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LOCALLY UNIFORMLY ROTUND RENORMING AND DECOMPOSITIONS OF BANACH SPACES

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A norm $|\cdot|$ of a Banach space X is called locally uniformly rotund if $\lim|x_n-x| = 0$ whenever x_n, $x \in X$, and $\lim 2|x|^2 + 2|x_n|^2 - |x+x_n|^2 = 0$. It is shown that such an equivalent norm exists on every Banach space X which possesses a projectional resolution $\{P_\alpha\}$ of the identity operator, for which all $(P_{\alpha+1}-P_\alpha)X$ admit such norms. This applies, for example, for the dual space of a space with Fréchet differentiable norm.

Projectional resolutions of the identity in nonseparable Banach spaces were first studied by Amir and Lindenstrauss [1]. It has proved to be a very powerful tool in the study of geometry of some nonseparable Banach spaces. For example, it was used to prove that every weakly compactly generated Banach space admits an equivalent locally uniformly rotund norm in [6], and that the dual space of a space with Fréchet smooth norm admits an equivalent strictly convex norm in [4]. In the projectional resolutions used in [6], the spaces $(P_{\alpha+1}-P_\alpha)X$ were all separable. Sometimes one

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cannot ensure this separability requirement, but only the fact that $(P_{\alpha+1}-P_\alpha)X$ all admit equivalent locally uniformly rotund norms. Then, a slight variant of the construction from [6], given here, works to give the result. This is, for example, the case mentioned in the abstract.

We will work in real Banach spaces. N will denote the set of all positive integers.

THEOREM 1. *Let* $(X, |\cdot|)$ *be a Banach space which possesses a family* $\{P_\alpha\}$ *,* $\alpha \in \Gamma$ *, of bounded linear operators* $P_\alpha : X \to X$ *, such that*

(i) the map T *defined on* X *by* $Tx(\gamma) = |P_\gamma x|$ *for* $x \in X$ *, maps* X *into* $c_0(\Gamma)$ *,*

(ii) if $x \in X$ *, then* $x \in \overline{\text{sp}}\{P_\alpha x\}$ *,*

(iii) for each $\alpha \in \Gamma$ *,* $P_\alpha X$ *admits an equivalent locally uniformly rotund norm.*

Then X *admits an equivalent locally uniformly rotund norm.*

An application of Theorem 1 is given below.

COROLLARY 1. *Suppose that* X *is a Banach space which admits a real valued continuously Fréchet differentiable function with bounded non-empty support. Then* X^* *admits an equivalent locally uniformly rotund norm.*

Proof. Using the method of [5], it was shown in [4] that under our assumption on X , the identity operator on X^* admits a projectional resolution $\{P_\alpha\}$ for which each $P_\alpha X^*$ is isometric to some X^* where $\text{dens } X_\alpha = \text{dens } X_\alpha^* < \text{dens } X = \text{dens } X^*$. So, using a transfinite induction argument on $\text{dens } X$, one can ensure the existence of operators needed in Theorem 1, by the projections $(P_{\alpha+1}-P_\alpha)$ on X^* . This proves Corollary 1 from Theorem 1.

Proof of Theorem 1. A variant of that in [6]. Assume, without loss of generality, that $|P_\alpha| \le 1$, $\alpha \in \Gamma$. Let $h_\alpha(x) = |P_\alpha x|_\alpha$ where $\alpha \in \Gamma$ and $x \in X$, and where $|\cdot|_\alpha$ is an equivalent locally uniformly rotund norm on $P_\alpha X$ such that $|\cdot| \le |\cdot|_\alpha \le 2|\cdot|$. Furthermore, for $k \in N$, let

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$r_j^k = \left(r_{j,1}^k, \ldots, r_{j,k}^k\right)$, $j = 1, 2, \ldots$ be a sequence of all (ordered) k-tuples of rational numbers. Let Q be a map which assigns to each finite subset $A \subset \Gamma$ an enumeration of A to an (ordered) sequence $Q(A) = \left(\alpha_1, \ldots, \alpha_n\right)$, α_i distinct.

