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A HOMOTOPICAL CONNER–RAYMOND THEOREM AND A QUESTION OF GOTTLIEB

BY

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ABSTRACT. A homotopy theoretic version is given of the following result of Conner and Raymond: If the circle acts on a space so that the orbit map induces an injection in homology, then the space fibres over the circle with finite structure group. This homotopical analogue is related to recent results pertaining to the effect of the fundamental group's structure on the Euler characteristic. It is also used in the construction of a compact, simple 7–manifold with trivial Gottlieb group which, together with an infinite dimensional example of Ganea, answers a question of Gottlieb.

Introduction. It was proved by D. Gottlieb [4] that, if the fundamental group of a finite aspherical polyhedron has nontrivial center, then the Euler characteristic of the polyhedron vanishes. The algebraic approach to Gottlieb's Theorem (due to Stallings [17]) in terms of finite projective complexes over group rings has led to recent generalizations by Rosset [16] and Eckmann [2]. In particular, Eckmann proves that, for a finite connected CW complex, if there exists a nontrivial torsionfree normal abelian subgroup of the fundamental group which acts nilpotently on the homology of the universal cover (by covering transformations), then the Euler characteristic vanishes.

In this paper we shall try to view such a situation in a fashion more akin to Gottlieb's topological outlook. More specifically, Corollary 7 is a particular case of Eckmann's theorem where the torsionfree normal abelian subgroup is generated by a Gottlieb element (§1) and the vanishing of the Euler characteristic is a "geometric" consequence. The motivation for the approach of Theorem 6 is a result of Conner and Raymond on the effect of the orbit map on the structure of an S^1–space. Indeed, the Conner–Raymond theorem itself may be derived from Theorem 6.

Finally, in [4], Gottlieb asked whether the Gottlieb group (see §1 for a definition) must necessarily be nontrivial in a simple space with nontrivial fundamental group. Ganea provided a negative answer to this question by constructing an infinite dimensional counterexample. The question remained of whether a finite counterexample could be found. In §4 such a space is constructed and, with the aid of Theorem 6, shown to be the desired counterexample.

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Throughout this paper, spaces will be connected and of the homotopy type of CW complexes. Hence, weak equivalences are homotopy equivalences and basepoints are nondegenerate.

After this paper was submitted, it was learned that Dan Gottlieb had independently discovered the torus splitting of Theorem 10 together with different and striking applications. His paper will appear in the Israel Journal of Mathematics [7]. In addition, Wolfgang Lück has proved a splitting theorem equivalent to theorem 6 case 1 [11]. His condition involves a splitting at the fundamental group level, whereas ours involves a splitting at the homology level.

1. **Preliminaries on the Gottlieb group.** In order to make this paper somewhat self–contained, in this section we recall basic definitions and properties of the Gottlieb group. We provide straightforward proofs of several properties essential to our main result. (also, see [4].)

Definition. *The Gottlieb Group of a space X, denoted $G(X)$, consists of all $\alpha \in \pi_1(X)$ such that there is an associated map $A : S^1 \times X \longrightarrow X$ and a homotopy commutative diagram,*

$$(*) \qquad \begin{tikzcd} S^1 \times X \arrow[r, "A"] & X \\ S^1 \vee X \arrow[u] \arrow[ur, "\alpha \vee 1_X"'] & \end{tikzcd}$$

Remark. (1) The types of spaces we are considering allow us, when convenient, to take the diagram above to be strictly commutative. Note also that we have abused notation by writing α for both the element of the fundamental group and a representing map.

(2) We have suppressed the basepoint because our spaces are connected and $G(X)$ is then independent of basepoints. Note, however, that $A(s_0, x_0) = x_0$.

(3) The fact that $G(X)$ is a subgroup of $\pi_1(X)$ is immediate from the following (Part (1)):

Theorem 1. *$G(X)$ is equal to: (1) $Im(ev_\# : \pi_1(X^X, 1_X) \longrightarrow \pi_1(X))$ where $ev : X^X \longrightarrow X$ is the evaluation map, $ev(f) = f(x_0)$ for a chosen basepoint $x_0 \in X$. (2) $\bigcup Im(\partial_\# : \pi_1(\Omega B) \longrightarrow \pi_1(X))$, where the union is taken over all fibrations $X \longrightarrow E \longrightarrow B$ and $\partial : \Omega B \longrightarrow X$ is the "transgression" in the Barratt–Puppe sequence of the fibration.*

