Adv. Appl. Prob. **21**, 935–937 (1989)
Printed in N. Ireland

ANOTHER VISIT TO TWO HALFLINES

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Abstract

We shall use three basic properties of Brownian motion to derive in an elegant and non-computational way the probability that standard Brownian motion, starting from 0, will ever cross the halflines $t \to \alpha t + \beta$ or $t \to \gamma t + \delta$ where $\gamma, \delta < 0 < \alpha, \beta$.

BOUNDARY CROSSING; BROWNIAN MOTION; KOLMOGOROV–SMIRNOV STATISTICS

1. Introduction

In a famous paper Doob (1949) derived the distributions of the Kolmogorov–Smirnov statistics from the probability that standard Brownian motion $W(t)$, starting at the origin, will ever cross the halflines indicated above (see also Durbin (1973)). Here we present an elementary and non-computational derivation of this result based on the following three properties of Brownian motion that are included in every basic course on this subject:

(i) The stochastic process $\tilde{W}(t)$, $t \geqq 0$, defined by

$$\tilde{W}(t) = \begin{cases} 0, & t = 0 \\ tW(t^{-1}), & t > 0 \end{cases}$$

is again standard Brownian motion.

(ii) The so-called scaling property: for each $\sigma > 0$

$$\sigma^{-1} W(\sigma^2 \cdot)$$

is again standard Brownian motion.

(iii) The explicit form of $g_{a,b}(u, y)$, the transition density for Brownian motion, started at 0 and killed on leaving $[b, a]$, where $b < 0 < a$ and $c = a - b$, given by

$$\begin{aligned} g_{a,b}(u, y) &= (2\pi u)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left[-\frac{(y + 2nc)^2}{2u} \right] - \exp\left[-\frac{(y - 2b + 2nc)^2}{2u} \right] \right\} \\ &= P\{b < \inf_{s<u} W(s) \leqq \sup_{s<u} W(s) < a,\ W(u) \in dy\}/dy\}/dy,\ y \in [b, a]. \end{aligned} \tag{1.1}$$

The precise result we shall prove is the following.

Theorem. For $\gamma, \delta < 0 < \alpha, \beta$

$$\begin{aligned} &P\{\sup_{t \geqq 0}(W(t) - (\alpha t + \beta)) < 0,\ \inf_{t \geqq 0}(W(t) - (\gamma t + \delta)) > 0\} \\ &\quad = (2\pi)^{\frac{1}{2}} \exp\left[\frac{1}{2} \left(\frac{\beta\gamma - \alpha\delta}{\sigma(\beta - \delta)} \right)^2 \right] g_{\alpha/\sigma, \gamma/\sigma}\left(1, \frac{\beta\gamma - \alpha\delta}{\sigma(\beta - \delta)}\right), \end{aligned} \tag{1.2}$$

where $\sigma := ((\alpha - \gamma)/(\beta - \delta))^{\frac{1}{2}}$.

Received 15 May 1989; revision received 14 June 1989.

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2. The derivation

Part $u=1$ and replace W by $\tilde{W}$ in (1.1) to obtain

$$P\Big\{b<\inf_{[0,1]}\tilde{W}(s)\leqq\sup_{[0,1]}\tilde{W}(s)<a,\ \tilde{W}(1)\in dy\Big\}=g_{a,b}(1,y)\,dy. \tag{2.1}$$

Since $\tilde{W}(0)=0$ a.s., we obtain

$$P\Big\{b<\inf_{(0,1]}sW(s^{-1})\leqq\sup_{(0,1]}sW(s^{-1})<a,\ W(1)\in dy\Big\}=g_{a,b}(1,y)\,dy. \tag{2.2}$$

We condition on the value of $W(1)$, which is a standard normal distributed random variable, and rewrite (2.2) as

$$P\Big\{b<\inf_{(0,1]}sW(s^{-1})\leqq\sup_{(0,1]}sW(s^{-1})<a\mid W(1)=y\Big\}=(2\pi)^{\frac{1}{2}}e^{y^2/2}g_{a,b}(1,y). \tag{2.3}$$

