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Global classical solution to the chemotaxis-Navier-Stokes system with some realistic boundary conditions

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In this paper, we consider the chemotaxis-Navier-Stokes model with realistic boundary conditions matching the experiments of Hillesdon, Kessler et al. in a two-dimensional periodic strip domain. For the lower boundary, we impose the usual homogeneous Neumann-Neumann-Dirichlet boundary condition. While, for the upper boundary, since it is open to the atmosphere, we consider three kinds of different mixed non-homogeneous boundary conditions, that is, (i) Neumann-Dirichlet-Navier slip boundary condition; (ii) Zero flux-Dirichlet-Navier slip boundary condition; (iii) Zero flux-Robin-Navier slip boundary condition. For boundary conditions (i) and (iii), the existence and uniqueness of global classical solutions for any initial data and any large chemotactic sensitivity coefficient is established, and for boundary condition (ii), the existence and uniqueness of global classical solutions for any initial data and small chemotactic sensitivity coefficient is proved.

Keywords: Chemotaxis-Navier-Stokes system; mixed nonhomogeneous boundary conditions; global classical solution; uniqueness

2020 *Mathematics Subject Classification:* 35M33; 35A09; 92C17

1. Introduction

When the well mixed suspension of *Bacillus subtilis* cells is placed in a chamber with the upper surface open to the atmosphere, this kind of aerobic bacteria consume oxygen, swim toward the direction of sufficient oxygen, that is, the surface of the water layer. Then they form a thin boundary layer with dense cells upstream. Below this layer, the cells in the suspension were severely depleted. Because that bacteria are about 10% denser than water, thus, the density of the mixed suspension becomes larger near the water surface than at the bottom. When the density of the upper boundary layer is too high, it becomes unstable and forms a descending bacterial plume. And finally evolved into various patterns [10, 11, 14]. Based on these experimental observations, Goldstein et al. [19] proposed the following chemotaxis-fluid model

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \Delta n - \chi\nabla\cdot(n\nabla c), & \text{in } Q,\\ c_t + \mathbf{u}\cdot\nabla c = \Delta c - cn, & \text{in } Q,\\ \mathbf{u}_t + \mathbf{u}\cdot\nabla \mathbf{u} = \Delta \mathbf{u} - \nabla\pi + n\nabla\varphi, & \text{in } Q,\\ \nabla\cdot\mathbf{u} = 0, & \text{in } Q, \end{cases} \tag{1.1}$$

where $Q = \Omega\times\mathbb{R}^+$, n, c represent the bacteria cell density, the oxygen concentration respectively, $\chi > 0$ is the sensitivity coefficient of aggregation induced by the concentration changes of oxygen, $J = n\nabla c$ is the chemotactic flux, $-cn$ is the consumption term of oxygen, that is more bacteria, more oxygen are consumed, $\mathbf{u}$, π are the fluid velocity and the associated pressure, the fluid couples to n and c through transports $\mathbf{u}\cdot\nabla n$, $\mathbf{u}\cdot\nabla c$ and the gravitational potential $n\nabla\varphi$.

In recent years, this kind of models has been widely studied by many authors. For the studies of Cauchy problem in $\mathbb{R}^N$, we refer to [**9**, **15**, **22**], and for the bounded domain with zero flux boundary value conditions for n, c, and no-slip boundary value condition for u, please refer to [**8**, **20**, **21**] or the references therein. From these results, one see that the solutions will convergence to the constant steady states, and there is no pattern formation. Matching the experiment descriptions, the following mixed boundary conditions is proposed [**6**, **16**]: the boundary condition at the top Γ_{top} describes the fluid-air surface, where there is no cell flux, the oxygen will be saturated with the air oxygen concentration c_{air} and the vertical fluid velocity and the tangential fluid stress are supposed to be zero

$$(\nabla n - \chi n\nabla c)\cdot\nu = 0, \quad c = c_{air}, \quad \mathbf{u}\cdot\nu = 0, \quad [D(\mathbf{u})\nu]_\tau = 0, \text{ on } \Gamma_T,$$

where ν denotes the outward unit normal vector of the boundary,τ denotes unit tangent vector of the boundary, and

$$[D(\mathbf{u})] = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T).$$

At the bottom of the domain Γ_B, the cell and oxygen fluxes and the fluid velocity are assumed to be zero:

$$(\nabla n - \chi n\nabla c)\cdot\nu = 0, \quad \nabla c\cdot\nu = 0, \quad \mathbf{u} = 0, \text{ on } \Gamma_B.$$