Outline of Proof. (See [4] and [6]). (1) Given A as in $(*)$, define $\tilde{A} : S^1 \longrightarrow (X^X, 1_X)$ by $\tilde{A}(s)(x) = A(s,x)$. then $ev \cdot \tilde{A}(s) = \tilde{A}(s)(x_0) = A(s, x_0) = \alpha(s)$. Hence $ev_\#[\tilde{A}] = \alpha$. Now suppose $\tilde{A} : S^1 \longrightarrow (X^X, 1_X)$ gives $ev \cdot \tilde{A} \simeq \alpha$. Define $A : S^1 \times X \longrightarrow X$ by $A(s,x) = \tilde{A}(s)(x)$ and note: (i) $A(s_0, x) = x$ since $\tilde{A}(s_0) = 1_X$, (ii) $A(s, x_0) = \tilde{A}(s)(x_0) = ev \cdot \tilde{A}(s) \simeq \alpha(s)$. Hence a diagram $(*)$ is obtained. (2) Every fibration with fibre

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X is classified by a map $B \longrightarrow BautX$, where $autX$ denotes the monoid of self–homotopy equivalences of X and $BautX$ is its classifying space. We obtain a homotopy commutative diagram of Barratt–Puppe sequences.

$$\begin{tikzcd}
\cdots & \Omega BautX \arrow[r] & X \arrow[r] & Baut.X \arrow[r] & BautX \\
\cdots & \Omega B \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "\simeq"'] & E \arrow[r] \arrow[u] & B \arrow[u]
\end{tikzcd}$$

where the top row arises from the universal fibration with fibre X, $X \longrightarrow Baut.X \longrightarrow BautX$. Now, $\pi_i(\Omega BautX, *) \cong \pi_i(autX, 1_X) \cong \pi_i(X^X, 1_X)$, so clearly $Im\partial_\# \subseteq Imev_\#$. The universal fibration itself furnishes an example with $Im\partial_\# = Imev_\#$, so $G(X) = \bigcup Im\partial_\#$.

Although we shall not use the following result, we shall refer to it later. Also, it is quite useful in computing $G(X)$ for certain X. A proof may be found in [4].

THEOREM 2. *Let $\tilde{X}$ denote the universal cover of X endowed with the π_1X–space structure given by the identification of π_1X with covering transformations. Then $G(X)$ consists of all covering transformations which are equivariantly homotopic to the identity.*

The following two results, due to Gottlieb [4], are essential to the proof of our main result, Theorem 6. For the first, recall that the fundamental group acts on the higher homotopy groups as follows: given $\alpha \in \pi_1X$ and $\xi \in \pi_nX$, the nondegeneracy of the basepoint $s_0 \in S^n$ (i.e. $s_0 \longrightarrow S^n$ is a cofibration) allows a solution F to the diagram

$$\begin{tikzcd}
s_0 \arrow[rr] \arrow[dr] \arrow[dd] & & s_0 \times I \arrow[dl, "\alpha"'] \arrow[dd] \\
& X & \\
S^n \arrow[rr] \arrow[ur, "\xi"] & & S^n \times I \arrow[ul, "F"']
\end{tikzcd}$$

Then $F_1 : S^n \longrightarrow X$ represents $\alpha \cdot \xi$. It can be shown that this process is well defined and that the action satisfies the formula $\alpha \cdot \xi = [\alpha, \xi] + \xi$, where $[\ \ ]$ denotes the Whitehead product. In particular, if $n = 1$, then $\alpha \cdot \xi = \alpha\xi\alpha^{-1}$. This discussion leads to the following equivalent definitions:

$$\begin{aligned} P(X) &= \{\alpha \in \pi_1X \mid \alpha \cdot \xi = \xi \text{ for all } \xi \in \pi_*X\} \\ &= \{\alpha \in \pi_1X \mid [\alpha, \xi] = 0 \text{ for all } \xi \in \pi_*X\}. \end{aligned}$$

THEOREM 3. $G(X) \subseteq P(X)$.

PROOF. Let $\alpha \in G(X)$, $\xi \in \pi_n(X)$ and note that we may write $(*)$ as

$$\begin{tikzcd}
I \times X \arrow[r, "A"] & X \\
I \vee X \arrow[u] \arrow[ur, "\alpha \vee 1_x"'] &
\end{tikzcd}$$

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where $A(0,x) = A(1,x) = x$ and $A(t,x_0) = \alpha(t)$ (thought of as a map $I \longrightarrow X$ with $\alpha(0) = \alpha(1) = x_0$). We obtain

$$\begin{tikzcd}
s_0 \arrow[rr] \arrow[dr] \arrow[dd] & & s_0 \times I \arrow[dl, "\alpha"'] \arrow[dd] \\
& X & \\
S^n \arrow[ur, "\xi"] \arrow[rr] & & S^n \times I \arrow[ul, "F"']
\end{tikzcd}$$

with $F = A(\xi \times 1)$. Commutativity is obvious. Hence, F_1 represents $\alpha \cdot \xi$. But $F_1(s) = F(s,1) = A(1,\xi(s)) = \xi(s)$, so $\alpha \cdot \xi = \xi$.

