big(-(i+1)\big)^{\alpha_{i-1}} \to \cdots \to 0$$

(with β_i possibly 0), where T denotes the coordinate ring of $\mathbb{P}^n$.

Where the double shift is expected, depends only on t (see [L3, Theorem 2.1]): for example, it is expected at the beginning (*i.e.* for $i = 0$), if and only if $t \geq \frac{2}{3}\binom{n+2}{2}$.

Furthermore, if β_i has the expected value, then, from the double shift on, the rest of the resolution also is forced to be the expected one (see §3 of [L2]): in particular, the whole resolution is the expected one if the double shift occurs at $i = 0$ and $\beta_0 = nt - 2\binom{n+2}{3}$.

In the case of our points, a direct computation shows that, when $k = d$ (whence also $n = d$), the number of cubics in the ideal of $t = \binom{n+2}{2} - n$ points equals the expected value of β_0, and so the whole resolution is the expected one. In other words:

COROLLARY 5.2. *$\binom{n+2}{2} - n$ points in $\mathbb{P}^n$ cut on V by general hyperplane sections, satisfy the minimal resolution conjecture.*

For this number of points the result was known under the hypothesis that the points be in "transversal uniform position" (see Definition 4.3 of [GM] and §3 of [L2]).

Note that, as $t = (n+1) + \binom{d}{2}$, with $\frac{n+1}{2} \leq d \leq n$, the number of points above (for $k = d = n$), is the maximum we can possibly obtain, with respect to n. But also $t = \binom{d+2}{2} - k$, with $1 \leq k \leq d$; and so that is also the minimum number of points we can obtain, with respect to d.

Unfortunately we cannot push this technique any further: we have already argued (Remark 4.6) that all the cubics are always needed, and this remains true also in the range where the ideal generation conjecture predicts generation by quadrics only, *i.e.* $t < \frac{2}{3}\binom{n+2}{2}$ (see also Example 3.1, which is the lowest case in which the ideal generation conjecture fails). Even in the remaining range, $t \geq \frac{2}{3}\binom{n+2}{2}$, the number of cubics is not the expected one (except, obviously, for the case covered by Corollary 5.2). The lowest case in this range in which the ideal generation conjecture fails is $t = 30$ (*i.e.* $d = 7$, $k = 6$): in this case no cubic is expected, while $I = I_V$ requires $\binom{6}{3} = 20$ cubics.

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