Finally, periodic boundary conditions at the sides of the domain are imposed in order to avoid any impact of these boundaries. And some numerical results are given in [**6**, **16**]. However, very little theoretical research in this regard has been carried out. Peng, Xiang [**18**] considered mixed boundary value problem in an unbounded strip domain of $\mathbb{R}^3$ with the consumption term cn being replaced by cn^γ $(\gamma \geqslant 2)$, and established the global existence and convergence of small strong solutions around an equilibrium state. Besides the above boundary value problem, such kind of mixed non-homogeneous boundary value problem

$$(\nabla n - \chi n\nabla c)\cdot\nu = 0, \quad \nabla c\cdot\nu = -a(x,t)c + b(x,t), \quad \mathbf{u} = 0, \text{ on } \partial\Omega$$

also be considered [**4**, **5**, **13**], in which, the global classical solution in two dimensional space [**4**, **5**] and time periodic solution in two and three dimensional space [**13**] are established respectively.

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Adopting the realistic boundary conditions mentioned above, in the present paper, we consider the model (1.1) in a two-dimensional strip periodic domain

$$\Omega = \{(x, y) \in \mathbb{R}^2; x \in \mathbb{R}, 0 \leqslant y \leqslant 1\}, \quad \Omega_l = \{(x, y) \in \mathbb{R}^2; 0 \leqslant x \leqslant l, 0 \leqslant y \leqslant 1\},$$

with period l and the following boundary conditions, that is

$$\begin{cases} n_y = 0,\ c = c_{air},\ v = 0,\ u_y = 0,\ (x, y) \in \Gamma_T, \\ n_y = c_y = 0,\ u = v = 0,\ (x, y) \in \Gamma_B, \end{cases} \tag{1.2}$$

or

$$\begin{cases} n_y - \chi n c_y = 0,\ \tau c_y = -c + c_{air},\ v = 0,\ u_y = 0,\ (x, y) \in \Gamma_T, \\ n_y = c_y = 0,\ u = v = 0,\ (x, y) \in \Gamma_B, \end{cases} \tag{1.3}$$

and periodic boundary conditions at the left and right sides of the domain Ω_l are imposed, where $\mathbf{u} = (u, v)$, Γ_T is the upper boundary of the rectangular area, Γ_B denotes the lower boundary, c_{air} is a positive constant, $\tau = 0$ or 1. It is easy to see that when $\tau = 0$, it is corresponding to the Dirichlet boundary condition, when $\tau = 1$, it is corresponding to the Robin boundary condition.

We also give the initial value as follows

$$\begin{aligned} n(x, y, 0) = n_0(x, y) \geqslant 0, \quad c(x, y, 0) = c_0(x, y) \geqslant 0, \quad \mathbf{u}(x, y, 0) \\ = \mathbf{u}_0(x, y), \quad (x, y) \in \Omega. \end{aligned} \tag{1.4}$$

Throughout this paper, we assume that

$$\begin{cases} n_0, c_0, \mathbf{u}_0 \in C^{2+\alpha}(\overline{\Omega}), \\ n_0, c_0 \geqslant 0, \\ n_0, c_0, \mathbf{u}_0 \text{ are periodic functions with respect to variable } x \text{ with period } l. \end{cases} \tag{1.5}$$

We give the global existence theorems as follows.

THEOREM 1.1. *Assume (1.5) holds. Then the problem (1.1), (1.2), and (1.4) admits a unique global bounded classical solution* $(n, c, \mathbf{u}, \pi)$ *(*π *is unique up to a constant) with* $(n, c, \mathbf{u}) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega}_l \times (0, +\infty)) \cap C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega}_l \times [0, +\infty))$, $\nabla\pi \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega}_l \times (0, +\infty))$, $n, c \geqslant 0$,

$$\|(n, c, \mathbf{u})\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(Q^l)} + \|\nabla\pi\|_{C^{\alpha, \frac{\alpha}{2}}(Q^l)} \leqslant C, \tag{1.6}$$

where the constant C *depends only on* n_0, c_0, $\mathbf{u}_0$, α, l, χ.

