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RESEARCH ARTICLE

Valuing vulnerable Asian options with liquidity risk under Lévy processes

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Abstract
In this paper, we study the pricing of vulnerable Asian options with liquidity risk. We employ general Lévy processes to capture the changes in the liquidity discount factors and the information processes of all assets. In the proposed pricing model, we obtain the closed-form pricing formula of vulnerable Asian options using the Fourier transform methods. Finally, the derived pricing formula is used to illustrate the effects of asymmetric jump risk, and the effects are relatively stable on (vulnerable) Asian options with different moneynesses.

1. Introduction

In this paper, we study the pricing problems of vulnerable Asian options with liquidity risk. In the option pricing literature, liquidity risk has been considered using different approaches. For example, Liu and Yong [27] study European contingent claims by assuming that the stock price is directly affected by the trading of option traders. Cruz and Ševčovič [10] investigate the price of European options by taking into account of feedback effects of a large trader on the underlying asset. Another strand of the literature is based on theory of conic finance (see, e.g., [28,29]). Leippold and Schärer [21] develop a stochastic liquidity model to generate term and skew structures of bid-ask spreads. There are also some other studies on options with liquidity risk, where the price of the illiquid stock is determined by equaling the demand and the supply of the stock, originating from Brunetti and Caldarera [4]. Li *et al.* [22] work in the model of Brunetti and Caldarera [4] and study the pricing problems of Asian options. Li *et al.* [23] extend the framework in Brunetti and Caldarera [4] by adding jumps in the information process and investigate the price of discrete barrier options. Wang [40] investigates European options with default risk in a model with both liquidity risk and jump risk. Different from these studies, we work in a more general pricing model including Brunetti and Caldarera [4], Li *et al.* [23] and Wang [40] as special cases, and focus on the pricing of Asian options with default risk.

Actually, Asian options with default risk have also been studied in the literature.[1] Tsao and Liu [36] derive the approximation formula for the arithmetic Asian options with default risk in the Black–Scholes model. Jeon *et al.* [16] employ the pricing formula of vulnerable European options with time-dependent coefficients to derive a closed-form formula of vulnerable geometric Asian options. Wang [38] obtains an explicit pricing formula of Asian options with default risk under a stochastic volatility model and illustrates the effects of systematic risk on Asian option prices. Wang [39] employs GARCH models to describe the dynamics of the underlying assets and obtains the closed form of the prices of vulnerable Asian options, where the reduced-form model is employed to capture default risk. All these studies are

[1] European options with default risk are investigated in Klein and Inglis [18,19], Liao and Huang [26], Liang and Ren [24], Yang *et al.* [43], Niu and Wang [32], Wang *et al.* [41], Yang *et al.* [44] and Liang and Wang [25].

based on the assumption that the underlying asset is completely liquid. The aim of the current paper is to relax this assumption and we work under the assumption that the underlying asset is an illiquid stock. In this sense, this paper contributes to the literature on Asian options with default risk and to the literature on default-free Asian options as well.[2]

In this paper, we study the pricing of vulnerable Asian options with liquidity risk, by relaxing the assumptions in the literature on vulnerable Asian options that the underlying stock is perfectly liquid. We follow Li *et al.* [22] and Wang [40], and determine the price of the illiquid stock by the market clearing condition. We use structural approaches to describe default risk and model the dynamics of option issuer's assets similarly. In addition, Lévy processes are used to capture jump changes in the liquidity discount factors and the information processes of all assets. In this pricing model, we study the pricing problems of Asian options with default risk, which are more complex than European options with default risk considered in Wang [40]. Because of the generality of Lévy processes, we use the Fourier transform methods to get closed forms of option prices. Moreover, we mainly focus on the impact of asymmetric jump risk on option prices. For example, when negative news comes, the decline of underlying stock prices is greater than the rise when positive news comes. Many studies provide evidence for the presence of asymmetric jump risk (see, e.g., Carr *et al.* [8], Cont and Tankov [9] and Frame and Ramezani [13]). For instance, Frame and Ramezani [13] calibrate the asymmetric affine jump-diffusion models using the time series data of market index and individual stocks. They find that the arrival frequency and jump amplitude of positive news are not equal to those of negative news. In order to characterize this asymmetric jump risk, in the numerical section, we use CGMY processes to show the difference of the asymmetry on option prices by changing different parameter values.

Comparing with the existing literature, this paper has several instructive characteristics. As mentioned above, the pricing model in this paper is constructed using similar methods as in Brunetti and Caldarera [4], Li *et al.* [23] and Wang [40], and includes them as special cases. Second, we take into account of default risk, while Brunetti and Caldarera [4] and Li *et al.* [23] work in a default-free model. Third, the pricing model in this paper uses more general Lévy processes, while Wang [40] employs compound Poisson processes to describe jumps. Fourth, we derive the pricing formula using Fourier transform methods, which are different from Wang [40]. Because compound Poisson processes are used in Wang [40], the pricing formula can be derived conditional on the exact numbers of Poisson jumps. However, this method is not applicable in general Lévy processes. Moreover, in this paper, we mainly focus on asymmetric jump risk, and the closed-form pricing formula is still available. Note that Wang [40] also illustrates asymmetric jumps on vulnerable European options using numerical methods (see footnote 3 therein), since the explicit pricing formula is not available there. Fifth, we consider vulnerable Asian options, while Wang [40] investigates vulnerable European options, and Li *et al.* [22] work in a default-free model. Lastly, the proposed pricing model can be extended to price vulnerable power exchange options (see, e.g., [33,42]).

The structure of this paper is as follows. In Section 2, we describe the model and get the closed-form pricing formula of vulnerable Asian options. In Section 3, we carry out numerical analysis. Section 4 concludes the paper. The detailed proof is given in the Appendix.

2. Vulnerable Asian options with liquidity risk

In this section, we consider vulnerable Asian options with liquidity risk. We first describe the pricing model, where both the underlying asset and option issuer's assets are exposed to liquidity risk and correlated in a general sense. More specifically, a common Lévy process is adopted to drive the information processes of all assets and the liquidity discount factors. In this proposed framework, we obtain the closed form of vulnerable Asian options with liquidity risk using Fourier transform methods.

[2]A partial list of the studies on default-free Asian options includes Fusai and Meucci [15], Cai and Kou [5], Cai *et al.* [6], Fusai and Kyriakou [14] and Song *et al.* [34].

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2.1. The model

Let P be a real-world probability measure. We consider an illiquid stock and use $S(t)$ to express its price at time t. Briefly speaking, market clearing gives the illiquid stock price, which means that the demand for the stock and the supply are equal. Following Brunetti and Caldarera [4], Li *et al.* [23] and Wang [40], we work with a fixed supply ($\bar{S}$), and assume that the demand for the illiquid stock has the following specific form,

$$D_1(S(t), I_1(t), L_1(t)) := g_1\left(\frac{I_1(t)^{\vartheta_1}}{L_1(t)S(t)}\right), \tag{2.1}$$

where $g_1(\cdot)$ is a smooth and strictly increasing function, and ϑ_1 is a constant. In addition, $I_1(t)$ represents the information process and $L_1(t)$ denotes the liquidity discount factor. Furthermore, the information process $I_1(t)$ is driven by the following process,

$$I_1(t) = I_1(0)\exp\{(\mu_1 - \tfrac{1}{2}\sigma_1^2 - k_1)t + \sigma_1 B_1(t) + \delta_1 X(t) + Y_1(t)\},$$

where μ_1 and σ_1 are constants, and $B_1(t)$ is a standard Brownian motion. The last two terms, $X(t)$ and $Y_1(t)$, are two independent pure jump Lévy processes, which describe the common and idiosyncratic jumps, respectively. Note that the above form of $I_1(t)$ contains the ones in Brunetti and Caldarera [4], Li *et al.* [23] and Wang [40] as special cases. We suppose the liquidity discount factor $L_1(t)$ has the following specific form,

$$L_1(t) = L_1(0)\exp\left\{-\beta_1\left(\int_0^t a(s)\,\mathrm{d}s + \int_0^t a(s)\,\mathrm{d}W(s) + \theta X(t)\right)\right\},$$

where $a(s)$ is a deterministic function of time, β_1 and θ are nonnegative constants, and $W(t)$ is a standard Brownian motion, independent of $B_1(t)$. It should also be noted that the above liquidity discount factor includes those in Brunetti and Caldarera [4], Li *et al.* [23] and Wang [40].

Based on the assumptions on the supply and the demand, the market clearing condition is as follows:

$$D_1(S(t), I_1(t), L_1(t)) = \bar{S},$$

which in turn implies that,

$$S(t) = S(0)\frac{(I_1(t)/I_1(0))^{\vartheta_1}}{L_1(t)/L_1(0)}.$$

Lastly, by substituting $I_1(t)$ and $L_1(t)$, we can get the following form:

$$\begin{aligned} S(t) = S(0)\exp\Big\{&\vartheta_1(\mu_1 - \frac{1}{2}\sigma_1^2 - k_1)t + \vartheta_1\sigma_1 B_1(t) + (\vartheta_1\delta_1 + \beta_1\theta)X(t) + \vartheta_1 Y_1(t) \\ &+\beta_1\left(\int_0^t a(s)\,\mathrm{d}s + \int_0^t a(s)\,\mathrm{d}W(s)\right)\Big\}. \end{aligned} \tag{2.2}$$

It is clear that the stock price is driven by the common jumps, idiosyncratic jumps and liquidity risk as well.

