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A note on zero-sets in the Stone-Čech compactification

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The ring $C(X)$ is the ring of all continuous real-valued functions on a completely regular Hausdorff space X, and βX is the Stone-Čech compactification of X.

The author proves a theorem which leads to a characterization of those zero-sets in X whose closures (in βX) are zero-sets in βX, and relates this characterization to the ideals in the ring $C(X)$.

Introduction

If X is a space in which $C(X)$ only has bounded members, that is, X is *pseudocompact*, then

(1) the uniform closure of any ideal in $C(X)$ is the same as its m-closure, and

(2) for any function $f \in C(X)$,

$$\mathrm{cl}_{\beta X}\{x \in X \mid f(x) = 0\} = \{p \in \beta X \mid f^*(p) = 0\} ,$$

where f^* denotes the extension of f to βX. (See [*1*, 7Q].)

Indeed, both (1) and (2) are each equivalent to pseudocompactness of X.

In this note we consider the case in which X is not pseudocompact. We then characterize those functions for which (2) does not hold and show that these are precisely the functions which "cause" (1) not to hold. We

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also obtain necessary and sufficient conditions for the closure (in βX) of a zero-set in X to be a zero-set in βX . (See [1, 6E.2].)

Preliminaries

The reader is referred to [1] and [2, §2] for background information and notations.

We shall let X denote an arbitrary completely regular Hausdorff space. (Of course, if X is pseudocompact, the assertions in this paper are vacuous.)

We point out that for $f \in C(X)$, $Z(f)$ denotes $\{x \in X \mid f(x) = 0\}$, and $Z(f^*)$ denotes $\{p \in \beta X \mid f^*(p) = 0\}$. If $f \in C^*(X)$, that is, if f is bounded, then f^* is denoted by $\hat{f}$.

For a subset S of βX , we denote $\mathrm{cl}_{\beta X}(S)$ by S^β .

For an ideal I of $C(X)$, I^u denotes the uniform closure of I , and I^m denotes the m-closure. (See [2, 2.4].)

DEFINITION 1.1. Let $f \in C(X)$. Then a subset A of X is said to be a near-zero set for f , if for any $\delta > 0$, there is an $a \in A$ with $|f(a)| < \delta$.

THEOREM 1.2. *Let* $f \in C(X)$. *Then the following are equivalent:*

(i) $Z(f^*) \neq Z(f)^\beta$;

(ii) there is a near-zero set for f *which is completely separated from* $Z(f)$;

(iii) there is a maximal ideal M *in* $C(X)$ *so that* $f \in M^u \setminus M$;

(iv) there is an ideal I *in* $C(X)$ *so that* $f \in I^u \setminus I^m$.

Proof. *(i)* → *(ii)*. Since $Z(f^*) \supseteq Z(f)^\beta$ ([1, 7.11]), the hypothesis implies that there is a $p \in Z(f^*) \setminus Z(f)^\beta$. Hence there is a neighborhood W of p so that $W^\beta \cap Z(f)^\beta = \emptyset$. Let $A = W \cap X$, and consider $\delta > 0$. Since $\{p \in \beta X \mid -\delta < f^*(p) < \delta\} \cap W$ is an open (in βX) neighborhood of p , it follows that $(f^*)^{-1}(-\delta, \delta) \cap (W \cap X) \neq \emptyset$, and hence A is a near-

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zero set for f . Also, since W^β and $Z(f)^\beta$ are completely separated in βX , A and $Z(f)$ are completely separated in X .

(ii) → (iii). Let A be a near-zero set for f which is completely separated from $Z(f)$. Then the closures A^β and $Z(f)^\beta$ are disjoint in βX . (See [1, 6.5 III].) For each $\delta > 0$, let

$$F_\delta = A^\beta \cap \{p \in \beta X \mid -\delta \le f^*(p) \le \delta\} .$$

By the compactness of βX , there is a $p \in \bigcap\{F_\delta \mid \delta > 0\}$, and this p has the property that $f^*(p) = 0$. Since $p \in A^\beta$, $p \notin Z(f)^\beta$, and we have, using [2, 2.4], that $f \in (M^p)^u \backslash M^p$.

(iii) → (i). If $f \in (M^p)^u \backslash M^p$, then $p \in Z(f^*) \backslash Z(f)^\beta$. The equivalence of *(iii)* and *(iv)* follows from [2, 5.2] and [1, 7Q.2].

COROLLARY 1.3. *Let* $f \in C(X)$. *Then* $Z(f)^\beta$ *is a zero-set in* βX *if and only if there is a* $g \in C^*(X)$ *so that* $Z(f) = Z(g)$ *, and no near-zero set for* g *is completely separated from* $Z(g)$ *. In this case,* $Z(f)^\beta = Z(\hat{g})$ *. Furthermore, given the zero-set* $Z(\hat{g})$ *in* βX *, it is of the form* $Z(f)^\beta$ *for some* $f \in C(X)$ *if and only if no near-zero set for* g *is completely separated from* $Z(g)$ *.*

Proof. Suppose $Z(f)^\beta$ is a zero-set in βX , say $Z(f)^\beta = Z(\hat{g})$. Then, intersecting with X , we have that $Z(f) = Z(g)$, from which it follows that $Z(g)^\beta = Z(\hat{g})$. By Theorem 1.2, no near-zero set for g can be completely separated from $Z(g)$.

Conversely, if $g \in C^*(X)$ which satisfies the hypotheses of the corollary, then it follows by Theorem 1.2 that $Z(g)^\beta = Z(\hat{g})$, and hence $Z(\hat{g}) = Z(f)^\beta$.

The rest of the corollary follows easily.

EXAMPLE 1.4. Let X denote the non-negative reals and define $f(x) = x$ on $0 \le x \le 1$ and $f(x) = 1/x$ for $x \ge 1$. Then $[1, \infty)$ is a near-zero set for f which is completely separated from $Z(f)$, and

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hence $Z(f)^\beta \neq Z(\hat{f})$. However $Z(f)^\beta = \{0\}$, a zero-set in βX . Thus $Z(f)^\beta$ can be a zero-set in βX , even if it is not the zero-set of f^* .

EXAMPLE 1.5. Consider the sine function on the non-negative reals. Since $Z(\text{sine})$ is countable and discrete, it follows that there is a near-zero set for sine which is completely separated from $Z(\text{sine})$, and hence sine $\in M^\mu \backslash M$ for some maximal ideal M . Also $Z(\text{sine})^\beta$ can not be a zero-set in βX , because if $Z(\text{sine}) = Z(g)$ for any $g \in C^*(X)$, then there would be a near-zero set for g which is completely separated from $Z(g)$.

REMARK 1.6. It is believed that the equivalence of condition (2) in the introduction with pseudocompactness of X is well known, but the author could find no direct reference. A proof could easily be written based on Theorem 1.2.

References

[1] Leonard Gillman and Meyer Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, New Jersey; Toronto; London; New York; 1960).

[2] David Rudd, "On isomorphisms between ideals in rings of continuous functions", *Trans. Amer. Math. Soc.* **159** (1971), 335-353.

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