THEOREM 1.2. *Assume (1.5) holds. When (i)* $\tau = 0$ *with appropriately small* χ*; or (ii)*$\tau = 1$ *with any large* χ*, the problem (1.1), (1.3), and (1.4) admits a unique global bounded classical solution* $(n, c, \mathbf{u}, \pi)$ *(*π *is unique up to a constant) with* $(n, c, \mathbf{u}) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega}_l \times (0, +\infty)) \cap C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega}_l \times [0, +\infty))$, $\nabla\pi \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega}_l \times (0, +\infty))$, $n, c \geqslant 0$,

$$\|(n, c, \mathbf{u})\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(Q^l)} + \|\nabla\pi\|_{C^{\alpha, \frac{\alpha}{2}}(Q^l)} \leqslant C, \tag{1.7}$$

where the constant C *depends only on* n_0, c_0, $\mathbf{u}_0$, α, l, χ.

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2. Preliminaries

Based on Gagliardo-Nirenberg interpolation inequality and Sobolev trace embedding inequality, we infer the following trace interpolation inequality.

Lemma 2.1. *For functions $u:\Omega\to\mathbb{R}$ defined on a bounded Lipschitz domain $\Omega\in\mathbb{R}^N$, we have*

$$\|u\|_{L^q(\partial\Omega)}\leqslant C\|Du\|^{1-\beta}_{L^p(\Omega)}\|u\|^{\beta}_{L^r(\Omega)}+C\|u\|_{L^r(\Omega)}, \tag{2.1}$$

where $0<r\leqslant p\leqslant q\leqslant \frac{(N-1)p}{(N-p)_+}$, $p\geqslant 1$, $\beta=\frac{r(N(p-q)+p(q-1))}{q(N(p-r)+pr)}$.

Proof.

(i) We first consider the case $p<N$. By Sobolev trace embedding inequality [1, 3], we have

$$\|u\|_{L^{p^*}(\partial\Omega)}\leqslant C_1\|Du\|_{L^p(\Omega)}+C_2\|u\|_{L^p(\partial\Omega)},$$

where $p^*=\frac{(N-1)p}{N-p}$. On the other hand, for any $p\leqslant q\leqslant p^*$,

$$\|u\|_{L^q(\partial\Omega)}\leqslant \|u\|^{1-\alpha}_{L^{p^*}(\partial\Omega)}\|u\|^{\alpha}_{L^p(\partial\Omega)}$$

with $\alpha=\frac{p(p^*-q)}{q(p^*-p)}=\frac{(N-1)p-(N-p)q}{q(p-1)}$ for $p<N$. Combining the above two inequalities, and noticing that $(a+b)^\alpha\leqslant a^\alpha+b^\alpha$ for any $a,b>0$, $0<\alpha\leqslant 1$, then we have

$$\begin{aligned}\|u\|_{L^q(\partial\Omega)}&\leqslant \left(C_1\|Du\|_{L^p(\Omega)}+C_2\|u\|_{L^p(\partial\Omega)}\right)^{1-\alpha}\|u\|^{\alpha}_{L^p(\partial\Omega)}\\&\leqslant\left(C_1^{1-\alpha}\|Du\|^{1-\alpha}_{L^p(\Omega)}+C_2^{1-\alpha}\|u\|^{1-\alpha}_{L^p(\partial\Omega)}\right)\|u\|^{\alpha}_{L^p(\partial\Omega)}\\&\leqslant C_1^{1-\alpha}\|Du\|^{1-\alpha}_{L^p(\Omega)}\|u\|^{\alpha}_{L^p(\partial\Omega)}+C_2^{1-\alpha}\|u\|_{L^p(\partial\Omega)}.\end{aligned} \tag{2.2}$$

By [7], for any $1\leqslant r\leqslant p<+\infty$,

$$\begin{aligned}\|u\|_{L^p(\partial\Omega)}&\leqslant C_3\left(\|Du\|_{L^p(\Omega)}+\|u\|_{L^r(\Omega)}\right)^{\theta}\|u\|^{1-\theta}_{L^r(\Omega)}\\&\leqslant C_3\|Du\|^{\theta}_{L^p(\Omega)}\|u\|^{1-\theta}_{L^r(\Omega)}+\|u\|_{L^r(\Omega)}\end{aligned} \tag{2.3}$$