Corollary 4. *$G(X) \subseteq Z\pi_1 X$, where $Z\pi$ denotes the center of π.*

Remark. It is easy to see that if $X = K(\pi, 1)$, then $G(X) = Z\pi$ [4]. For any $\alpha \in Z\pi$ simply take the map $\psi : \pi \times \mathbb{Z} \longrightarrow \pi$ defined by $\psi(x,n) = x\alpha^n$. The map ψ is a homomorphism precisely because $\alpha \in Z\pi$. Now realize ψ on the space level (where $S^1 = K(\mathbb{Z}, 1)$),

$$\begin{tikzcd}
K(\pi,1) \times S^1 \arrow[r, "\psi"] & K(\pi,1) \\
K(\pi,1) \vee S^1 \arrow[u] \arrow[ur, "1 \vee \alpha"'] &
\end{tikzcd}$$

and note that this is a diagram of type $(*)$.

Theorem 5. [4] *Let $p : \bar{X} \longrightarrow X$ be a covering map. If $p_{\#}(\alpha) \in G(X)$, then $\alpha \in G(\bar{X})$.*

Proof. Let $A : X \times I \longrightarrow X$ be a map associated to $p \cdot \alpha$. That is, A provides a diagram $(*)$, where again we assume $A(x,0) = A(x,1) = x$. Because p is (in particular) a fibration and $x_0 \in X$ is nondegenerate, the homotopy lifting property provides a solution $\bar{A}$ to the diagram.

$$\begin{tikzcd}
\bar{X} \times 0 \cup \bar{x}_0 \times I \arrow[rr, "1_{\bar{X}} \cup \alpha"] \arrow[d] & & \bar{X} \arrow[d, "p"] \\
\bar{X} \times I \arrow[urr, "\bar{A}"] \arrow[r, "p \times 1"'] & X \times I \arrow[r, "A"'] & X
\end{tikzcd}$$

That is, $\bar{A}(\bar{x},0) = \bar{x}$, $\bar{A}(\bar{x}_0,t) = \alpha(t)$ and $p\bar{A} = A \cdot (p \times 1)$. Now, $p\bar{A}_1 = A_1 p = p$ and $\bar{A}(\bar{x}_0, 1) = \alpha(1) = \bar{x}_0$, so $A_1 : \bar{X} \longrightarrow \bar{X}$ is a lifting of 1_X with $A_1(\bar{x}_0) = \bar{x}_0$. But $1_{\bar{X}}$ is also such a lifting and the uniqueness of liftings for covering maps implies $A_1 = 1_{\bar{X}}$. Hence we obtain,

$$\begin{tikzcd}
\bar{X} \times I \arrow[r, "A"] & \bar{X} \\
\bar{X} \vee I \arrow[u] \arrow[ur, "1_{\bar{X}} \vee \alpha"'] &
\end{tikzcd}$$

with $\bar{A}_0 = \bar{A}_1 = 1_{\bar{X}}$ which shows $\alpha \in G(\bar{X})$.

EXAMPLES

(1) If X is an H–space, then $G(X) = \pi_1 X$. Given $\alpha \in \pi_1 X$, an associated map $A : X \times S^1 \longrightarrow X$ is obtained via the multiplication: $X \times S^1 \xrightarrow{1\times\alpha} X \times X \xrightarrow{\mu} X$. For example, an n–torus T^n has $G(T^n) = \bigoplus_{i=1}^{n} \mathbb{Z}$.

(2) Applications of Theorem 2 give: $G(\mathbb{R}P^{2n}) = \{1\}$; $G(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}/2$; $G(L(p,q)) \cong \mathbb{Z}/p$, where $L(p,q)$ is the 3–dimensional lens space of type (p,q); G (Poincaré homology sphere) $\cong \mathbb{Z}/2$. For details see [4] and [10].

(3) The author recently has proven the following result, the details of which are too long to present here: If H is a finite group which acts freely and orientation preservingly on a sphere S^n, then $G(S^n/H) = ZH$. (Compare [10]).

2. **The Homopotical Conner–Raymond Theorem.** In this section we give our main result Theorem 6 and some immediate consequences. Furthermore, although our later application to Gottlieb's question will only require Theorem 6, we also extend the result to torus–splitting in Theorem 10.