Now, we turn to default risk. Here, we follow Klein [17] and Wang [37], and adopt structural approaches. Let $V(t)$ be the value of option issuer's assets and we assume that it has a similar form to the stock, that is,

$$\begin{aligned} V(t) = V(0)\exp\Big\{&\vartheta_2(\mu_2 - \frac{1}{2}\sigma_2^2 - k_2)t + \vartheta_2\sigma_2 B_2(t) + (\vartheta_2\delta_2 + \beta_2\theta)X(t) + \vartheta_2 Y_2(t) \\ &+\beta_2\left(\int_0^t a(s)\,\mathrm{d}s + \int_0^t a(s)\,\mathrm{d}W(s)\right)\Big\}, \end{aligned} \tag{2.3}$$

where $B_2(t)$ is a standard Brownian motion independent of $W(t)$ and $Y_2(t)$ is a pure jump Lévy process, independent of $X(t)$ and $Y_1(t)$. Recall that $B_1(t)$ is a standard Brownian motion, driving the information process of the illiquid stock. Similarly, $B_2(t)$ drives the information process of option issuer's assets. To work in a more general pricing model, here we assume that $B_1(t)$ and $B_2(t)$ are correlated and have a correlation coefficient ρ. Moreover, both the stock and option issuer's assets are affected by liquidity risk and common jumps.

In the pricing model described above, we will value vulnerable Asian options with liquidity risk in the coming subsection. As mentioned before, different from Wang [40], we will derive the pricing formula using Fourier transform methods, since the method in Wang [40] is not applicable in general Lévy processes. Moreover, in this paper, we mainly focus on asymmetric jump risk. In order to capture asymmetric jump risk, in the numerical section, we will employ CGMY processes, albeit the pricing formula is available in general Lévy processes. CGMY processes are proposed by Carr *et al.* [8] to capture jump changes of assets, including both finite or infinite activities and finite or infinite variation. There are four parameters C, G, M and Y, which characterize different properties of the process. We refer interested readers to Carr *et al.* [8] for more details. In this paper, because we are mainly interested in asymmetric jump risk, we pay more attention to the parameters G and M, which characterize the exponential decay rate on the left and right of the Lévy density. Specially, when $G = M$, the density function is symmetric. In addition, the CGMY process has been used commonly in the option pricing literature (see, e.g., Madan and Yor [30], Ballotta and Kyriakou [2], Figueroa-López *et al.* [12] and references therein) and its characteristic function is given as follows:

$$\phi_{\mathrm{CGMY}}(u,t;C,G,M,Y) = \exp\{tC\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y]\}, \tag{2.4}$$

where $C > 0,\ G \geqslant 0,\ M \geqslant 0, Y < 2$, and $\Gamma(\cdot)$ is the gamma function.

2.2. *The pricing formula*

In this subsection, we use Fourier transform methods to derive the pricing formula of vulnerable Asian options with liquidity risk. Let $G(T)$ denote discrete geometric average price of the illiquid stock $S(t)$, and it has the following form:

$$G(T) = \left[\prod_{j=1}^{n} S(t_j)\right]^{1/n}, \tag{2.5}$$

with $0 < t_1 < \cdots < t_n = T$ and $t_j = jT/n$. The payoff of Asian options with fixed strike prices is therefore given by,

$$\max\{G(T) - K, 0\},$$

where K is the fixed strike price. When $n = 1$, the above payoff equals the standard European call options. Recall that we use structural approaches to capture default risk. Assuming that the value of option issuer's debt is D and the recovery rate is α, we then have the payoff of Asian options with default risk as follows:

$$\max\{G(T) - K, 0\}\left(\mathbf{1}(V(T) \geqslant D) + \frac{\alpha V(T)}{D}\mathbf{1}(V(T) < D)\right).$$

According to the risk-neutral pricing theory, we have the following price of Asian options with default risk,

$$C_0 = e^{-rT}\mathrm{E}^{Q}\left[\max\{G(T) - K, 0\}\left(\mathbf{1}(V(T) \geqslant D) + \frac{\alpha V(T)}{D}\mathbf{1}(V(T) < D)\right)\right], \tag{2.6}$$

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where Q is a risk-neutral probability measure. Based on the above price of vulnerable Asian options, we can obtain the difference between vanilla Asian options and vulnerable Asian options, that is,

$$e^{-rT}\mathrm{E}^Q\left[\max\{G(T)-K,0\}(1-\frac{\alpha V(T)}{D})\mathbf{1}(V(T)<D)\right].$$

This difference is the credit value adjustment (CVA) of Asian options (see, e.g., [1]) and is an adjustment to the risk-free value in order to include default risk. In addition, investment banks have built trading desks and complex models around managing CVA (see, e.g., [3]).

In order to obtain the closed form of C_0 listed above, we need to work under a risk-neutral probability measure, which could be found based on the Esscher transform (see, e.g., [11]). Here, we give the dynamics of the illiquid stock and the issuer's assets under Q directly,

$$\begin{aligned} S(t) = S(0)\exp&\left\{rt+\vartheta_1\sigma_1\tilde{B}_1(t)-\frac{1}{2}\vartheta_1^2\sigma_1^2 t+\beta_1\int_0^t a(s)\,\mathrm{d}\tilde{W}(s)-\frac{1}{2}\beta_1^2\int_0^t a(s)^2\,\mathrm{d}s\right.\\ &\left.+(\vartheta_1\delta_1+\beta_1\theta)\tilde{X}(t)+\vartheta_1\tilde{Y}_1(t)-\bar{k}_1 t\right\}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} V(t) = V(0)\exp&\left\{rt+\vartheta_2\sigma_2\tilde{B}_2(t)-\frac{1}{2}\vartheta_2^2\sigma_2^2 t+\beta_2\int_0^t a(s)\,\mathrm{d}\tilde{W}(s)-\frac{1}{2}\beta_2^2\int_0^t a(s)^2\,\mathrm{d}s\right.\\ &\left.+(\vartheta_2\delta_2+\beta_2\theta)\tilde{X}(t)+\vartheta_2\tilde{Y}_2(t)-\bar{k}_2 t\right\}, \end{aligned} \tag{2.8}$$

where r is risk-free interest rate, $\bar{k}_1 = \ln(\mathrm{E}^Q[e^{(\vartheta_1\delta_1+\beta_1\theta)X(1)+\vartheta_1 Y_1(1)}])$ and $\bar{k}_2 = \ln(\mathrm{E}^Q[e^{(\vartheta_2\delta_2+\beta_2\theta)X(1)+\vartheta_2 Y_2(1)}])$. We can easily verify that the discounted prices of two assets are martingales under Q.

In what follows, we use Fourier transform methods to derive the pricing formula. To this end, we rewrite C_0 in (2.6) as follows,

$$C_0 = C_1(k,h)+C_2(k,\tilde{h}),$$

where

$$\begin{aligned} C_1(k,h) &= e^{-rT}\,\mathrm{E}^Q\left[\max\{G(T)-K,0\}(\mathbf{1}(V(T)\geqslant D))\right]\\ &= e^{-rT}\,\mathrm{E}^Q\left[\max\{e^{Y(T)}-e^k,0\}(\mathbf{1}(e^{H(T)}\geqslant e^h))\right],\\ C_2(k,\tilde{h}) &= e^{-rT}\,\mathrm{E}^Q\left[\max\{G(T)-K,0\}\left(\frac{\alpha V(T)}{D}\mathbf{1}(V(T)<D)\right)\right]\\ &= e^{-rT}\,\mathrm{E}^Q\left[\max\{e^{Y(T)}-e^k,0\}\left(\frac{\alpha e^{H(T)}}{e^{-\tilde{h}}}\mathbf{1}(e^{H(T)}<e^{-\tilde{h}})\right)\right], \end{aligned}$$

where $Y(T)=\ln G(T)$, $H(T)=\ln V(T)$, $k=\ln K$, $h=\ln D$ and $\tilde{h}=-\ln D$.

Additionally, we can obtain the respective Fourier transforms of $C_1(k,h)$ and $C_2(k,\tilde{h})$ shown below,

$$\mathscr{F}[C_1](u,v)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\varrho_1 k+\varrho_2 h}C_1(k,h)\,e^{2\pi i(uk+vh)}\,\mathrm{d}k\mathrm{d}h, \tag{2.9}$$

$$\mathscr{F}[C_2](u,v)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\varrho_1 k}C_2(k,\tilde{h})\,e^{2\pi i(uk+v\tilde{h})}\,\mathrm{d}k\mathrm{d}\tilde{h}, \tag{2.10}$$

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where ϱ_1 and ϱ_2 are positive constants. It should be noted that ϱ_1 and ϱ_2 are the damping factors, which are introduced in order to ensure the existence of the integrals (see, e.g., [7]).

Substituting the concrete forms of $C_1(k,h)$ and $C_2(k,\tilde{h})$ into the above expression, we get the following results:

$$\begin{aligned}
\mathscr{F}[C_1](u,v) &= \frac{e^{-rT}}{(2\pi iu+\varrho_1)(2\pi iu+\varrho_1+1)(2\pi iv+\varrho_2)}\mathrm{E}^Q[e^{(2\pi iu+\varrho_1+1)Y(T)}\,e^{(2\pi iv+\varrho_2)H(T)}]\\
&= \frac{e^{-rT}}{(2\pi iu+\varrho_1)(2\pi iu+\varrho_1+1)(2\pi iv+\varrho_2)}\zeta(2\pi iu+\varrho_1+1, 2\pi iv+\varrho_2),\\
\mathscr{F}[C_2](u,v) &= \frac{\alpha\, e^{-rT}}{(2\pi iu+\varrho_1)(2\pi iu+\varrho_1+1)(2\pi iv+1)}\mathrm{E}^Q[e^{(2\pi iu+\varrho_1+1)Y(T)}\,e^{-(2\pi iv)H(T)}]\\
&= \frac{\alpha\, e^{-rT}}{(2\pi iu+\varrho_1)(2\pi iu+\varrho_1+1)(2\pi iv+1)}\zeta(2\pi iu+\varrho_1+1, -2\pi iv).
\end{aligned}$$

The detailed calculations are shown in the Appendix and $\zeta(\cdot,\cdot)$ is given in Proposition 2.1. Finally, $C_1(k,h)$ and $C_2(k,\tilde{h})$ are expressed by the inverse Fourier transforms,

$$\begin{aligned}
C_1(k,h) &= e^{-\varrho_1 k-\varrho_2 h}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathscr{F}[C_1](u,v)\,e^{-i2\pi(uk+vh)}\,\mathrm{d}u\mathrm{d}v,\\
C_2(k,\tilde{h}) &= e^{-\varrho_1 k}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathscr{F}[C_2](u,v)\,e^{-i2\pi(uk+v\tilde{h})}\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Therefore, we obtain the pricing formula of vulnerable Asian options with liquidity risk, that is, $C_0 = C_1(k,h)+C_2(k,\tilde{h})$. Note that we have obtained the pricing formula expressed by the inverse Fourier transforms. In Section 3, we will set the lower and upper limits for the infinite integrals to −10,000 and 10,000 in order to obtain the prices of (vulnerable) Asian options. Additionally, we use the functions '$quad(\cdot)$' and '$dblquad(\cdot)$' in Matlab to evaluate Asian options and vulnerable Asian options, respectively.