with $\theta=\frac{N(p-r)+r}{N(p-r)+pr}$. Combining (2.2) and (2.3), we get that

$$\begin{aligned}\|u\|_{L^q(\partial\Omega)}\leqslant{}& C_1^{1-\alpha}\|Du\|^{1-\alpha}_{L^p(\Omega)}\left(C_3\|Du\|^{\theta}_{L^p(\Omega)}\|u\|^{1-\theta}_{L^r(\Omega)}+\|u\|_{L^r(\Omega)}\right)^{\alpha}\\&+C_2^{1-\alpha}\left(C_3\|Du\|^{\theta}_{L^p(\Omega)}\|u\|^{1-\theta}_{L^r(\Omega)}+\|u\|_{L^r(\Omega)}\right)\\\leqslant{}& C_1^{1-\alpha}C_3^{\alpha}\|Du\|^{1-\alpha+\theta\alpha}_{L^p(\Omega)}\|u\|^{\alpha(1-\theta)}_{L^r(\Omega)}+C_1^{1-\alpha}\|Du\|^{1-\alpha}_{L^p(\Omega)}\|u\|^{\alpha}_{L^r(\Omega)}\end{aligned}$$

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$$
\begin{aligned}
&+ C_2^{1-\alpha} C_3 \|Du\|_{L^p(\Omega)}^{\theta} \|u\|_{L^r(\Omega)}^{1-\theta} + C_2^{1-\alpha}\|u\|_{L^r(\Omega)} \\
&= C_1^{1-\alpha} C_3^{\alpha} \|Du\|_{L^p(\Omega)}^{1-\alpha+\theta\alpha} \|u\|_{L^r(\Omega)}^{\alpha(1-\theta)} \\
&+ C_1^{1-\alpha} \|Du\|_{L^p(\Omega)}^{1-\alpha} \|u\|_{L^r(\Omega)}^{\frac{\alpha(1-\alpha)(1-\theta)}{1-\alpha+\theta\alpha}} \|u\|_{L^r(\Omega)}^{\frac{\alpha\theta}{1-\alpha+\theta\alpha}} \\
&+ C_2^{1-\alpha} C_3 \|Du\|_{L^p(\Omega)}^{\theta} \|u\|_{L^r(\Omega)}^{\alpha(1-\theta)\frac{\theta}{1-\alpha+\theta\alpha}} \|u\|_{L^r(\Omega)}^{\frac{(1-\theta)(1-\alpha)}{1-\alpha+\theta\alpha}} \\
&+ C_2^{1-\alpha}\|u\|_{L^r(\Omega)} \\
&\leqslant C_4 \|Du\|_{L^p(\Omega)}^{1-\alpha+\theta\alpha} \|u\|_{L^r(\Omega)}^{\alpha(1-\theta)} + C_5\|u\|_{L^r(\Omega)}.
\end{aligned}
$$

Let $\beta = \alpha(1-\theta)$, by a direct calculation, we see that

$$
\beta = \frac{r(N(p-q)+p(q-1))}{q(N(p-r)+pr)}, \quad \text{for } p < N.
$$

Then (2.1) is proved for $p < N$, $r \geqslant 1$.
Next, we show that (2.1) also holds for any $r > 0$. By the prove above, we see that

$$
\|u\|_{L^q(\partial\Omega)} \leqslant C\|Du\|_{L^p(\Omega)}^{1-\tilde{\beta}} \|u\|_{L^{\tilde{r}}(\Omega)}^{\tilde{\beta}} + C\|u\|_{L^{\tilde{r}}(\Omega)}, \tag{2.4}
$$

for $1 \leqslant \tilde{r} \leqslant p \leqslant q$, $\tilde{\beta} = \frac{\tilde{r}(N(p-q)+p(q-1))}{q(N(p-\tilde{r})+p\tilde{r})}$ for $p < N$. While by Gagliardo-Nirenberg inequality [**17**], for any $0 < r < 1$,

$$
\|u\|_{L^{\tilde{r}}(\Omega)} \leqslant C_6\|Du\|_{L^p(\Omega)}^{\alpha_1} \|u\|_{L^r(\Omega)}^{1-\alpha_1} + C_7\|u\|_{L^r(\Omega)},
$$

with $\frac{1}{\tilde{r}} = (\frac{1}{p} - \frac{1}{N})\alpha_1 + \frac{1-\alpha_1}{r}$. Substituting it into (2.4), we arrive at