THEOREM 6. *Let X be a space with $H_1(X;\mathbb{Z})$ finitely generated. If there exists $\alpha \in G(X)$ with Hurewicz image $h(\alpha)$ of infinite order, then there is a finite cyclic cover $\bar{X}$ of X with $\bar{X} \simeq Y \times S^1$.*

LEMMA. *Let $f : \pi \longrightarrow \mathbb{Z}$ be a group homomorphism and suppose there exists $\alpha \in Z\pi$ such that $Im(f) = n\mathbb{Z}$ is generated by $f(\alpha) = n$. Then $\pi \cong \mathbb{Z} \times K$, where $\mathbb{Z} = \langle\alpha\rangle$ and $K = Ker\, f$.*

PROOF. Define $\sigma : n\mathbb{Z} \longrightarrow \pi$ by letting $\sigma(n) = \alpha$ and then extending freely. Then $f\sigma(n) = n$, so we obtain a split short exact sequence of groups,

$$\begin{tikzcd} K \arrow[r] & \pi \arrow[r, "f"] & n\mathbb{Z} \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

Hence, π has a semidirect product structure, $\pi \cong n\mathbb{Z} \rtimes K \cong \sigma(n\mathbb{Z}) \rtimes K$, where the action of $\sigma(n\mathbb{Z}) = \langle\alpha\rangle$ on K is given by conjugation in π. However, $\alpha \in Z\pi$, so the action is trivial and the semidirect product reduces to a product $\pi \cong \langle\alpha\rangle \times K$.

PROOF OF THEOREM 6. Let $H_1(X;\mathbb{Z}) = A \oplus T$, where the free part A has basis $a_1, \ldots a_k$ and T is finite. Then $h(\alpha)$ has the form (not all λ_i zero),

$$h(\alpha) = \lambda_1 a_1 + \ldots + \lambda_k a_k + t.$$

CASE 1. Suppose some $\lambda_i = 1$; without loss of generality let $i = 1$. Identify $H_1(S^1;\mathbb{Z})$ with $\mathbb{Z}$ and define $\Phi : H_1(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$ by $\Phi(a_1) = 1$, $\Phi(a_i) = 0$ for $1 > 1$ and $\Phi(T) = 0$. The bijections $Hom(H_1(X;\mathbb{Z}),\mathbb{Z}) \cong H^1(X;\mathbb{Z}) \cong [X, K(\mathbb{Z},1)] \cong [X, S^1]$

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provide a map $\phi : X \longrightarrow S^1$ with $\phi_* = \Phi$. Now, we again write $\alpha : S^1 \longrightarrow X$ to represent $\alpha \in G(X)$ and h to denote the Hurewicz map. We obtain

$$1 = \Phi h(\alpha) = \Phi\alpha_*(1) = \phi_*\alpha_*(1) = (\phi\alpha)_*(1).$$

Now, $[S^1, S^1] \cong Hom(H_1S^1, H_1S^1) \cong Hom(\mathbb{Z}, \mathbb{Z})$, so $\phi \cdot \alpha \simeq 1_{S^1}$. Therefore, there is a fibration ϕ with homotopy section α and homotopy fibre Y,

$$\begin{tikzcd} Y \arrow[r, "i"] & X \arrow[r, "\phi"] & S^1 \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

The splitting α provides isomorphisms $i_\# : \pi_i Y \cong \pi_i X$ for $i > 2$ and $\pi_1 X \cong \pi_1 Y \rtimes \mathbb{Z}$ (where $\rtimes$ denotes the semidirect product). However, by Corollary 4, $\alpha \in Z\pi_1 X$, so the action in the semidirect product is trivial. Hence, $\pi_1 Y \times \mathbb{Z} \cong \pi_1 X$ via $(a, b) \longmapsto i_\#(a)\alpha_\#(b)$.

By Theorem 1, since $\alpha \in G(X)$ there exists a fibration $X \longrightarrow E \longrightarrow B$ and $\beta \in \pi_1(\Omega B)$ with $\partial_\#(\beta) = \alpha$. We may now use the "action" of ΩB on X to "add up" Y and S^1. Specifically, recall (see [8]) that the action $c : X \times \Omega B \longrightarrow X$ induces $c_\#(x, y) = x\partial_\#(y)$ on π_1 (and $x + \partial_\#(y)$ on π_i, $i > 1$). Hence the composition

$$Y \times S^1 \xrightarrow{i \times \beta} X \times \Omega B \xrightarrow{c} X$$

gives $c_\#(i_\# \times \beta_\#) = i_\# \cdot \partial_\#\beta_\# = i_\# \cdot \alpha_\#$, since $\partial_\#(\beta) = \alpha$. Thus, $c(i \times \beta)$ induces isomorphisms on homotopy groups and so is a homotopy equivalence, $Y \times S^1 \simeq X$.

CASE 2. Suppose $\lambda_i \neq 1$ for all i. By hypothesis some $\lambda_i = n \neq 0$. Without loss of generality, assume $i = 1$. Define Φ and obtain $\phi : X \longrightarrow S^1$ as in Case 1. Note that in this case we have,

$$(\phi\alpha)_*(1) = \Phi\alpha_*(1) = \Phi h(\alpha) = n.$$

Let $p : \mathbb{Z} \longrightarrow \mathbb{Z}/n$ be projection, $H = Ker(p\Phi h)$ and $\bar{X}$ be the cover of X corresponding to H.