We end this subsection by giving the explicit expression of $\zeta(\cdot,\cdot)$ defined below,

$$\zeta(p,q) = \mathrm{E}^Q[e^{pY(T)+qH(T)}],$$

where p and q are complex numbers, $Y(T) = \ln G(T)$ and $H(T) = \ln V(T)$. From the dynamics of the illiquid stock and the issuer's assets under Q, one gets that

$$\begin{aligned}
Y(T) &= \ln(G(T)) \qquad (2.11)\\
&= \frac{1}{n}\sum_{j=1}^{n}\ln(S(t_j))\\
&= \frac{1}{n}\sum_{j=1}^{n}\Big[\ln(S(0)) + (r-\frac{1}{2}\vartheta_1^2\sigma_1^2-\bar{k}_1)t_j - \frac{1}{2}\beta_1^2\int_0^{t_j}a(s)^2\,\mathrm{d}s + \vartheta_1\sigma_1\tilde{B}_1(t_j)\\
&\quad +\beta_1\int_0^{t_j}a(s)\,\mathrm{d}\tilde{W}(s) + (\vartheta_1\delta_1+\beta_1\theta)\tilde{X}(t_j)+\vartheta_1\tilde{Y}_1(t_j)\Big]\\
&= \ln(S(0)) + \left(r-\frac{1}{2}\vartheta_1^2\sigma_1^2-\bar{k}_1\right)\frac{n+1}{2n}T - \frac{\beta_1^2}{2n}\sum_{j=1}^{n}\int_0^{t_j}a(s)^2\,\mathrm{d}s + \frac{\vartheta_1\sigma_1}{n}\sum_{j=1}^{n}\tilde{B}_1(t_j)
\end{aligned}$$

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$$+\frac{\beta_1}{n}\sum_{j=1}^{n}\int_0^{t_j} a(s)\,\mathrm{d}\tilde{W}(s)+\frac{\vartheta_1\delta_1+\beta_1\theta}{n}\sum_{j=1}^{n}\tilde{X}(t_j)+\frac{\vartheta_1}{n}\sum_{j=1}^{n}\tilde{Y}_1(t_j), \tag{2.12}$$

and

$$\begin{aligned}H(T) &= \ln(V(T))\\ &= \ln(V(0))+\left(r-\frac{1}{2}\vartheta_2^2\sigma_2^2-\bar{k}_2\right)T-\frac{1}{2}\beta_2^2\int_0^T a(s)^2\,\mathrm{d}s+\vartheta_2\sigma_2\tilde{B}_2(T)+\beta_2\int_0^T a(s)\,\mathrm{d}\tilde{W}(s)\\ &\quad+(\vartheta_2\delta_2+\beta_2\theta)\tilde{X}(T)+\vartheta_2\tilde{Y}_2(T),\end{aligned} \tag{2.13}$$

where $0 < t_1 < \cdots < t_n = T$ and $t_j = jT/n$. In order to give the explicit expression of $\zeta(\cdot,\cdot)$, we introduce the following notations,

$$\begin{aligned}\Delta\tilde{X}_j &= \tilde{X}(t_j)-\tilde{X}(t_{j-1}),\\ \mathrm{E}^Q[e^{iw\Delta\tilde{X}_1}] &= \psi_{\Delta\tilde{X}}(w),\\ \mathrm{E}^Q[e^{iw\Delta\tilde{Y}_1}] &= \psi_{\Delta\tilde{Y}_1}(w),\\ \mathrm{E}^Q[e^{iw\tilde{Y}_2}] &= \psi_{\tilde{Y}_2}(w).\end{aligned}$$

Proposition 2.1. *In the proposed pricing model, we have the closed form of $\zeta(\cdot,\cdot)$ as follows:*

$$\zeta(p,q)=A_1(p,q)\cdot A_2(p,q)\cdot A_3(p,q)\cdot A_4(p,q)\cdot A_5(p,q)\cdot A_6(p,q),$$

where

$$\begin{aligned}A_1(p,q) &= e^{p(\ln(S(0))+(r-\frac{1}{2}\vartheta_1^2\sigma_1^2-\bar{k}_1)\frac{n+1}{2n}T-\frac{\beta_1^2}{2n}\sum_{j=1}^n\int_0^{t_j}a(s)^2\,\mathrm{d}s)+q(\ln(V(0))+(r-\frac{1}{2}\vartheta_2^2\sigma_2^2-\bar{k}_2)T-\frac{1}{2}\beta_2^2\int_0^T a(s)^2\,\mathrm{d}s)},\\ A_2(p,q) &= \exp\left\{\frac{1}{2}\sum_{j=1}^{n}\left[p\frac{\vartheta_1\sigma_1}{n}(n+1-j)+q\vartheta_2\sigma_2\rho\right]^2\frac{T}{n}+\frac{1}{2}q^2\vartheta_2^2\sigma_2^2(1-\rho^2)T\right\},\\ A_3(p,q) &= \exp\left\{\frac{1}{2}\sum_{j=1}^{n}\left[p\frac{\beta_1}{n}(n+1-j)+q\beta_2\right]^2\int_{t_{j-1}}^{t_j}a(s)^2\,\mathrm{d}s\right\},\\ A_4(p,q) &= \prod_{j=1}^{n}\psi_{\Delta X}\left(-ip(\vartheta_1\delta_1+\beta_1\theta)\frac{n+1-j}{n}-iq(\vartheta_2\delta_2+\beta_2\theta)\right),\\ A_5(p,q) &= \prod_{j=1}^{n}\psi_{\Delta Y}\left(-ip\frac{\vartheta_1}{n}(n+1-j)\right),\\ A_6(p,q) &= \psi_{Y_2}(-iq\vartheta_2).\end{aligned}$$

Proof. See the Appendix. □

By far, we have obtained the closed-form pricing formula of vulnerable Asian options in the proposed pricing model. In the coming section, we will illustrate vulnerable Asian option prices and focus on the difference of the asymmetric jump risk.

3. Numerical results

In this section, we use the derived pricing formula to focus on the effects of asymmetric jump risk on the prices of (vulnerable) Asian options. In Subsection 3.1, we report Asian option values in the proposed pricing model, and the corresponding pricing formula can be obtained by discarding default

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Table 1. *Asian option prices in three models. Model* 1 *is the one proposed in this paper, which reduces to Model* 2 *by discarding default risk and further to Model* 3 *by assuming liquidity risk away.*

Parameters		Model 1	Model 2	Model 3
Base case		2.1494	2.3315	1.4998
T	1	1.6574	1.7538	1.0777
	1.5	1.9392	2.0808	1.3098
K	8	2.7797	3.1133	2.5398
	12	1.6584	1.7587	0.8467
$S(0)$	8	1.2437	1.3130	0.5844
	12	3.1969	3.5587	2.8030
n	1	3.5926	3.7691	2.2722
	5	1.9134	2.0968	1.3609
Y	0.3757	1.9046	2.0965	1.3280
	0.7757	2.5040	2.6792	1.7749
G	9.3750	2.5406	2.7171	1.8192
	37.5000	1.9304	2.1203	1.3415
M	16.4750	2.3849	2.5602	1.6714
	65.9000	2.0471	2.2332	1.4286

risk. In Subsection 3.2, the prices of vulnerable Asian options are illustrated. As mentioned before, here we choose a particular Lévy process, that is, the CGMY process, to describe common jumps $X(t)$ and investigate the impact of asymmetric jumps on option prices. In order to directly observe this impact, here we assume away idiosyncratic jumps of both assets. In this way, the asymmetric effects are only caused by common jumps, rather than the combination of common and idiosyncratic jumps. Additionally, the effects of idiosyncratic jumps are quite intuitive (see, e.g., [40]).

In order to obtain option prices, we need to set the parameter values. As we know, there are four parameters in the CGMY process (see Subsection 2.1 for a brief introduction or refer to Carr *et al.* [8] for more details). We set four parameters to be $C_0 = 6.51$, $G_0 = 18.75$, $M_0 = 32.95$ and $Y_0 = 0.5757$ as a base case. These parameter values are from Carr *et al.* [8]. The other parameter values are borrowed from Wang [40] and take the following values: $r = 0.02$, $\vartheta_1 = 1.0$, $\sigma_1 = 0.25$, $\beta_1 = 0.75$, $\delta_1 = 0.80$, $\theta = 0.80$ and $a(t) = 0.50$. In addition, we assume that the option is at the money ($S(0) = K = 10$) and has a maturity of 2.0 years, and three observations are made during the whole period, which means that $n = 3$. Finally, we set the damping factors $\varrho_1 = \varrho_2 = 1.10$. We also calculate option prices with other values and the results are the same.

Before illustrating the effects of asymmetric jump risk on the prices of (vulnerable) Asian options, here we compare option prices in the proposed framework with those derived in pricing models without default risk or liquidity risk. The results are shown in Table 1, and option prices are close to those in previous literature (see, e.g., [22]). It should be remarked that option prices in our models are higher than those in Li *et al.* [22], because there are jumps in our models. The effects of the parameters G and M can also be observed from Table 1. It can be seen that option prices decrease when the value of the parameter G or M increases, which is consistent with Merton [31], Kou and Wang [20] and Tian *et al.* [35]. Furthermore, these effects are the focus of this paper and will be investigated in great detail in the following subsections.