$$
\begin{aligned}
\|u\|_{L^q(\partial\Omega)} &\leqslant C\|Du\|_{L^p(\Omega)}^{1-\tilde{\beta}} \left(C_6\|Du\|_{L^p(\Omega)}^{\alpha_1} \|u\|_{L^r(\Omega)}^{1-\alpha_1} + C_7\|u\|_{L^r(\Omega)} \right)^{\tilde{\beta}} \\
&\quad + C\left(C_6\|Du\|_{L^p(\Omega)}^{\alpha_1} \|u\|_{L^r(\Omega)}^{1-\alpha_1} + C_7\|u\|_{L^r(\Omega)} \right) \\
&\leqslant CC_6^{\tilde{\beta}} \|Du\|_{L^p(\Omega)}^{1-\tilde{\beta}+\alpha_1\tilde{\beta}} \|u\|_{L^r(\Omega)}^{\tilde{\beta}(1-\alpha_1)} + CC_7^{\tilde{\beta}} \|Du\|_{L^p(\Omega)}^{1-\tilde{\beta}} \|u\|_{L^r(\Omega)}^{\tilde{\beta}} \\
&\quad + CC_6\|Du\|_{L^p(\Omega)}^{\alpha_1} \|u\|_{L^r(\Omega)}^{1-\alpha_1} + CC_7\|u\|_{L^r(\Omega)} \\
&\leqslant C_8\|Du\|_{L^p(\Omega)}^{1-\tilde{\beta}+\alpha_1\tilde{\beta}} \|u\|_{L^r(\Omega)}^{\tilde{\beta}(1-\alpha_1)} + C_9\|u\|_{L^r(\Omega)}.
\end{aligned} \tag{2.5}
$$

By a direct calculation, we see that $\tilde{\beta}(1-\alpha_1) = \frac{r(N(p-q)+p(q-1))}{q(N(p-r)+pr)}$. Combining with (2.4), we prove (2.1) for any $r > 0$.

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(ii) Next, we prove the case $p \geqslant N$. For any $q \geqslant p$, there exists $\tilde{p} < N$ such that $q < \frac{(N-1)p}{N-\tilde{p}}$, namely $\frac{q\tilde{p}}{p} < \frac{(N-1)\tilde{p}}{N-\tilde{p}}$, then by the result of (i), we have

$$\begin{aligned}
\|u\|_{L^q(\partial\Omega)} &= \|u^{\frac{p}{\tilde{p}}}\|^{\frac{\tilde{p}}{p}}_{L^{\frac{q\tilde{p}}{p}}(\partial\Omega)} \leqslant C_{10}\|Du^{\frac{p}{\tilde{p}}}\|^{\frac{\tilde{p}}{p}(1-\gamma)}_{L^{\tilde{p}}(\Omega)}\|u^{\frac{p}{\tilde{p}}}\|^{\frac{\tilde{p}}{p}\gamma}_{L^{\tilde{p}}(\Omega)} + C_{11}\|u^{\frac{p}{\tilde{p}}}\|^{\frac{\tilde{p}}{p}}_{L^{\tilde{p}}(\Omega)} \\
&\leqslant C_{10}\left\|\frac{p}{\tilde{p}}u^{\frac{p-\tilde{p}}{\tilde{p}}}Du\right\|^{\frac{\tilde{p}}{p}(1-\gamma)}_{L^{\tilde{p}}(\Omega)}\|u\|^{\gamma}_{L^p(\Omega)} + C_{11}\|u\|_{L^p(\Omega)} \\
&\leqslant C_{12}\|u\|^{\frac{p-\tilde{p}}{p}(1-\gamma)}_{L^p(\Omega)}\|Du\|^{\frac{\tilde{p}}{p}(1-\gamma)}_{L^p(\Omega)}\|u\|^{\gamma}_{L^p(\Omega)} + C_{11}\|u\|_{L^p(\Omega)} \\
&= C_{12}\|u\|^{\frac{p-\tilde{p}}{p}(1-\gamma)+\gamma}_{L^p(\Omega)}\|Du\|^{\frac{\tilde{p}}{p}(1-\gamma)}_{L^p(\Omega)} + C_{11}\|u\|_{L^p(\Omega)}
\end{aligned}$$

where $\gamma = \frac{(N-1)p-q(N-\tilde{p})}{q\tilde{p}}$, it is easy to see that $\frac{p-\tilde{p}}{p}(1-\gamma)+\gamma = \frac{p(q-1)+N(p-q)}{pq}$. We denote $\tilde{\beta} = \frac{p(q-1)+N(p-q)}{pq}$. The above inequality is equivalent to

$$\|u\|_{L^q(\partial\Omega)} \leqslant C_{12}\|u\|^{\tilde{\beta}}_{L^p(\Omega)}\|Du\|^{1-\tilde{\beta}}_{L^p(\Omega)} + C_{11}\|u\|_{L^p(\Omega)} \tag{2.6}$$