Clearly $\alpha \in H$ and, by Theorem 5, $\alpha \in G(\bar{X})$ as well. By the definition of H, $\Phi h : H \longrightarrow \mathbb{Z}$ has $Im(\Phi h) = n\mathbb{Z} = \langle \Phi h(\alpha) \rangle$. By the Lemma, $H \cong \mathbb{Z} \times K$, where $\mathbb{Z} = \langle \alpha \rangle$ and $K = Ker(\Phi h)$. Because $\pi_1 \bar{X} = H$, we have $H_1(\bar{X}; \mathbb{Z}) = \mathbb{Z} \times K_{ab}$, where $\mathbb{Z} = \langle h(\alpha) \rangle$. That is, $H_1(\bar{X}; \mathbb{Z})$ has a free factor generated by the Hurewicz image of a Gottlieb element. This is sufficient to apply Case 1. (Indeed, the assumption that $H_1(X; \mathbb{Z})$ be finitely generated was made to ensure this!) Hence $\bar{X} \simeq Y \times S^1$.

REMARKS. (1) Note that the result of Case 1 is interesting in itself: If $\alpha \in G(X)$ and $h(\alpha)$ is a generator in the free part of $H_1(X)$, then $X \simeq Y \times S^1$.

(2) A higher dimensional localized version of Theorem 6 is given in [12]: If $\alpha \in G_n(X)$ (the n^{th} Gottlieb group) and $h(\alpha)$ has infinite order, then $X \simeq_p Y \times S^n$ for almost all primes p.

(3) See [13] and [14] for rational splittings of this type.

COROLLARY 7. *If X is a finite complex which satisfies the hypotheses of Theorem 6, then the Euler characteristic of X vanishes.*

PROOF. First, note that, because $\bar{X} \longrightarrow X$ is a finite covering, we have the formula: $\chi(\bar{X}) = n\chi(X)$. But $\chi(\bar{X}) = \chi(Y) \cdot \chi(S^1) = \chi(Y) \cdot 0 = 0$. Hence, $\chi(X) = 0$ as well.

REMARK. An α as in Theorem 6 generates a torsionfree normal abelian subgroup which acts nilpotently (in fact, trivially by Theorem 2) on $H_*(\tilde{X})$. Hence, Theorem 6 and Corollary 7 together may be thought of as a special case of Eckmann's result (see the Introduction), where the underlying reason for the vanishing of the euler characteristic is geometrically evident.

Theorem 6 also furnishes an amusing proof of,

COROLLARY 8. *If X is an H–space and $H_1(X;\mathbb{Z})$ has a $\mathbb{Z}$–summand, then $X \simeq Y \times S^1$.*

PROOF. By example (1) following Theorem 5, we have $G(X) = \pi_1 X$. Since the Hurewicz map is surjective in degree 1, the generator of the $\mathbb{Z}$–summand is hit by a Gottlieb element. Remark (1) following Theorem 6 now applies.

We also have the following group theoretic consequence of Theorem 6.

COROLLARY 9. *Let π be a group with finitely generated abelianization. If there is an element in the center of π whose image under abelianization has infinite order, then π contains a subgroup of finite index $\tilde{\pi}$ with $\tilde{\pi} \cong \mathbb{Z} \times K$.*

We now give an extension of Theorem 6 to torus splitting. Because the "covering argument" notation becomes unwieldy, we restrict ourselves to the following version:

THEOREM 10. *If $h(G(X))$ contains a free summand of $H_1(X;\mathbb{Z})$ of rank n, then $X \simeq Y \times T^n$.*

PROOF. Let $\alpha_1, \dots \alpha_n \in G(X)$ with $\langle h(\alpha_1), \dots h(\alpha_n)\rangle = \bigoplus_{i=1}^{n} \mathbb{Z}$ a direct summand of $H_1(X)$. For each α_j, define Φ_j as in the proof of Theorem 6 and obtain $\phi_j : X \longrightarrow S^1$ with $\phi_j\alpha_j \simeq 1_{S^1}$ (where we think of $\alpha_j : S^1 \longrightarrow X$). The product map is then,

$$Y \longrightarrow X \xrightarrow{\phi} T^n = S^1 \times \cdots \times S^1 \quad (n\text{–times}),$$

where Y = homotopy fibre of ϕ.