Recall that the parameters G and M characterize the exponential decay rate on the left and right of the Lévy density. To illustrate the effects of asymmetric jump risk, in what follows, we will show (vulnerable) Asian option prices in the following cases: fixed M, $G = 0.5G_0$; fixed M, $G = 2G_0$; fixed G, $M = 0.5M_0$; fixed G, $M = 2M_0$. In this way, we can observe the difference in (vulnerable) Asian option prices caused by asymmetric jump risk.

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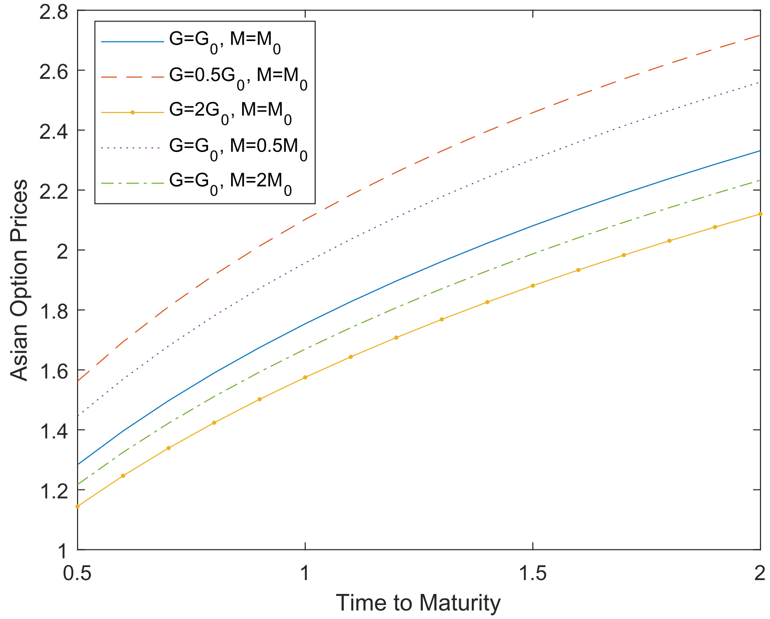

Figure 1. *Prices of Asian call options against time to maturities. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with ($G = G_0,\ M = M_0$), prices of Asian options with ($G = 0.5G_0,\ M = M_0$), prices of Asian options with ($G = 2G_0,\ M = M_0$), prices of Asian options with ($G = G_0,\ M = 0.5M_0$) and prices of Asian options with ($G = G_0,\ M = 2M_0$), respectively.*

3.1. Asian options

In this subsection, the prices of Asian options in the proposed pricing model are illustrated, and the results are shown in Figures 1–5. Figure 1 shows the prices of Asian options with different maturities. First, in all cases, Asian option prices rise with an increase of time to maturity. Next, we focus on the effects of the parameter M. Note that the parameter M characterizes the exponential decay rate on the right of the Lévy density. The solid line corresponds to the base case, that is, $G = G_0,\ M = M_0$. The dotted line and the dot-dashed line illustrate Asian option prices with ($G = G_0,\ M = 0.5M_0$) and ($G = G_0,\ M = 2M_0$), respectively. Intuitively, a lower value of the parameter M corresponds to a slower rate of exponential decay on the right of the Lévy density, and hence a higher Asian option price. Moreover, the differences between the base case and the case with ($G = G_0,\ M = 0.5M_0$) are larger than those between the base case and the case with ($G = G_0,\ M = 2M_0$).

In what follows, we focus on the effect of the parameter G. Once again, note that the parameter G characterizes the exponential decay rate on the left of the Lévy density. When the value of the parameter G decreases, the Lévy density on the left decays more slowly, meaning that negative jumps occur more likely. Option prices should fall when the value of the parameter G drops. However, an interesting finding in Figure 1 is that option prices increase when the value of the parameter G decreases. These results are a little counterintuitive. The reason is that the parameter G also affects the risk compensation term $\bar{k}_1$ under the risk neutral probability measure. It can be seen clearly from Figure 2 that when the parameter G decreases, $\bar{k}_1$ decreases rapidly, resulting in the rise of underlying asset prices and option prices as well. The similar conclusions can be found in Merton [31], Kou and Wang [20] and Tian *et al.* [35]. In Figure 1, it can also be seen that option prices increase more when we reduce the value of the parameter G by half than when we reduce the value of the parameter M by half. Similarly, when we double the values of G and M, negative jumps have a more pronounced effect.

Figure 3 depicts option prices with alternative strike prices. Intuitively, option prices drop with an increase of strike prices, and the prices of Asian options with ($G = 0.5G_0,\ M = M_0$) are largest. By

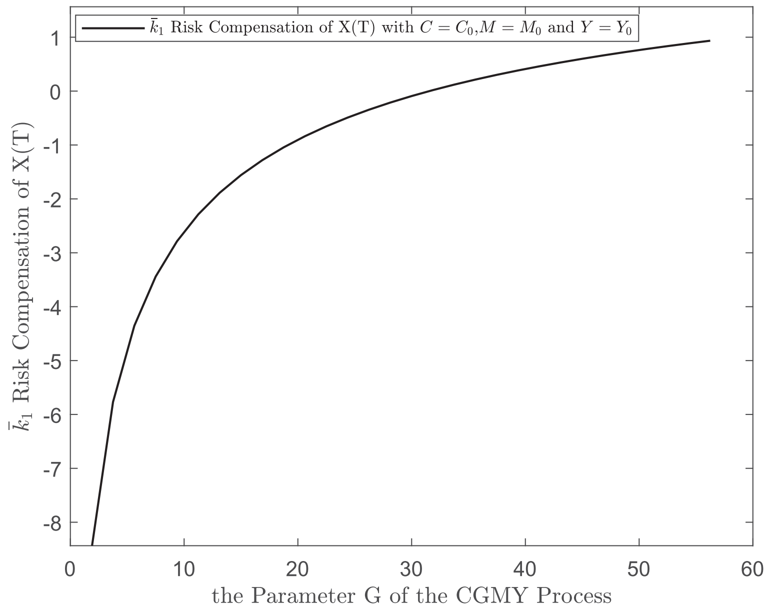

Figure 2. *Risk compensation $\bar{k}_1$ against the parameter G in the CGMY process with $C = C_0$, $M = M_0$, $Y = Y_0$.*

comparing the solid line and the dotted line (the dashed line), we see that the distance between the two lines almost keeps unchanged when strike price changes. This result tells us that the effects of these two parameters are relatively stable for alternative moneynesses.

Figure 4 shows the influence of the change of θ in the liquidity discount factor on option prices. Note that θ captures the impact of jumps in the liquidity discount factor. With a larger value of θ, we have a lower level of the liquidity discount factor and hence a higher underlying asset price, resulting in a higher option price. This is consistent with the intuition. In addition, with an increase of θ, the gaps between the five lines are growing, indicating that the impact of asymmetric jump risk becomes greater.

Figure 5 illustrates the effect of the parameter Y in the CGMY process on Asian option prices. The parameter Y controls the fine structure of the stochastic process (see, e.g., [8]). From Figure 5, we can observe that option prices rise with an increase of Y, but the difference between the five situations does not change significantly.

3.2. Vulnerable Asian options

In this subsection, we report the values of vulnerable Asian options in the proposed pricing model. To obtain the prices of the options with default risk, we need the parameter values in the dynamics of option issuer's assets. For simplicity, here we set them the same as those in the dynamics of the illiquid stock and the correlation coefficient is assumed to be $\rho = -0.50$. Moreover, the initial value of issuer's assets is $V(0) = 100$, the value of the debt is $D = 80$ and the recovery rate is $\alpha = 0.40$. The results are shown in Figures 6–13.

Figures 6 and 7 illustrate the values of vulnerable Asian options with alternative maturities and different strike prices, respectively. These patterns are similar to the ones we observed in the last subsection and could be understood in a similar way. In addition, the price difference between Figure 1 (Figure 3) and Figure 6 (Figure 7) shows the effects of default risk. Due to the existence of default risk, vulnerable Asian option prices are lower. For example, in the base case, the prices of Asian options and vulnerable Asian options are 2.3315 and 2.1494, respectively.

Figures 8 and 9 show vulnerable Asian option prices against different values of θ and the parameter Y in the CGMY process, respectively. Once again, these patterns are similar to the ones we observed

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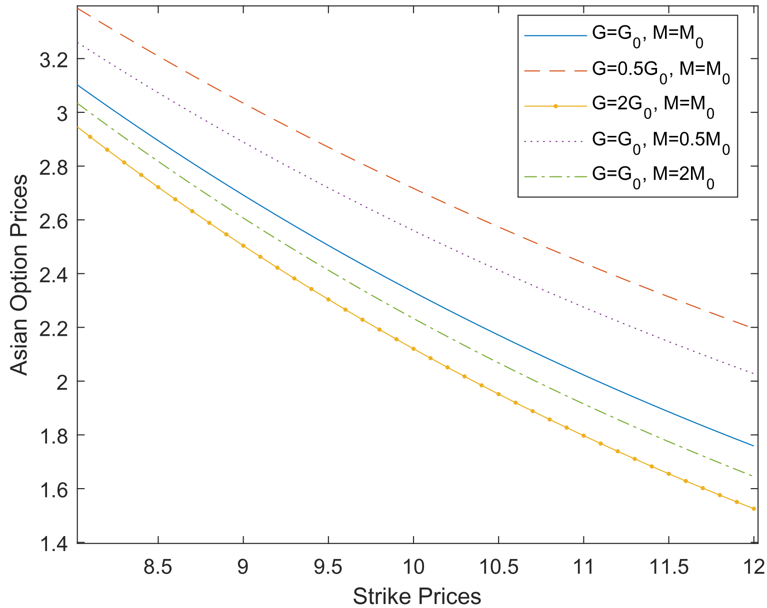

Figure 3. *Prices of Asian call options against strike prices. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with* $(G = G_0,\ M = M_0)$*, prices of Asian options with* $(G = 0.5G_0,\ M = M_0)$*, prices of Asian options with* $(G = 2G_0,\ M = M_0)$*, prices of Asian options with* $(G = G_0,\ M = 0.5M_0)$ *and prices of Asian options with* $(G = G_0,\ M = 2M_0)$*, respectively.*