Similar to the proof of (2.5), using Gagliardo-Nirenberg inequality [**17**], for any $0 < r < p$,

$$\|u\|_{L^p(\Omega)} \leqslant C_{13}\|Du\|^{\beta_1}_{L^p(\Omega)}\|u\|^{1-\beta_1}_{L^r(\Omega)} + C_{14}\|u\|_{L^r(\Omega)},$$

with $\frac{1}{p} = \left(\frac{1}{p} - \frac{1}{N}\right)\beta_1 + \frac{1-\beta_1}{r}$. Substituting it into (2.6), we arrive at

$$\begin{aligned}
\|u\|_{L^q(\partial\Omega)} &\leqslant C_{12}\left(C_{13}\|Du\|^{\beta_1}_{L^p(\Omega)}\|u\|^{1-\beta_1}_{L^r(\Omega)} + C_{14}\|u\|_{L^r(\Omega)}\right)^{\tilde{\beta}}\|Du\|^{1-\tilde{\beta}}_{L^p(\Omega)} \\
&\quad + C_{11}\left(C_{13}\|Du\|^{\beta_1}_{L^p(\Omega)}\|u\|^{1-\beta_1}_{L^r(\Omega)} + C_{14}\|u\|_{L^r(\Omega)}\right) \\
&\leqslant C_{12}C_{13}^{\tilde{\beta}}\|Du\|^{\beta_1\tilde{\beta}+1-\tilde{\beta}}_{L^p(\Omega)}\|u\|^{\tilde{\beta}(1-\beta_1)}_{L^r(\Omega)} + C_{12}C_{14}^{\tilde{\beta}}\|Du\|^{1-\tilde{\beta}}_{L^p(\Omega)}\|u\|^{\tilde{\beta}}_{L^r(\Omega)} \\
&\quad + C_{11}C_{13}\|Du\|^{\beta_1}_{L^p(\Omega)}\|u\|^{1-\beta_1}_{L^r(\Omega)} + C_{11}C_{14}\|u\|_{L^r(\Omega)} \\
&\leqslant C_{15}\|Du\|^{\beta_1\tilde{\beta}+1-\tilde{\beta}}_{L^p(\Omega)}\|u\|^{\tilde{\beta}(1-\beta_1)}_{L^r(\Omega)} + C_{16}\|u\|_{L^r(\Omega)}.
\end{aligned}$$

By a direct calculation, we get that $\tilde{\beta}(1-\beta_1) = \frac{r(N(p-q)+p(q-1))}{q(N(p-r)+pr)}$. We complete the proof.

□

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By [12], we have the following lemma.

LEMMA 2.2. *Let $T > 0$, $\tau \in (0,T)$, $\alpha > 0$, $\beta > 0$, and suppose that $f : [0,T) \to [0,\infty)$ is absolutely continuous, and satisfies*

$$f'(t) - g(t)f(t) + f^{1+\sigma}(t) \leqslant h(t), t \in \mathbb{R}^+,$$

where $\sigma > 0$ is a constant, $g(t), h(t) \geqslant 0$ with $g(t), h(t) \in L^1_{loc}([0,T))$ and

$$\sup_{t\in[\tau,T)} \int_{t-\tau}^{t} g(s)\,\mathrm{d}s \leqslant \alpha, \qquad \sup_{t\in[\tau,T)} \int_{t-\tau}^{t} h(s)\,\mathrm{d}s \leqslant \beta.$$