For $s\in(0,1)$ we have $s^{-1}\geqq1$ and so by the weak Markov property $\{W(s^{-1})-W(1),\ s^{-1}\geqq1\}$ is independent of $W(1)$. Hence the left-hand side of (2.3) is equal to

$$P\Big\{b<\inf_{(0,1]}(sW(s^{-1})-sW(1)+sy)\leqq\sup_{(0,1]}(sW(s^{-1})-sW(1)+sy)<a\Big\}. \tag{2.4}$$

Furthermore $\{W(s^{-1})-W(1),\ s^{-1}\geqq1\}$ has the same distribution as $\{W(1-s)/s),\ s^{-1}\geqq1\}$, and so the simple time substitution $t=(1-s)/s$ shows that

$$\begin{aligned}&P\Big\{\sup_{t\geqq0}(W(t)-(at+a-y))<0,\ \inf_{t\geqq0}(W(t)-(bt+b-y))>0\Big\}\\&=(2\pi)^{\frac{1}{2}}e^{y^2/2}g_{a,b}(1,y).\end{aligned} \tag{2.5}$$

Finally we take $\sigma=[(\alpha-\gamma)/(\beta-\delta)]^{\frac{1}{2}}$, $a=\alpha/\sigma$, $b=\gamma/\sigma$, $y=(\gamma\beta-\alpha\delta)/\sigma(\beta-\delta)\in[\gamma/\sigma,\ \alpha/\sigma]$ and apply (ii) to obtain

$$\begin{aligned}&(2\pi)^{\frac{1}{2}}e^{y^2/2}g_{\alpha/\sigma,\gamma/\sigma}(1,y)\\&=P\Big(\sup_{t\geqq0}\Big(W(t)-\Big(\frac{\alpha}{\sigma}t+\frac{\alpha}{\sigma}-y\Big)\Big)<0,\ \inf_{t\geqq0}\Big(W(t)-\Big(\frac{\gamma}{\sigma}t+\frac{\gamma}{\sigma}-y\Big)\Big)>0\Big\}\\&=P\Big\{\sup_{s\geqq0}\Big(\frac{1}{\sigma}W(\sigma^2s)-\Big(\alpha s+\frac{\alpha}{\sigma^2}-\frac{y}{\sigma}\Big)\Big)<0,\ \inf_{s\geqq0}\Big(\frac{1}{\sigma}W(\sigma^2s)-\Big(\gamma s+\frac{y}{\sigma^2}-\frac{y}{\sigma}\Big)\Big)>0\Big\}\\&=P\Big\{\sup_{s\geqq0}\Big(W(s)-\Big(\alpha s+\frac{\alpha}{\sigma^2}-\frac{y}{\sigma}\Big)\Big)<0,\ \inf_{s\geqq0}\Big(W(s)-\Big(\gamma s+\frac{\gamma}{\sigma^2}-\frac{y}{\sigma}\Big)\Big)>0\Big\},\end{aligned} \tag{2.6}$$

which yields (1.2) after verifying that $y=\alpha/\sigma-\beta\sigma=\gamma/\sigma-\delta\sigma$.

Relation (2.5) was also noted in Hooghiemstra (1987), but there the proof was a bit more involved. Note that (2.3) together with (2.4) is sufficient for the most general Kolmogorov–Smirnov statistic, because it implies (once more using (i)),

$$\begin{aligned}&P\Big\{b<\inf_{[0,1]}(W(t)-tW(1))\leqq\sup_{[0,1]}(W(t)-tW(1))<a\Big\}\\&=P\Big\{b<\inf_{[0,1]}(\tilde{W}(t)-t\tilde{W}(1))\leqq\sup_{[0,1]}(\tilde{W}(t)-t\tilde{W}(1))<a\Big\}\\&=P\Big\{b<\inf_{(0,1]}t\Big(W\Big(\frac{1}{t}\Big)-W(1)\Big)\leqq\sup_{(0,1]}t\Big(W\Big(\frac{1}{t}\Big)-W(1)\Big)<a\Big\}\\&=(2\pi)^{\frac{1}{2}}g_{a,b}(1,0).\end{aligned} \tag{2.7}$$

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