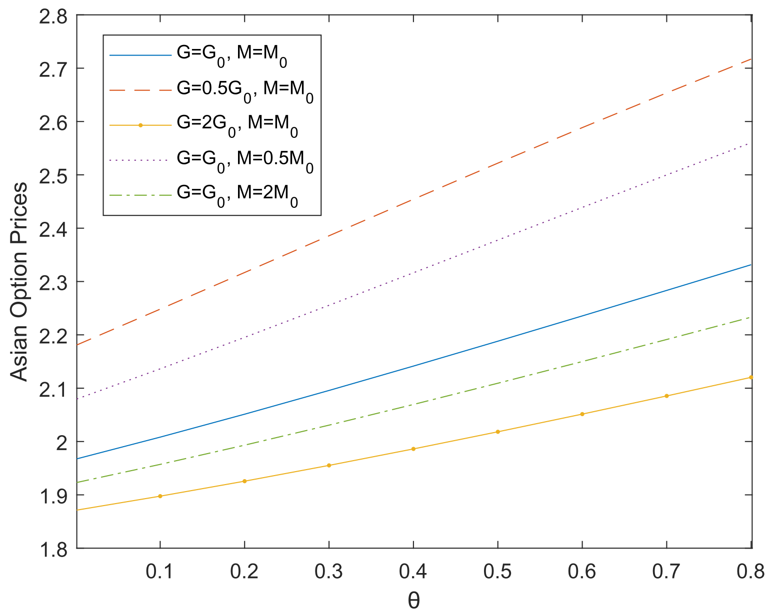

Figure 4. *Prices of Asian call options against the values of* θ*. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with* $(G = G_0,\ M = M_0)$*, prices of Asian options with* $(G = 0.5G_0,\ M = M_0)$*, prices of Asian options with* $(G = 2G_0,\ M = M_0)$*, prices of Asian options with* $(G = G_0,\ M = 0.5M_0)$ *and prices of Asian options with* $(G = G_0,\ M = 2M_0)$*, respectively.*

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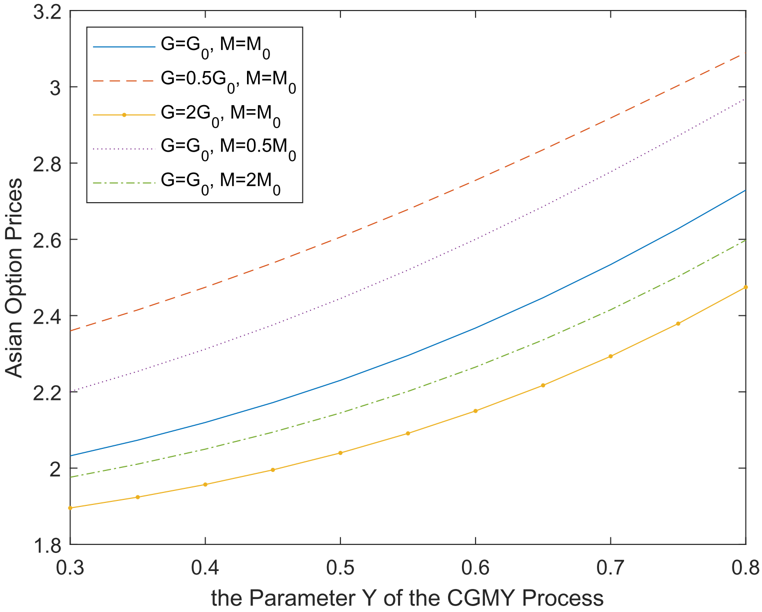

Figure 5. *Prices of Asian call options against the values of the parameter Y in the CGMY process. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with* ($G = G_0,\ M = M_0$)*, prices of Asian options with* ($G = 0.5G_0,\ M = M_0$)*, prices of Asian options with* ($G = 2G_0,\ M = M_0$)*, prices of Asian options with* ($G = G_0,\ M = 0.5M_0$) *and prices of Asian options with* ($G = G_0,\ M = 2M_0$)*, respectively.*

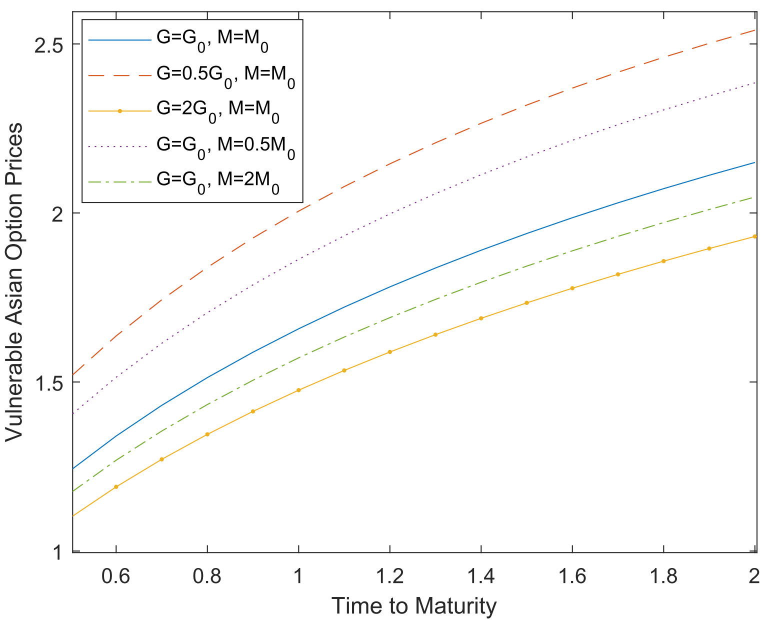

Figure 6. *Prices of vulnerable Asian call options against maturities. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with* ($G = G_0,\ M = M_0$)*, prices of Asian options with* ($G = 0.5G_0,\ M = M_0$)*, prices of Asian options with* ($G = 2G_0,\ M = M_0$)*, prices of Asian options with* ($G = G_0,\ M = 0.5M_0$) *and prices of Asian options with* ($G = G_0,\ M = 2M_0$)*, respectively.*

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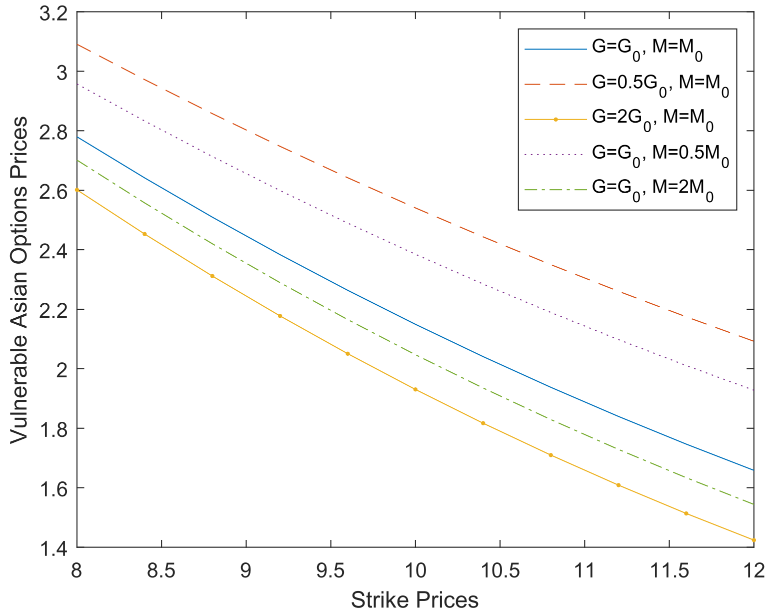

Figure 7. *Prices of vulnerable Asian call options against strike prices. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with* $(G = G_0,\ M = M_0)$*, prices of Asian options with* $(G = 0.5G_0,\ M = M_0)$*, prices of Asian options with* $(G = 2G_0,\ M = M_0)$*, prices of Asian options with* $(G = G_0,\ M = 0.5M_0)$ *and prices of Asian options with* $(G = G_0,\ M = 2M_0)$*, respectively.*

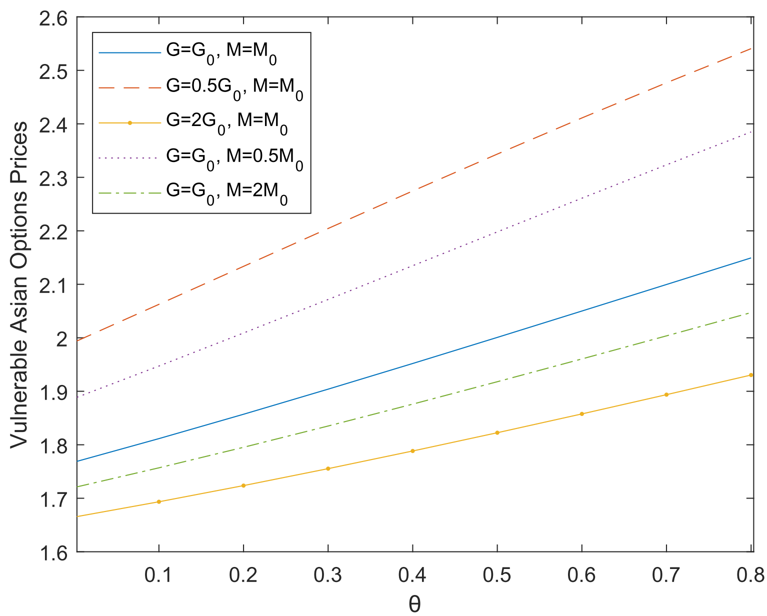

Figure 8. *Prices of vulnerable Asian call options against the values of* θ*. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with* $(G = G_0,\ M = M_0)$*, prices of Asian options with* $(G = 0.5G_0,\ M = M_0)$*, prices of Asian options with* $(G = 2G_0,\ M = M_0)$*, prices of Asian options with* $(G = G_0,\ M = 0.5M_0)$ *and prices of Asian options with* $(G = G_0,\ M = 2M_0)$*, respectively.*