As before, we need a splitting $T^n \longrightarrow X$. Consider $\alpha_1 \vee \dots \vee \alpha_n : S^1 \vee \dots \vee S^1 \longrightarrow X$. Since each $\alpha_i \in G(X)$, we may apply Theorem 1 to obtain $\tilde{\alpha}_i \in \pi_1(\Omega BautX)$ with $\partial_\#(\tilde{\alpha}_i) = \alpha_i$. This allows the formation of a homotopy commutative diagram (with $B = BautX$),

$$\begin{array}{ccccc}
T^n = S^1 \times \cdots \times S^1 & \xrightarrow{A} & \Omega B \times \cdots \times \Omega B & \xrightarrow{m} & \Omega B \\
\uparrow & & & & \downarrow \partial \\
S^1 \vee \cdots \vee S^1 & & \xrightarrow{\alpha_1 \vee \cdots \vee \alpha_n} & & X
\end{array}$$

where $A = \tilde{\alpha}_1 \times \ldots \times \tilde{\alpha}_n$ and m is the multiplication of ΩB. Hence, $\partial m A = \alpha$ is the required splitting, $\phi\alpha \simeq 1_{T^n}$. (An equivalent formulation is to say that the higher order Whitehead product associated to $\alpha_1 \vee \ldots \vee \alpha_n$ vanishes: compare [18].)

As before, we may now use the holonomy of the universal fibration to "add up" the homotopy groups of T^n and Y:

$$T^n \times Y \longrightarrow \Omega BautX \times X \longrightarrow X$$

induces an isomorphism on π_*, so $X \simeq Y \times T^n$.

3. **The Conner–Raymond Theorem.** In this section we shall give a proof of the Conner–Raymond Theorem based on Theorem 6 (and the following Remark). Recall,

THEOREM 11 (Conner–Raymond [1]). *If S^1 acts on X so that the orbit map $\omega : S^1 \longrightarrow X$ induces an injection $\omega_* : H_1(S^1) \longrightarrow H_1(X)$ onto a $\mathbb{Z}$–summand, then*

$$X \cong_{\text{homeo}} (X/S^1) \times S^1$$

and the action is on the 2^{nd} factor.

Before we give Gottlieb's proof of Theorem 11, we recall how the orbit map is related to $G(X)$.

Let G act on X and let $\omega : G \longrightarrow X$ be defined by $\omega(g) = g \cdot x_0$, for fixed $x_0 \in X$. The action gives rise to the associated Borel fibration and, hence, to the Barratt–Puppe sequence,

$$\cdots \longrightarrow \Omega BG \xrightarrow{\partial} X \longrightarrow X \times_G EG \longrightarrow BG.$$

The following Lemma shows that $\omega_\#(\pi_* G) = \partial_\#(\pi_* \Omega BG)$, so $Im\omega_{\#1} \subseteq G(X)$ by Theorem 1.

LEMMA 12. *The following diagram homotopy commutes:*

$$\begin{tikzcd} \Omega BG \arrow[r, "\partial"] \arrow[d, "\simeq"'] & X \\ G \arrow[ur, "\omega"'] & \end{tikzcd}$$

PROOF. In the following diagram each column is a fibration and the maps $G \longrightarrow G \times X$, $EG \longrightarrow EG \times X$ are the obvious inclusions into the first factors with second coordinate $x_0 \in X$. Commutativity is easily checked. ($\bar{\omega}$ denotes the action.)

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G \times X \arrow[r, "\bar{\omega}"] \arrow[d] & X \arrow[d] \\ EG \arrow[r] \arrow[d] & EG \times X \arrow[r] \arrow[d] & EG \times_G X \arrow[d] \\ BG \arrow[r, "1"] & BG \arrow[r, "1"] & BG \end{tikzcd}$$

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Observe that the composition of maps in the top row is ω. The mappings of fibrations now provide a homotopy commutative diagram one step back in the Barratt–Puppe sequence:

$$\begin{tikzcd} \Omega BG \arrow[r, "1"] \arrow[d, "\simeq"'] & \Omega BG \arrow[r, "1"] \arrow[d] & \Omega BG \arrow[d, "\partial"] \\ G \arrow[r] \arrow[rr, bend right, "\omega"'] & G \times X \arrow[r, "\bar{\omega}"] & X \end{tikzcd}$$

PPROOF OF THEOREM 11. The orbit of the action is clearly S^1/Isotropy. But since ω_* is *onto* a $\mathbb{Z}$–summand (generated by the orbit), it must be the case that S^1 acts freely. Hence, we obtain a principal bundle $S^1 \longrightarrow X \longrightarrow X/S^1$. Now, by Theorem 6, we have $X \simeq Y \times S^1$, where the S^1–factor is split off via the orbit map ω. This means that in the principal bundle, X homotopy retracts onto S^1 and $\pi_*(X) \cong \pi_*(X/S^1) \times \pi_*(S^1)$ compatible with $\pi_*(X) \cong \pi_*(Y) \times \pi_*(S^1)$. Hence the composition $Y \longrightarrow X \longrightarrow X/S^1$ induces isomorphisms on homotopy groups and consequently, $Y \simeq X/S^1$. This provides a homotopy section of the bundle, $X/S^1 \longrightarrow X$. The homotopy lifting property may now be applied to obtain a true section and, therefore, the bundle is trivial. Thus $X \cong (X/S^1) \times S^1$ and the action is translation on the 2^{nd} factor.

