**On 107.03: Nick Lord writes:** I enjoyed this neat triple integral evaluation of the volume of the ungula. But it is worth noting that there is a single integral derivation which, although slightly longer, makes the calculation more accessible to Sixth Formers. The Figure shows the horizontal and vertical cross-sections of the ungula at height $h$, $0 \leqslant h \leqslant 2a$.

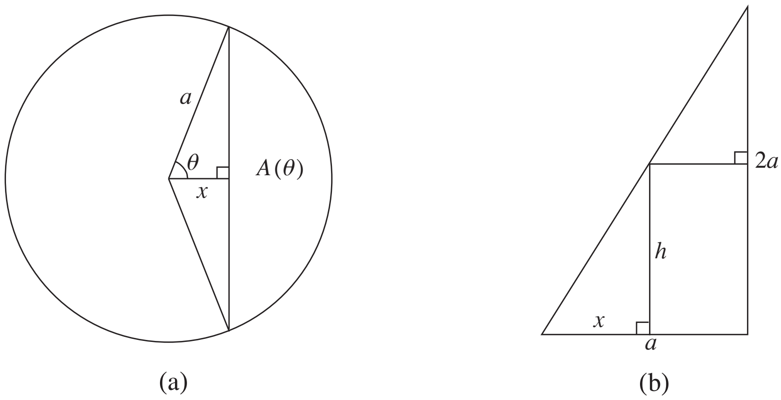


FIGURE: Cross-sections of the ungula: (a) horizontal segment, (b) vertical triangle.

The volume, $V$, is given by $V = \int_0^{2a} A(\theta)\,dh$ where, from the Figure, $h = 2x = 2a\cos\theta$ and $A(\theta) = \frac{1}{2}a^2(2\theta - \sin 2\theta) = a^2(\theta - \sin\theta\cos\theta)$.

Thus

$$V = 2a^3 \int_0^{\pi/2} (\theta - \sin\theta\cos\theta)\sin\theta\,d\theta$$

$$= 2a^3\left[-\theta\cos\theta + \sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \frac{4}{3}a^3.$$

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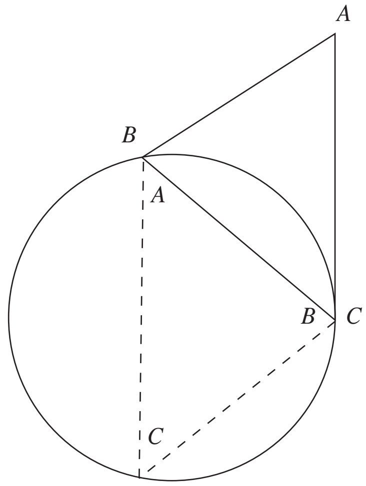
**On 107.09: Nick Lord writes:** There is a quick geometric proof of the theorem in this note, which gave the radius $r$ of the circle which is tangent to side $AC$ and passes through vertex $B$ of triangle $ABC$ as

$$r^2 = \frac{a^4b^2}{2(a^2b^2 + a^2c^2 + b^2c^2) - a^4 - b^4 - c^4}.$$

From the expanded form of Heron's formula for the area, $\Delta$, of triangle $ABC$, this formula is equivalent to $r = \dfrac{a^2b}{4\Delta}$. In the Figure, the alternate segment theorem guarantees that a triangle similar to $ABC$ may be inscribed in the circle with $r = \frac{a}{c}R$, where $R$ is the circumradius of $ABC$. Since $\Delta = \frac{1}{2}bc\sin A = \dfrac{abc}{4\Delta}$, it follows that $r = \dfrac{a}{c}\cdot\dfrac{abc}{4\Delta} = \dfrac{a^2b}{4R}$.

It is also worth noting that we can alternatively merge the two proofs to give an unusual derivation of Heron's formula.

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**On 107.27: Graham Jameson writes:** the author's reasoning requires the additional assumption that $f_j > 0$ for $1 \leqslant j \leqslant S$. This condition appears in the working at the top of page 346, but is not included in the initial statement of the result. It is clearly essential for the proof: if $f' = 0$, then one cannot conclude that $u_t < M$. In fact, a trivial example shows that the theorem can fail without this assumption: take $f_1 = 0$ and $f_2 = 1$, so that $u_n = u_{n-2}$, hence $u_n$ equals 1 for even $n$ and 0 for odd $n$. The author's method can be compared with [1], where the same result is established in non-probabilistic language.

There are important applications of the renewal theorem, notably in population dynamics, in which the above assumption does not hold. The condition that is needed for the conclusion to hold is actually as follows [2, p. 330]: if $K(f)$ is the set of $j$ for which $f_j > 0$, then the greatest common divisor of the members of $K(f)$ must be 1. This more general version requires considerably more work: the proof in [2] is not easy.

*References*

1. H. Flanders, Averaging sequences again, *Math. Gaz.* **80** (March 1996), pp. 219-222.
2. W. Feller, *An Introduction to Probability Theory and its Applications*, Wiley (1971).

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