Then for any $t > t_0$, we have

$$f(t) \leqslant f(t_0)\,\mathrm{e}^{\int_{t_0}^{t} g(s)\,\mathrm{d}s} + \int_{t_0}^{t} h(\tau)\,\mathrm{e}^{\int_{\tau}^{t} g(s)\,\mathrm{d}s}\,\mathrm{d}\tau,$$

and

$$\sup_{t\in(0,T)} f(t) \leqslant \sigma\left(\frac{2A}{1+\sigma}\right)^{\frac{1+\sigma}{\sigma}} + 2B, \quad \sup_{t\in[\tau,T)} \int_{t-\tau}^{t} f^{1+\sigma}(s)\,\mathrm{d}s \leqslant (1+\alpha) \sup_{t\in(0,T)} \{f(t)\} + \beta,$$

where

$$A = \tau^{-\frac{1}{1+\sigma}}(1+\alpha)^{\frac{1}{1+\sigma}}\,\mathrm{e}^{2\alpha}, \quad B = \tau^{-\frac{1}{1+\sigma}}\beta^{\frac{1}{1+\sigma}}\,\mathrm{e}^{2\alpha} + 2\beta\,\mathrm{e}^{2\alpha} + f(0)\,\mathrm{e}^{\alpha}.$$

3. Global classical solution: Neumann-Dirichlet-Navier slip boundary value condition

We first give some notations, which will be used throughout this paper.

Notations: $Q^l = \Omega_l \times \mathbb{R}^+$, $Q^l_T = \Omega_l \times (0,T)$, $\|f\|_{L^q} := \|f\|_{L^q(\Omega_l)}$.

In this section, we pay our attention to the global existence of classical solutions to the problem (1.1), (1.2) and (1.4). We use Leray-Schauder's fixed point framework to show the local existence of classical solutions. For this purpose, let's consider the following linear problem

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \Delta n - \chi\nabla\cdot(n_+\nabla c), & \text{in } Q, \\ c_t - \Delta c + \mathbf{u}\cdot\nabla c = -\tilde{n}_+ c, & \text{in } Q, \\ \mathbf{u}_t + \tilde{\mathbf{u}}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} - \nabla\pi + \sigma\tilde{n}\nabla\varphi, & \text{in } Q, \\ \nabla\cdot\mathbf{u} = 0, & \text{in } Q, \\ n_y = 0,\ c = \sigma c_{air},\ v = 0,\ u_y = 0, & (x,y)\in\Gamma_T, \\ n_y = c_y = 0,\ u = v = 0, & (x,y)\in\Gamma_B, \\ n(x,y,0) = \sigma n_0(x,y) \geqslant 0,\ c(x,y,0) & \\ \quad = \sigma c_0(x,y) \geqslant 0,\ \mathbf{u}(x,y,0) = \sigma\mathbf{u}_0(x,y), & x\in\Omega, \end{cases} \tag{3.1}$$

for any $T > 1$ and for any given $\tilde{\mathbf{u}} \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \cap W^{1,0}_2(Q^l_T), \tilde{n} \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T)$ with $\nabla\cdot\tilde{\mathbf{u}} = 0$ and $\tilde{\mathbf{u}}\cdot\nu|_{\Gamma_T} = 0$. By classical theory of linear parabolic equations, we

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have $\mathbf{u} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}^l \times (0,T]) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T)$, $\nabla\pi \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}^l \times (0,T])$, and we further have $c \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}^l \times (0,T]) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T)$, and $n \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}^l \times (0,T]) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T)$.

On the other hands, by comparison lemma, it is easy to obtain $c \geqslant 0$. Next, we show that $n \geqslant 0$. Let $n_- = \min\{0, n\}$. Multiplying the first equation of (3.1) by n_1, and integration it over $\Omega_l \times (0,t)$ yields

$$\begin{aligned}
\int_{\Omega_l} |n_-(\cdot,t)|^2 \,\mathrm{d}x\,\mathrm{d}y &= -\int_0^t \int_{\Omega_l} \mathbf{u}\cdot\nabla n n_- \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau + \int_0^t \int_{\Omega_l} n_-\Delta n \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \\
&\quad - \chi \int_0^t \int_{\Omega_l} n_-\nabla\cdot(n_+\nabla c)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \\
&= -\frac{1}{2}\int_0^t \int_{\partial J(\tau)} |n_-|^2 \mathbf{u}\cdot\nu \,\mathrm{d}S\,\mathrm{d}\tau + \int_0^t \int_{\partial J(\tau)} n_- \frac{\partial n}{\partial \nu}\,\mathrm{d}S\,\mathrm{d}\tau \\
&\quad - \int_0^t \int_{\Omega_l} |\nabla n_-|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \\
&= -\int_0^t \int_{\Omega_l} |\nabla n_-|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \leqslant 0,
\end{aligned}$$

where $J(t) = \{x \in \Omega_l; n(x,t) \leqslant 0\}$, which implies that $\int_{\Omega_l} |n_-(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}y = 0$, that is $n \geqslant 0$.

