The Relation of Morley's Theorem to the Hessian Axis and Circumcentre.

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1. In a previous paper (*q.v.*) by Mr W. L. Marr and the present writer, it was shown that, in accordance with Morley's Theorem, if the angles $A+2p\pi$, $B+2q\pi$, $C+2r\pi$ of the triangles ABC be trisected, the three groups of six lines at the vertices give rise to 27 triangles DEF, the biangular coordinates of D with respect to BC being $(B/3+2q\pi/3, C/3+2r\pi/3)$ or (β_q, γ_r), and similarly for E and F with respect to CA and AB.

These 27 triangles, 18 of which are equilateral, were arranged in 9 triads, those of each triad being two by two in perspective, and the three poles being collinear. The coordinates of these poles were represented by

$$(pqr),\ (p+1,\ q+1,\ r+1),\ (p+2,\ q+2,\ r+2) \quad\text{..........(A)}$$

where $(pqr) \equiv (\cos\alpha_p, \cos\beta_q, \cos\gamma_r)$.

Further, these 27 poles were shown to be points of concurrence of lines AL, BM, CN; where L, M, N are the isogonal conjugates of D, E, F, being in fact the meets of pairs of the second trisectors of the generalised angles $A+2p\pi$, etc.

The 9 lines DD each contain a triad of points (pqr); similarly for EE and FF.

We shall denote a progressive triad such as (A) by $\overline{pqr}$. In discussing the properties of these triads we depend entirely on Ptolemy's Theorem expressed in the forms

$$\Sigma\sin\lambda\sin(\mu-\nu)=0\,; \quad\text{..............................(1)}$$

$$\Sigma\cos\lambda\sin(\mu-\nu)=0. \quad\text{..............................(2)}$$

2. *The nine triads of poles.*

From (2) it follows that the points (A) lie on the line

$$\alpha\sin(\beta_q-\gamma_r)+\beta\sin(\gamma_r-\alpha_p)+\gamma\sin(\alpha_p-\beta_q)=0 \quad\text{..........(3)}$$

where $\quad \beta_q-\gamma_r=\beta-\gamma+(q-r)2\pi/3 \equiv (\beta-\gamma)_{q-r} \equiv L_l,$ say.

(3) may therefore be written

$$\alpha\sin L_l + \beta\sin M_m + \gamma\sin N_n = 0. \quad\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Since $l+m+n=0$, there are only 9 different lines of this series formed by varying l, m, n, as is otherwise obvious.

3. *Three triads of concurrent lines.*

Again, a triad of lines $\overline{lmn}$ will by (1), pass through the point

$$[\sin(M_m - N_n), \sin(N_n - L_l), \sin(L_l - M_m)]$$

or

$$(\sin D_d, \sin E_e, \sin F_f)\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where $D = M - N = \beta + \gamma - 2\alpha = \pi/3 - A = \pi - (A + 2\pi/3) \equiv \pi - A_1$,

and $\quad d = m - n = q + r - 2p = p + q + r - 3p \equiv p + q + r \pmod{3}$.

Hence $d \equiv e \equiv f \pmod{3}$

and $\quad D_d = (\pi - A_1)_d = \pi - A_{1-d}$.

The points (5) are therefore

$$(\sin A_{1-d}, \sin B_{1-d}, \sin C_{1-d}) \quad (d = 0, 1, 2)\ldots\ldots\ldots\ldots(6)$$

These three points are

$(\sin A)$, the Lemoine or symmedian point,

$[\sin(A + 2\pi/3)]$, the negative Hessian point,

$[\sin(A + 4\pi/3)]$, the positive Hessian point.

It will be observed that when d, or $p+q+r$, is 1, the point is the Lemoine point. This corresponds to the group of 9 non-equilaterals DEF (see previous paper).

It is a curious fact, proved by Mr Marr, that the lines of the triads which meet at the Hessian points make angles $2\pi/3$ with each other in both cases.

4. *One triad of collinear points.*

Finally from (1) follows the well-known fact that these three points lie on the line

$$\alpha\sin(B - C) + \beta\sin(C - A) + \gamma\sin(A - B) = 0,$$

which is termed the Hessian axis.

5. *The lines DD, EE, FF.*

The line $D_{qr}, D_{q+1, r+1}$ was shown in the previous paper to contain the triad of points

$$(n+1, 4-n-r, 4-n-q) \quad (n = 0, 1, 2).$$

There are 9 collinear triads, as may be seen by varying q and r, and as all the 27 points (pqr) are accounted for, we may, without confusion, express each triad in the form

$$(pqr),\ (p+1,\ q-1,\ r-1),\ (p+2,\ q-2,\ r-2).$$

Now let $\alpha=\alpha',\ \beta=-\beta',\ \gamma=-\gamma'$

$$p=p',\ q=-q',\ r=-r'.$$

Then $\cos\beta_q=\cos\beta'_{q'}$; $\cos\beta_{q-1}=\cos\beta'_{q'+1}$; $\cos\beta_{q-2}=\cos\beta'_{q'+2}$; and similarly for γ_r.

Hence the triad becomes the progressive triad $\overline{p'q'r'}$.

Proceeding as before, we obtain the nine lines DD, namely,

$$\Sigma\alpha\sin L'_{l'}=0$$

passing three by three through three points

$$(\sin D'_{d'},\ \sin E'_{d'},\ \sin F'_{d'})\quad (d'=0,\ 1,\ 2).$$

And these three points lie on the line

$$\Sigma\alpha\sin(E'-F')=0$$

where $\quad E'-F'=M'+N'-2L'=L'+M'+N'-3L'=-3L'$

$$=-3(\beta'-\gamma')=B-C$$

$$F'-D'=-3M'=-3(\gamma'-\alpha')=C+A=\pi-B$$

$$D'-E'=-3N'=-3(\alpha'-\beta')=-(A+B)=C-\pi.$$

The line is therefore

$$\alpha\sin(B-C)+\beta\sin B-\gamma\sin C=0,$$

which is the line through the circumcentre O perpendicular to BC.

Similarly the lines EE, FF lead to the lines through O perpendicular to CA and AB respectively.

6. *Summary.*

The 27 points (pqr) may be classified in the following ways:—

I. 9 triads of collinear poles of the 9 triads DEF; giving
   3 triads of concurrent lines; giving
   1 triad of collinear points, viz. the Lemoine and Hessian points.

II. 9 triads of collinear points (viz. the lines DD); giving
   3 triads of concurrent lines; giving
   1 triad of collinear points on the line through O $\perp^r$ to BC.

III. and IV. Similarly for EE and FF.

In addition to these triads we have

(*a*) 9 collinear triads AL, since every AL contains three points.

(*b*), (*c*) Similarly for BL, CL.

Since, in I., the Hessian axis passes through O, we have finally four lines I. to IV. passing through O, each line leading eventually by triads back to the 27 points (pqr).

It may be of interest to note that Ptolemy's Theorem may be used to find the equation of $D_{qr} D_{q+1, r+1}$, the coordinates of D_{qr} being written as

$$(\cos\pi/3, \cos\gamma_r, \cos\beta_q) \text{ or } \{\cos(-\pi/3), \cos\gamma_r, \cos\beta_q\}.$$

Also the parallelism of the lines EF, etc., will follow from the same theorem.