REMARK. Of course, the torus version of Theorem 11 holds as well.

4. **An Answer to a Question of Gottlieb.** In ([4] p. 846) Gottlieb asked whether the inclusion $G(X) \subseteq P(X)$ (with notation as in §1) is, in fact, an equality. Tudor Ganea provided a negative answer to this question in [3] by constructing the following infinite dimensional example and showing that $G(X) = \{1\}$ and $P(X) = \pi_1 X = \mathbb{Z}/2$.

EXAMPLE. The Ganea space X is the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] & PK \arrow[d] \\ \mathbb{R}P(\infty) \arrow[r, "i"] & K(\mathbb{Z}/2,3) \end{tikzcd}$$

where $PK \longrightarrow K(\mathbb{Z}/2,3)$ is the path fibration and i is the nontrivial class in $H^3(\mathbb{R}P(\infty); \mathbb{Z}/2) \cong \mathbb{Z}/2$.

Of course, the one unsatisfying aspect of Ganea's example is its infinite dimensionality. Indeed, it can be shown that any finite approximation of X loses its salient properties. It seemed conceivable, then, that the restrictiveness of requiring finiteness might force $G(X) = P(X)$. The following result, whose proof makes essential use of Theorem 6, provides a finite example with $G(X) \neq P(X)$.

THEOREM 13. *There exists a compact 7–dimensional manifold* X *with* $P(X) = \pi_1 X = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ *and* $G(X) = \{1\}$.

REMARK. It was originally thought that a finite nilpotent complex X with $\pi_1 X \neq \{1\}$ would also have $G(X) \neq \{1\}$. This notion is easily disposed of in several ways. The

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most straightforward I owe to Bob Oliver: Let L_p denote the standard 3–dimensional lens space and M_p the Moore space constructed as the cofibre of a degree p map $S^3 \longrightarrow S^3$. Let

$$X = L_p \vee M_p$$

and note that $\pi_1 \cong \mathbb{Z}/p$ by Van Kampen's Theorem. Also, the universal cover $\tilde{X}$ has the form of S^3 with p copies of M_p attached at points in a $\mathbb{Z}/p$–orbit of the basepoint. Hence,

$$\tilde{H}_i(\tilde{X}) = \begin{cases} \mathbb{Z} + (\mathbb{Z}/p)^p & i = 3 \\ 0 & \text{otherwise.} \end{cases}$$

The action of $\pi_1 X$ on $\tilde{H}_i(\tilde{X})$ is trivial on the $\mathbb{Z}$–factor (since the covering transformations on $\tilde{L}_p = S^3$ are homotopic to the identity) and the nontrivial action on $(\mathbb{Z}/p)^p$ is by permutation of factors. Now, any action of a p–group on another p–group is nilpotent. Hence, X is a finite nilpotent complex with $\pi_1 X = \mathbb{Z}/p$, but $G(X) = \{1\}$ (by Theorem 2, since no element of $\pi_1 X$ acts trivially on $\tilde{H}_i(X)$).

PROOF OF THEOREM 13. Let T^4 denote the 4–torus $S^1 \times S^1 \times S^1 \times S^1$ and let $k : T^4 \longrightarrow S^4$ be a map of degree 1. Note that such a map exists by the Hopf Classification and that we may assume k is smooth since any homotopy class of maps between manifolds has a smooth representative. We define X to be the principal bundle induced via k from the Hopf fibration $S^7 \longrightarrow S^4$. We then have a principal bundle $S^3 \longrightarrow X \longrightarrow T^4$.

Now, the action of $\pi_1 X \cong \pi_1 T^4 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ on $\pi_i X \cong \pi_i S^3$ is trivial because S^7 is simply connected and the action on the homotopy groups of the fibre S^3 is natural with respect to the map of fibrations $X \longrightarrow S^7$.

Hence, X is a simple space, i.e. $P(X) = \pi_1 X$.

Of course, X is compact because it is a closed subspace of the compact manifold $T^4 \times S^7$. The usual local triviality arguments show that it is in fact a 7–dimensional manifold (since k is assumed smooth).

In order to complete the proof, we must show that $G(X) = \{1\}$. Suppose this is not the case. Then there exists $\alpha \in G(X)$ and, because $h : \pi_1 X \xrightarrow{\cong} H_1(X; \mathbb{Z})$, $h(\alpha)$ has infinite order. By our main result Theorem 6, there is a finite covering $\bar{X} \longrightarrow X$ with $\bar{X} \simeq Y \times S^1$.