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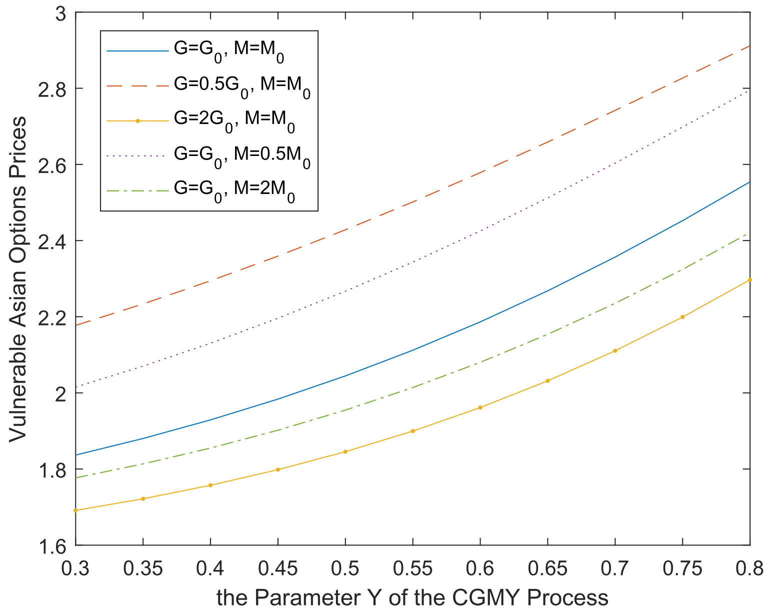

Figure 9. *Prices of vulnerable Asian call options against the parameter Y in the CGMY process. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with* ($G = G_0,\ M = M_0$)*, prices of Asian options with* ($G = 0.5G_0,\ M = M_0$)*, prices of Asian options with* ($G = 2G_0,\ M = M_0$)*, prices of Asian options with* ($G = G_0,\ M = 0.5M_0$) *and prices of Asian options with* ($G = G_0,\ M = 2M_0$)*, respectively.*

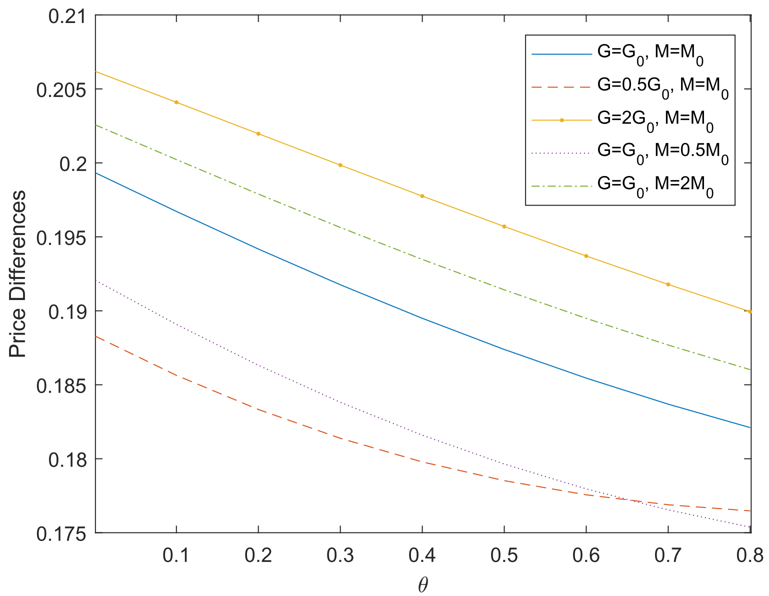

Figure 10. *Price differences between Asian call options and vulnerable Asian call options against against the values of* θ*. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to* ($G = G_0,\ M = M_0$), ($G = 0.5G_0,\ M = M_0$), ($G = 2G_0,\ M = M_0$), ($G = G_0,\ M = 0.5M_0$) *and* ($G = G_0,\ M = 2M_0$)*, respectively.*

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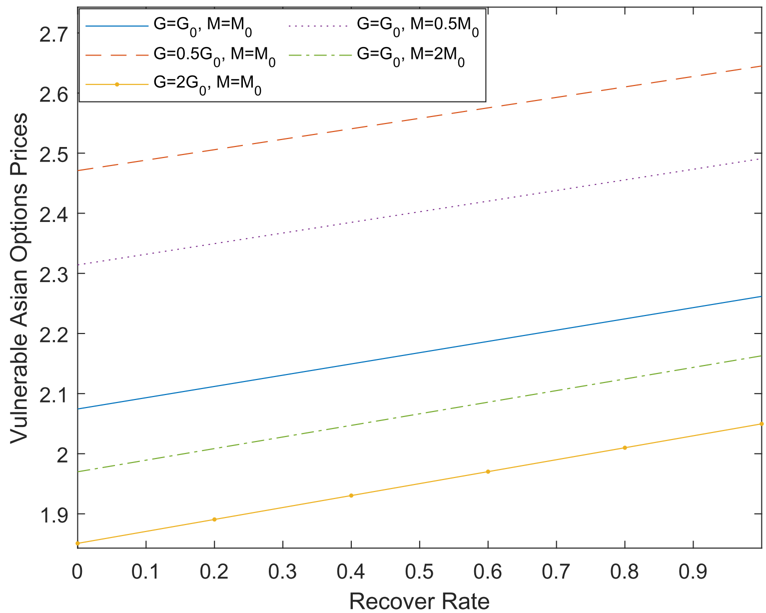

Figure 11. *Prices of vulnerable Asian call options against the recover rate α. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with $(G = G_0,\ M = M_0)$, prices of Asian options with $(G = 0.5G_0,\ M = M_0)$, prices of Asian options with $(G = 2G_0,\ M = M_0)$, prices of Asian options with $(G = G_0,\ M = 0.5M_0)$ and prices of Asian options with $(G = G_0,\ M = 2M_0)$, respectively.*

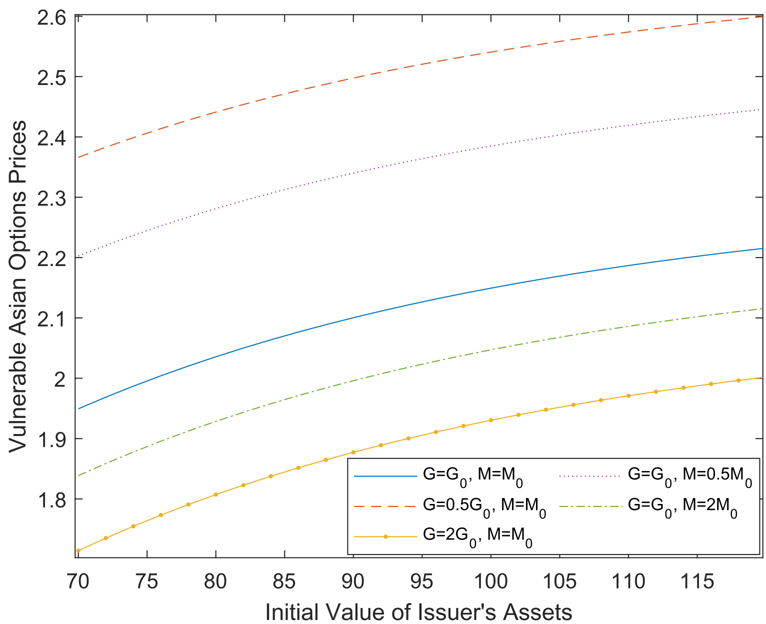

Figure 12. *Prices of vulnerable Asian call options against initial values of issuer's assets $V(0)$. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with $(G = G_0,\ M = M_0)$, prices of Asian options with $(G = 0.5G_0,\ M = M_0)$, prices of Asian options with $(G = 2G_0,\ M = M_0)$, prices of Asian options with $(G = G_0,\ M = 0.5M_0)$ and prices of Asian options with $(G = G_0,\ M = 2M_0)$, respectively.*

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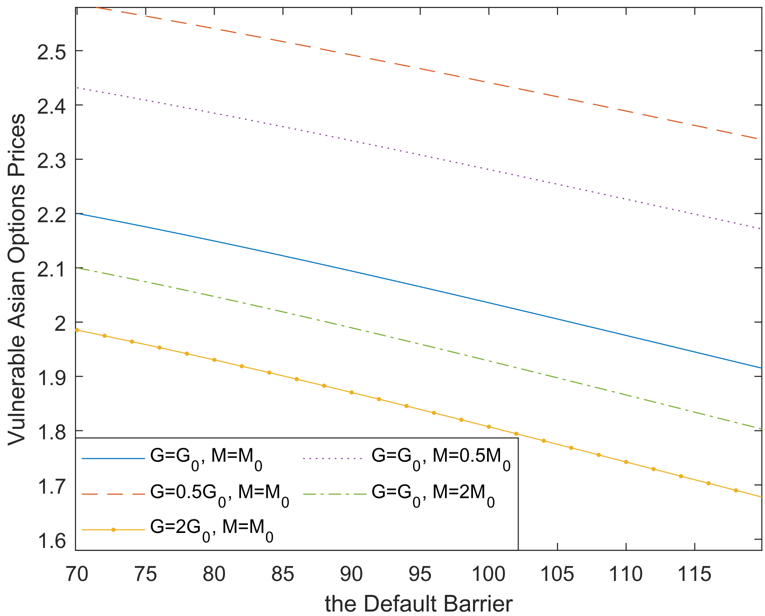

Figure 13. *Prices of vulnerable Asian call options against the default barrier D. The solid, dashed, dot-solid, dotted and dot-dashed lines correspond to prices of Asian options with ($G = G_0,\ M = M_0$), prices of Asian options with ($G = 0.5G_0,\ M = M_0$), prices of Asian options with ($G = 2G_0,\ M = M_0$), prices of Asian options with ($G = G_0,\ M = 0.5M_0$) and prices of Asian options with ($G = G_0,\ M = 2M_0$), respectively.*

in the last subsection. However, it should be noted that the values of θ and the parameter Y also affect issuer's assets (default risk). We take the change of θ as example. When the values of θ increase, we have a lower level of the liquidity discount factor. Therefore, we have a higher underlying asset price, and a higher value of issuer's assets, implying that vulnerable Asian options increase faster than those of Asian options without default risk. Indeed, from Figure 10, we can see that the price differences between Asian options and vulnerable Asian options narrow, as the value of θ rises.