If $A \subset \Gamma$, where card $A = n$ and $Q(A) = \left(\alpha_1, \ldots, \alpha_n\right)$ and if $j, l \in N$ define

$$E(A, j, l)(x) = \left(\sum_{\alpha\in A} h_\alpha^2(x) + \frac{1}{l} \cdot 1/\left(\sum_{i=1}^{n} \left|r_{j,i}^n\right|+1\right)^2 \left|x - \sum_{i=1}^{n} r_{j,i}^n P_{\alpha_i} x\right|^2\right)^{\frac{1}{2}}$$

for $x \in X$. Furthermore, if $j, l, n \in N$ are fixed, let

$$G(j, l, n)(x) = \sup\{E(A, j, l)(x), A \subset \Gamma, \text{card } A = n\}$$

for $x \in X$.

Finally, let

$$\|x\| = \left(|x|^2 + \sum \frac{1}{2^{j+l+n}} G^2(j, l, n)(x)\right)^{\frac{1}{2}}$$

for $x \in X$. Then it is easy to see that $\|\cdot\|$ is an equivalent norm on X .

We shall show that it is locally uniformly rotund.

To this end suppose x_k , $x \in X$, are such that $|x| = 1$ and

$$\text{(1)} \qquad \lim 2\|x\|^2 + 2\|x_k\|^2 - \|x+x_k\|^2 = 0 .$$

We need show that $\lim|x_k-x| = 0$. Thus, given $\varepsilon > 0$, we shall show that beginning with some index k_0 , $|x_k-x| < 4\varepsilon$. For that, first find a set $A \subset \{\alpha \in \Gamma, h_\alpha(x) \neq 0\}$, $Q(A) = \left(\alpha_1, \ldots, \alpha_n\right)$ such that $\rho\left(x, \text{sp}\{P_\alpha x, \alpha \in A\}\right) < \varepsilon$.

Assume without loss of generality that

$$\min_{\alpha\in A} h_\alpha^2(x) - \max_{\alpha\notin A} h^2(x) = d > 0 .$$

Let

(2) $$\left| x - \sum_{s=1}^{n} r_s P_{\alpha_s} x \right| < \varepsilon$$

where $(r_1, \ldots, r_n)$ is a sequence of rationals.

Choose $j \in N$ so that $r_j^n = \left(r_{j,1}^n, \ldots, r_{j,n}^n\right) = (r_1, \ldots, r_n)$. Finally let $l > 4/d$. Fix these n, j, l. From (1) we have that

(3) $$a_k = 2G^2(j, l, n)(x) + 2G^2(j, l, n)(x_k) - G^2(j, l, n)(x+x_k) \xrightarrow[k]{} 0 .$$

Let $A_k \subset \Gamma$, card $A_k = n$, $Q(A_k) = \left(\alpha_1^k, \ldots, \alpha_n^k\right)$, be so that

(4) $$0 \le c_k = G^2(j, l, n)(x+x_k) - E^2(A_k, j, l)(x+x_k) \xrightarrow[k]{} 0 .$$

Then

$$\begin{aligned} a_k &\ge 2E^2(A_k, j, l)(x) + 2E^2(A_k, j, l)(x_k) - E^2(A_k, j, l)(x+x_k) - c_k \\ &= b_k - c_k \end{aligned}$$

for some nonnegative b_k and thus, since $\lim a_k = \lim c_k = 0$, we have that $\lim b_k = 0$ as well. Thus

$$\begin{aligned} b_k = 2 \sum_{\alpha\in A_k} h_\alpha^2(x) &+ \frac{2}{l}\left[1/\left(\sum_{i=1}^{n} \left|r_{j,i}^n\right| + 1\right)^2\right] \left| x - \sum_{i=1}^{n} r_{j,i}^n P_{\alpha_i^k} x \right|^2 \\ &+ 2 \sum_{\alpha\in A_k} h_\alpha^2(x_k) + \frac{2}{l}\left[1/\left(\sum_{i=1}^{n} \left|r_{j,i}^n\right| + 1\right)^2\right] \left| x_k - \sum_{i=1}^{n} r_{j,i}^n P_{\alpha_i^k} x_k \right|^2 \\ &- \sum_{\alpha\in A_k} h_\alpha^2(x+x_k) \\ &- \frac{1}{l} \cdot 1/\left(\sum_{i=1}^{n} \left|r_{j,i}^n\right| + 1\right)^2 \left| x+x_k - \sum_{i=1}^{n} r_{j,i}^n P_{\alpha_i^k}(x+x_k) \right|^2 \xrightarrow[k]{} 0 . \end{aligned}$$