Now $\bar{X}$ is a covering of the simple space X, so it is simple as well. Furthermore, the covering map provides $\pi_1 \bar{X} \cong \pi_i X$ for $i \geqq 2$ and a short exact sequence $\pi_1 \bar{X} \longrightarrow \pi_1 X \longrightarrow \mathbb{Z}/n$. After tensoring with $\mathbb{Q}$, we obtain $\pi_i \bar{X} \otimes \mathbb{Q} \cong \pi_i X \otimes \mathbb{Q}$. for all i. Hence, the rationalizations of $\bar{X}$ and X are homotopy equivalent (see [9], for example, for an exposition of localization theory); $\bar{X}_0 \simeq X_0$. Hence, $X_0 \simeq Y_0 \times S_0^1$ since localization commutes with products. The homotopy groups of Y_0 are then: $\pi_1 Y_0 \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$, $\pi_3 Y_0 \cong \mathbb{Q}$ and $\pi_i Y_0 = 0$ otherwise (since $\pi_i S^3 \otimes \mathbb{Q} = 0$ for $i \neq 3$). Therefore, a Postnikov Tower for Y_0 is given by,

$$\begin{array}{ccc} Y_0 & \longrightarrow & PK \\ \downarrow & & \downarrow \\ T_0^3 & \xrightarrow{\ell} & K(\mathbb{Q}, 4) \end{array}$$

However, $\ell \in [T_0^3, K(\mathbb{Q},4)] \cong H^4(T_0^3;\mathbb{Q}) = 0$, so $\ell \simeq$ constant. Hence, $Y_0 \simeq T_0^3 \times K(\mathbb{Q},3) \simeq T_0^3 \times S_0^3$. But this then implies,

$$X_0 \simeq T_0^3 \times S_0^3 \times S_0^1 \simeq T_0^4 \times S_0^3.$$

This could only happen if the defining map $k : T^4 \longrightarrow S^4$ were rationally trivial. This, of course, is not the case since k has degree 1, so we have arrived at a contradiction. Therefore, $G(X) = \{1\}$.

REMARK. (1) Note that the universal cover of X is homotopy equivalent to S^3 and that the covering transformations are all homotopic to the identity. Thus $G(X) = \{1\}$ does not immediately follow as in Oliver's example (following Theorem 13).

(2) The interested reader should consult [18] for further examples relating Gottlieb groups, Whitehead products and H-space structures.

QUESTION. Besides usual examples such as H-spaces, $K(\pi, 1)$'s with $\mathbb{Z}\pi \neq \{1\}$, Lens spaces or total spaces of principal bundles over simply connected bases (see [15]), is there a class of finite complexes characterized by homotopical conditions whose members have nontrivial Gottlieb groups?

REFERENCES

1. P. Conner and F. Raymond, *Injective operations of the toral groups*, Top. 10 (1971), 283–296.
2. B. Eckmann, *Nilpotent group action and euler characteristic*, Springer Lecture Notes Math. vol. 1298 (1987), 120–123.
3. T. Ganea, *Cyclic homotopies*, Ill. J. Math. 12 (1968), 1–4.
4. D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. 87 (1965), 840–856.
5. ———, *Evaluation subgroups of homotopy groups*, Amer. J. Math. 91 (1969), 729–755.
6. ———, *On fibre spaces and the evaluation map*, Ann. of Math. (2) 87 No. 1 (1968), 42–55.
7. ———, *Splitting off tori and the evaluation subgroup*, Israel Journal of Mathematics, Vol. 66, (1989), 216–222.
8. P. Hilton, *Homotopy theory and duality*, Gordon & Breach, New York 1965.
9. P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North Holland, Amsterdan 1975.
10. G. Lang, *Evaluation subgroups of factor spaces*, Pac. J. Math. 42 No. 3 (1972), 701–709.
11. Wolfgang Lück, *The transfer maps induced in the algebraic K_0– and K_1–groups by a fibration II*, J. Pure Appl. Algebra 45(1987), 143–169.
12. J. Oprea, *Decompositions of localized fibres and cofibres*, Canad. Bull. Math. vol. 31(4) (1988), 424–431.
13. ———, *Decomposition theorems in rational homotopy theory*, Proc. Amer. Math. Soc. 96 (1986), 505–512.
14. ———, *The Samelson space of a fibration*, Mich. Math. J. 34 (1987), 127–141.
15. J. Pak, *On the fibered Jiang spaces*, Comtemporary Math. 72(1988), 179–181.
16. S. Rosset, *A vanishing theorem for euler characteristics*, Math. Zeit. 185 (1984), 211–215.
17. J. Stallings, *Centerless groups – an algebraic formulation of Gottlieb's theorem*, Top. **4** (1965), 129–134.
18. J. Siegel, *G-spaces, H-spaces and W-spaces*, Pac. J. Math. 31 no. 1 (1969), 209–214.

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