Figure 11 illustrates the effect of the recovery rate α on the price of vulnerable Asian options. The recovery rate α means that when the issuers default, option holders can get a certain proportion of compensation. In the pricing formula, the effect of recovery rates on vulnerable Asian option prices is linear, which is also reflected in Figure 11. Figures 12 and 13 show the impact of initial values of issuer's assets $V(0)$ and the default barrier D, respectively. Both of them are related to the probability of default. The larger the $V(0)$ or the smaller the D, the less likely the issuers are to default, and the higher vulnerable Asian option prices. These effects are not linear, and the effects of asymmetric jump risk keep relatively stable.

4. Conclusion

This paper studies the pricing of vulnerable Asian options in a pricing model with liquidity risk, where general Lévy processes are used to describe jumps in the information processes of both assets and the liquidity discount factors. The proposed pricing model includes some existing ones such as Brunetti and Caldarera [4], Li *et al.* [23] and Wang [40]. Employing the Fourier transform methods, we obtain the closed-form pricing formula of vulnerable Asian options. Finally, we illustrate the effects of asymmetric jump risk using CGMY processes to describe the jumps. More specifically, we investigate the effects of the parameters G and M in the CGMY process, which characterize the exponential decay rate on the left and right of the Lévy density. The numerical results show that option prices increase when the value of the parameter G or M decreases.

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Compliance with ethical standards. The authors declare that they have no conflict of interests.

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Appendix. Derivations of Fourier transforms of $C_1(k,h)$ and $C_2(k,\tilde{h})$

Here, we show the detailed calculations of Fourier transforms of $C_1(k,h)$ and $C_2(k,\tilde{h})$. In order to guarantee the existence of the integrals, we employ the method in Carr and Madan [7] and introduce the damping factors ϱ_1 and ϱ_2. We have the Fourier transform of $C_1(k,h)$ as follows:

$$
\begin{aligned}
\mathscr{F}[C_1](u,v) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\varrho_1 k+\varrho_2 h} C_1(k,h)\, e^{2\pi i(uk+vh)}\,\mathrm{d}k\mathrm{d}h \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\varrho_1 k+\varrho_2 h}\, e^{-rT}\mathrm{E}^{Q}[\max\{e^{Y(T)}-e^{k},0\}(\mathbf{1}(e^{H(T)}\geqslant e^{h}))]\cdot e^{i\cdot 2\pi(uk+vh)}\,\mathrm{d}k\mathrm{d}h \\
&= e^{-rT}\mathrm{E}^{Q}\left[\int_{-\infty}^{+\infty} e^{(2\pi iv+\varrho_2)h}\mathbf{1}(H(T)\geqslant h)\int_{-\infty}^{Y(T)} e^{(2\pi iu+\varrho_1)k}(e^{Y(T)}-e^{k})\,\mathrm{d}k\mathrm{d}h\right] \\
&= e^{-rT}\mathrm{E}^{Q}\left[\int_{-\infty}^{+\infty} e^{(2\pi iv+\varrho_2)h}\mathbf{1}(H(T)\geqslant h)\left(\frac{e^{(2\pi iu+\varrho_1)k+Y(T)}}{2\pi iu+\varrho_1}\Bigg|_{-\infty}^{Y(T)} - \frac{e^{(2\pi iu+\varrho_1+1)k}}{2\pi iu+\varrho_1+1}\Bigg|_{-\infty}^{Y(T)}\right)\mathrm{d}h\right] \\
&= e^{-rT}\mathrm{E}^{Q}\left[\int_{-\infty}^{+\infty} e^{(2\pi iv+\varrho_2)h}\mathbf{1}(H(T)\geqslant h)\left(\frac{e^{(2\pi iu+\varrho_1+1)Y(T)}}{2\pi iu+\varrho_1} - \frac{e^{(2\pi iu+\varrho_1+1)Y(T)}}{2\pi iu+\varrho_1+1}\right)\mathrm{d}h\right] \\
&= e^{-rT}\mathrm{E}^{Q}\left[\frac{e^{(2\pi iu+\varrho_1+1)Y(T)}}{(2\pi iu+\varrho_1)(2\pi iu+\varrho_1+1)}\int_{-\infty}^{+\infty} e^{(2\pi iv+\varrho_2)h}\mathbf{1}(H(T)\geqslant h)\,\mathrm{d}h\right]
\end{aligned}
$$

$$= e^{-rT}\mathrm{E}^Q\left[\left(\frac{e^{(2\pi iu+\varrho_1+1)Y(T)}}{(2\pi iu+\varrho_1)(2\pi iu+\varrho_1+1)}\right)\left(\frac{e^{(2\pi iv+\varrho_2)h}}{2\pi iv+\varrho_2}\Bigg|_{-\infty}^{H(T)}\right)\right]$$
$$= e^{-rT}\mathrm{E}^Q\left[\frac{e^{(2\pi iu+\varrho_1+1)Y(T)}}{(2\pi iu+\varrho_1)(2\pi iu+\varrho_1+1)}\times\frac{e^{(2\pi iv+\varrho_2)H(T)}}{2\pi iv+\varrho_2}\right].$$

Note that we have used that ϱ_1 and ϱ_2 are positive.

As for the Fourier transform of $C_2(k,\tilde{h})$, we only need to introduce one damping factor ϱ_1, because there is a real part in front of $\tilde{h}$ which ensures that the integral exists at infinity. The detailed calculations of $\mathscr{F}[C_2](u,v)$ are shown below:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\varrho_1 k}C_2(k,\tilde{h})\,e^{2\pi i(uk+v\tilde{h})}\,\mathrm{d}k\mathrm{d}\tilde{h}$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\varrho_1 k}\,e^{-rT}\,\mathrm{E}^Q\left[\max\{e^{Y(T)}-e^k,0\}\left(\frac{\alpha e^{H(T)}}{e^{-\tilde{h}}}\mathbf{1}(e^{H(T)}<e^{-\tilde{h}})\right)\right]\cdot e^{i\cdot 2\pi(uk+v\tilde{h})}\,\mathrm{d}k\mathrm{d}\tilde{h}$$
$$= \alpha e^{-rT}\,\mathrm{E}^Q\left[\int_{-\infty}^{+\infty} e^{(2\pi iv+1)\tilde{h}}e^{H(T)}\mathbf{1}(H(T)<-\tilde{h})\int_{-\infty}^{+\infty} e^{(2\pi iu+\varrho_1)k}\max\{e^{Y(T)}-e^k,0\}\,\mathrm{d}k\mathrm{d}\tilde{h}\right]$$
$$= \alpha e^{-rT}\,\mathrm{E}^Q\left[\int_{-\infty}^{+\infty} e^{(2\pi iv+1)\tilde{h}}e^{H(T)}\mathbf{1}(\tilde{h}<-H(T))\int_{-\infty}^{Y(T)} e^{(2\pi iu+\varrho_1)k}(e^{Y(T)}-e^k)\,\mathrm{d}k\mathrm{d}\tilde{h}\right]$$
$$= \alpha e^{-rT}\,\mathrm{E}^Q\left[\int_{-\infty}^{+\infty} e^{(2\pi iv+1)\tilde{h}}e^{H(T)}\mathbf{1}(\tilde{h}<-H(T))\left(\frac{e^{(2\pi iu+\varrho_1)k+Y(T)}}{2\pi iu+\varrho_1}\Bigg|_{-\infty}^{Y(T)}-\frac{e^{(2\pi iu+\varrho_1+1)k}}{2\pi iu+\varrho_1+1}\Bigg|_{-\infty}^{Y(T)}\right)\mathrm{d}\tilde{h}\right]$$
$$= \alpha e^{-rT}\,\mathrm{E}^Q\left[\int_{-\infty}^{+\infty} e^{(2\pi iv+1)\tilde{h}}e^{H(T)}\mathbf{1}(\tilde{h}<-H(T))\left(\frac{e^{(2\pi iu+\varrho_1+1)Y(T)}}{2\pi iu+\varrho_1}-\frac{e^{(2\pi iu+\varrho_1+1)Y(T)}}{2\pi iu+\varrho_1+1}\right)\mathrm{d}\tilde{h}\right]$$
$$= \alpha e^{-rT}\,\mathrm{E}^Q\left[\frac{e^{(2\pi iu+\varrho_1+1)Y(T)}}{(2\pi iu+\varrho_1)(2\pi iu+\varrho_1+1)}\int_{-\infty}^{-H(T)} e^{(2\pi iv+1)\tilde{h}}\,e^{H(T)}\,\mathrm{d}\tilde{h}\right]$$
$$= \alpha e^{-rT}\,\mathrm{E}^Q\left[\frac{e^{(2\pi iu+\varrho_1+1)Y(T)}}{(2\pi iu+\varrho_1)(2\pi iu+\varrho_1+1)}\left(\frac{e^{(2\pi iv+1)\tilde{h}+H(T)}}{2\pi iv+1}\Bigg|_{-\infty}^{-H(T)}\right)\right]$$
$$= \alpha e^{-rT}\,\mathrm{E}^Q\left[\frac{e^{(2\pi iu+\varrho_1+1)Y(T)}}{(2\pi iu+\varrho_1)(2\pi iu+\varrho_1+1)}\times\frac{e^{-(2\pi iv)H(T)}}{2\pi iv+1}\right].$$

We can find that $\mathscr{F}[C_1](u,v)$ and $\mathscr{F}[C_2](u,v)$ have similar structures.