Thus by the convexity argument,

(5) $$2 \sum_{\alpha\in A_k} h_\alpha^2(x) + 2 \sum_{\alpha\in A_k} h_\alpha^2(x_k) - \sum_{\alpha\in A_k} h_\alpha^2(x+x_k) \xrightarrow[k]{} 0$$

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and

$$(6) \qquad \left| x - \sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i^k} x \right| - \left| x_k - \sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i^k} x_k \right| \to 0 .$$

We now show that, beginning with some index k_1 , $A_k = A$ and so then $\left\{\alpha_1^k, \ldots, \alpha_n^k\right\} = \left\{\alpha_1, \ldots, \alpha_n\right\}$. Indeed, if this were not the case, we would have for infinitely many k , $A_k \neq A$. But for these k we would have

$$\begin{aligned}
a_k \geq\ & 2 \sum_{\alpha\in A} h^2(x) + \frac{2}{l}\left(1/\left(\sum_{i=1}^{n}\left|r_{j,i}^{n}\right|+1\right)^2\right)\left|x - \sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i} x\right|^2 + 2\sum_{\alpha\in A_k} h^2(x_k) \\
& + \frac{2}{l}\left(1/\left(\sum_{i=1}^{n}\left|r_{j,i}^{n}\right|+1\right)^2\right)\left|x_k - \sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i^k} x_k\right|^2 - \sum_{\alpha\in A_k} h_\alpha^2(x+x_k) \\
& - \frac{1}{l}\left(1/\left(\sum_{i=1}^{n}\left|r_{j,i}^{n}\right|+1\right)^2\right)\left|x + x_k - \sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i^k}(x+x_k)\right|^2 - c_k \\
=\ & 2\left(\sum_{\alpha\in A} h_\alpha^2(x) - \sum_{\alpha\in A_k} h_\alpha^2(x)\right) \\
& + \frac{2}{l}\left(1/\left(\sum_{i=1}^{n}\left|r_{j,i}^{n}\right|+1\right)^2\right)\left(\left|x - \sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i} x\right|^2 - \left|x - \sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i^k} x\right|^2\right) \\
& + 2\sum_{\alpha\in A_k} h_\alpha^2(x) + \frac{2}{l}\left(1/\left(\sum_{i=1}^{n}\left|r_{j,i}^{n}\right|+1\right)^2\right)\left|x - \sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i^k} x\right|^2 \\
& + 2\sum_{\alpha\in A_k} h_\alpha^2(x_k) + \frac{2}{l}\left(1/\left(\sum_{i=1}^{n}\left|r_{j,i}^{n}\right|+1\right)^2\right)\left|x_k - \sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i^k} x_k\right|^2 \\
& - \sum_{\alpha\in A_k} h_\alpha^2(x+x_k) - \frac{1}{l}\left(1/\left(\sum_{i=1}^{n}\left|r_{j,i}^{n}\right|+1\right)^2\right)\left|x + x_k - \sum r_{j,i}^{n} P_{\alpha_i^k}(x+x_k)\right|^2 - c_k \\
\geq\ & 2d - \frac{4}{l} - c_k \geq d - c_k ,
\end{aligned}$$

which is a contradiction with the fact that $\lim a_k = \lim c_k = 0$, $d > 0$.

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Therefore, beginning with some index k_1, we must have that $A_k = A$ and thus $\left(\alpha_1^k, \ldots, \alpha_n^k\right) = \left(\alpha_1, \ldots, \alpha_n\right)$.

Now from (5) we have that, for each $\alpha \in A$,

$$2h_\alpha^2(x) + 2h_\alpha^2\left(x_k\right) - h_\alpha^2\left(x+x_k\right) \xrightarrow[k]{} 0$$

and thus from the locally uniformly rotund of $|\cdot|_\alpha$, we have that $P_\alpha\left(x_k-x\right) \xrightarrow[k]{} 0$. Therefore

$$\left|\sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i} x - \sum_{i=1}^{n} r_{j,i}^{n} P_{\alpha_i} x_k\right| \xrightarrow[k]{} 0$$

and using this and (2) and (6) we have that beginning with some index k_0, we have that $\left|x-x_k\right| < 4\varepsilon$. The proof is finished.

REMARK 1. A version of Corollary 1 was published in [2]. However, we are afraid that the proof given in [2] is not correct.

REMARK 2. The locally uniformly rotund norm constructed in Corollary 1 on X^* cannot generally be made to be a dual one (see [5]).

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