Proof of Proposition 2.1. Recall the forms of $Y(T)$ and $H(T)$,

$$Y(T) = \ln(S(0)) + \left(r-\frac{1}{2}\vartheta_1^2\sigma_1^2-\bar{k}_1\right)\frac{n+1}{2n}T - \frac{\beta_1^2}{2n}\sum_{j=1}^{n}\int_0^{t_j} a(s)^2\,\mathrm{d}s + \frac{\vartheta_1\sigma_1}{n}\sum_{j=1}^{n}\tilde{B}_1(t_j)$$
$$+ \frac{\beta_1}{n}\sum_{j=1}^{n}\int_0^{t_j} a(s)\,\mathrm{d}\tilde{W}(s) + \frac{\vartheta_1\delta_1+\beta_1\theta}{n}\sum_{j=1}^{n}\tilde{X}(t_j) + \frac{\vartheta_1}{n}\sum_{j=1}^{n}\tilde{Y}_1(t_j),$$

and

$$H(T) = \ln(V(0)) + (r-\frac{1}{2}\vartheta_2^2\sigma_2^2-\bar{k}_2)T - \frac{1}{2}\beta_2^2\int_0^T a(s)^2\,\mathrm{d}s + \vartheta_2\sigma_2\tilde{B}_2(T) + \beta_2\int_0^T a(s)\,\mathrm{d}\tilde{W}(s)$$
$$+ (\vartheta_2\delta_2+\beta_2\theta)\tilde{X}(T) + \vartheta_2\tilde{Y}_2(T),$$

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where $0 < t_1 < \cdots < t_n = T$ and $t_j = jT/n$. Substituting the above forms of $Y(T)$ and $H(T)$, we separate the items according to their relevance,

$$\mathrm{E}[e^{pY(T)+qH(T)}] = A_1(p,q)\cdot A_2(p,q)\cdot A_3(p,q)\cdot A_4(p,q)\cdot A_5(p,q)\cdot A_6(p,q).$$

We put all deterministic items in $A_1(p,q)$, that is,

$$A_1(p,q) = e^{p(\ln(S(0))+(r-\frac{1}{2}\vartheta_1^2\sigma_1^2-\bar{k}_1)\frac{n+1}{2n}T-\frac{\beta_1^2}{2n}\sum_{j=1}^n\int_0^{t_j}a(s)^2\,\mathrm{d}s)+q(\ln(V(0))+(r-\frac{1}{2}\vartheta_2^2\sigma_2^2-\bar{k}_2)T-\frac{1}{2}\beta_2^2\int_0^T a(s)^2\,\mathrm{d}s)},$$

and the other terms are listed below,

$$\begin{aligned}
A_2(p,q) &= \mathrm{E}^Q[e^{p\frac{\vartheta_1\sigma_1}{n}\sum_{j=1}^n\tilde{B}_1(t_j)+q\vartheta_2\sigma_2\tilde{B}_2(T)}],\\
A_3(p,q) &= \mathrm{E}^Q[e^{p\frac{\beta_1}{n}\sum_{j=1}^n\int_0^{t_j}a(s)\,\mathrm{d}\tilde{W}(s)+q\beta_2\int_0^T a(s)\,\mathrm{d}\tilde{W}(s)}],\\
A_4(p,q) &= \mathrm{E}^Q[e^{p(\vartheta_1\delta_1+\beta_1\theta)\frac{1}{n}\sum_{j=1}^n\tilde{X}(t_j)+q(\vartheta_2\delta_2+\beta_2\theta)\tilde{X}(T)}],\\
A_5(p,q) &= \mathrm{E}^Q[e^{p\vartheta_1\frac{1}{n}\sum_{j=1}^n\tilde{Y}_1(t_j)}],\\
A_6(p,q) &= \mathrm{E}^Q[e^{q\vartheta_2\tilde{Y}_2(T)}].
\end{aligned}$$

In the following, we deal with $A_2(p,q)$-$A_6(p,q)$ in turn. In $A_2(p,q)$, there are two correlated Brownian motions $\tilde{B}_1(t)$ and $\tilde{B}_2(t)$ with correlation coefficient ρ. We write $\tilde{B}_2(t)$ by introducing another standard Brownian motion $\tilde{B}_3(t)$, independent of $\tilde{B}_1(t)$. Based on the form of $\tilde{B}_2(t) = \rho\tilde{B}_1(t) + \sqrt{1-\rho^2}\tilde{B}_3(t)$, one gets that

$$\begin{aligned}
A_2(p,q) &= \mathrm{E}^Q[e^{p\frac{\vartheta_1\sigma_1}{n}\sum_{j=1}^n\tilde{B}_1(t_j)+q\vartheta_2\sigma_2\tilde{B}_2(T)}]\\
&= \mathrm{E}^Q[e^{p\frac{\vartheta_1\sigma_1}{n}\sum_{j=1}^n(n+1-j)\Delta\tilde{B}_1(t_j)+q\vartheta_2\sigma_2\sum_{j=1}^n(\rho\Delta\tilde{B}_1(t_j)+\sqrt{1-\rho^2}\Delta\tilde{B}_3(t_j))}]\\
&= \mathrm{E}^Q[e^{\sum_{j=1}^n[p\frac{\vartheta_1\sigma_1}{n}(n+1-j)+q\vartheta_2\sigma_2\rho]\Delta\tilde{B}_1(t_j)+\sum_{j=1}^n q\vartheta_2\sigma_2\sqrt{1-\rho^2}\Delta\tilde{B}_3(t_j)}]\\
&= \prod_{j=1}^n\mathrm{E}^Q[e^{[p\frac{\vartheta_1\sigma_1}{n}(n+1-j)+q\vartheta_2\sigma_2\rho]\Delta\tilde{B}_1(t_j)}]\cdot\mathrm{E}^Q[e^{q\vartheta_2\sigma_2\sqrt{1-\rho^2}\tilde{B}_3(T)}]\\
&= \prod_{j=1}^n e^{\frac{1}{2}[p\frac{\vartheta_1\sigma_1}{n}(n+1-j)+q\vartheta_2\sigma_2\rho]^2\frac{T}{n}}\cdot e^{\frac{1}{2}q^2\vartheta_2^2\sigma_2^2(1-\rho^2)T}\\
&= e^{\frac{1}{2}\sum_{j=1}^n[p\frac{\vartheta_1\sigma_1}{n}(n+1-j)+q\vartheta_2\sigma_2\rho]^2\frac{T}{n}+\frac{1}{2}q^2\vartheta_2^2\sigma_2^2(1-\rho^2)T}.
\end{aligned}$$

In $A_3(p,q)$, the term $\int_0^T a(s)\,\mathrm{d}\tilde{W}(s)$ is normally distributed in nature, and hence, its expectation and variance are zero and $\int_0^T a(s)^2\,\mathrm{d}s$, respectively. Therefore, it is easy to obtain the following result:

$$\begin{aligned}
A_3(p,q) &= \mathrm{E}^Q[e^{p\frac{\beta_1}{n}\sum_{j=1}^n\int_0^{t_j}a(s)\mathrm{d}\tilde{W}(s)+q\beta_2\int_0^T a(s)\,\mathrm{d}\tilde{W}(s)}]\\
&= \mathrm{E}^Q[e^{\sum_{j=1}^n p\frac{\beta_1}{n}(n+1-j)\int_{t_{j-1}}^{t_j}a(s)\mathrm{d}\tilde{W}(s)+\sum_{j=1}^n q\beta_2\int_{t_{j-1}}^{t_j}a(s)\,\mathrm{d}\tilde{W}(s)}]\\
&= \prod_{j=1}^n\mathrm{E}^Q[e^{[p\frac{\beta_1}{n}(n+1-j)+q\beta_2]\int_{t_{j-1}}^{t_j}a(s)\,\mathrm{d}\tilde{W}(s)}]\\
&= e^{\frac{1}{2}\sum_{j=1}^n[p\frac{\beta_1}{n}(n+1-j)+q\beta_2]^2\int_{t_{j-1}}^{t_j}a(s)^2\,\mathrm{d}s}.
\end{aligned}$$

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In $A_4(p,q)$ and $A_5(p,q)$, we meet the similar forms. Write $e^{\sum_{j=1}^{n}\tilde{X}(t_j)}$ in the form of $e^{\sum_{j=1}^{n}(n+1-j)\Delta\tilde{X}_j}$, and through the property of stationary and independent increments of Lévy processes, we get that

$$
\begin{aligned}
A_4(p,q) &= \mathrm{E}^{Q}\left[e^{p(\vartheta_1\delta_1+\beta_1\theta)\frac{1}{n}\sum_{j=1}^{n}\tilde{X}(t_j)+q(\vartheta_2\delta_2+\beta_2\theta)\tilde{X}(T)}\right] \\
&= \mathrm{E}^{Q}\left[e^{p(\vartheta_1\delta_1+\beta_1\theta)\frac{1}{n}\sum_{j=1}^{n}(n+1-j)\Delta\tilde{X}_j+q(\vartheta_2\delta_2+\beta_2\theta)\sum_{j=1}^{n}\Delta\tilde{X}_j}\right] \\
&= \prod_{j=1}^{n}\mathrm{E}^{Q}\left[e^{i[-ip(\vartheta_1\delta_1+\beta_1\theta)\frac{n+1-j}{n}-iq(\vartheta_2\delta_2+\beta_2\theta)]\Delta\tilde{X}_j}\right] \\
&= \prod_{j=1}^{n}\psi_{\Delta X}\left(-ip(\vartheta_1\delta_1+\beta_1\theta)\frac{n+1-j}{n}-iq(\vartheta_2\delta_2+\beta_2\theta)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
A_5(p,q) &= \mathrm{E}^{Q}\left[e^{p\vartheta_1\frac{1}{n}\sum_{j=1}^{n}\tilde{Y}_1(t_j)}\right] \\
&= \mathrm{E}^{Q}\left[e^{\sum_{j=1}^{n}p\frac{\vartheta_1}{n}(n+1-j)\Delta\tilde{Y}_j}\right] \\
&= \prod_{j=1}^{n}\psi_{\Delta Y}\left(-ip\frac{\vartheta_1}{n}(n+1-j)\right).
\end{aligned}
$$

Lastly, $A_6(p,q)$ can be simply written in the form of its characteristic function,

$$
\begin{aligned}
A_6(p,q) &= \mathrm{E}^{Q}\left[e^{q\vartheta_2 Y_2(T)}\right] \\
&= \psi_{Y_2}(-iq\vartheta_2).
\end{aligned}
$$

This completes the proof of Proposition 2.1. □

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