We define the mapping

$$\begin{aligned}
\mathcal{T} : C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \cap W^{1,0}_2(Q^l_T) \times C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \times [0,1] \\
\to C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \cap W^{1,0}_2(Q^l_T) \times C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T), \\
\mathcal{T} : (\tilde{\mathbf{u}}, \tilde{n}, \sigma) \to (\mathbf{u}, n).
\end{aligned}$$

From the above analysis, we see that $\mathbf{u}, n \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}^l \times (0,T]) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T)$, and noticing that $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}^l_T) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \hookrightarrow C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \cap W^{1,0}_2(Q^l_T)$, then the operator $\mathcal{T}$ is completely continuous. It is easy to verify that

$$\mathcal{T} : (\tilde{\mathbf{u}}, \tilde{n}, 0) \equiv \mathbf{0}.$$

In fact, it is easy to see that when $\sigma = 0$, it follows $c = 0$, $\mathbf{u} = 0$, then one can further derive that $n = 0$.

By Leray-Schauder's fixed point theorem, to show the local existence of classical solutions, we only need to show that $\|\mathbf{u}, n\|_{C^{\alpha,\frac{\alpha}{2}}} + \|\mathbf{u}\|_{W^{1,0}_2(Q^l_T}} \leqslant C$ if $(\mathbf{u}, n, \sigma)$ is a classical solution of $\mathcal{T} : (\mathbf{u}, n, \sigma) = (\mathbf{u}, n)$. For this purpose, in what follows, we pay our attention to the energy estimates.

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LEMMA 3.1. *Let* $\mathcal{T} : (\mathbf{u}, n, \sigma) = (\mathbf{u}, n)$. *Then we have*

$$n \geqslant 0, \quad 0 \leqslant c \leqslant \sigma c_{air},$$

$$\|n(\cdot,t)\|_{L^1} + \chi \int_0^t \int_{\Gamma_T} n|c_y| \,\mathrm{d}S \,\mathrm{d}\tau = \|n_0\|_{L^1}, \tag{3.2}$$

and

$$\begin{aligned}&\sup_{0<t<T} \int_{\Omega_l} \left(|\nabla\sqrt{c}|^2 + n\ln n + |\mathbf{u}|^2\right) \,\mathrm{d}x \,\mathrm{d}y \\ &\quad + \sup_{0<t<T-1} \int_t^{t+1} \mathrm{d}\tau \int_{\Omega_l} \left(|\nabla \mathbf{u}|^2 + \frac{|\nabla n|^2}{n} + c|D^2 \ln c|^2 + n|\nabla\sqrt{c}|^2\right) \,\mathrm{d}x \,\mathrm{d}y \\ &\quad + \sup_{0<t<T-1} \int_t^{t+1} \mathrm{d}\tau \int_{\Gamma_T} \left(|c_y|^3 + |c_y n \ln n|\right) \,\mathrm{d}S \leqslant C, \end{aligned} \tag{3.3}$$

where C *depends only on* $n_0, c_0, \mathbf{u}_0$, χ, *and* c_{air}, *and it is independent of* T.

Proof. By comparison lemma, it is easy to obtain that

$$0 \leqslant c \leqslant \sigma c_{air}.$$

The positivity of n has been established in the above analysis. By a direct integration, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_l} n(x,y,t) \,\mathrm{d}x \,\mathrm{d}y + \chi \int_{\Gamma_T} n\partial_y c \,\mathrm{d}S = 0, \tag{3.4}$$

which implies (3.2) since $\partial_y c \geqslant 0$ on Γ_T. By the second equation, and using a direct calculation, we see that

$$\begin{aligned}\partial_t|\nabla\sqrt{c}|^2 + \frac{1}{2}c|D^2\ln c|^2 &= \Delta|\nabla\sqrt{c}|^2 - \nabla\cdot(\mathbf{u}|\nabla\sqrt{c}|^2) - n|\nabla\sqrt{c}|^2 \\ &\quad - \frac{1}{2}\nabla n\nabla c - 2\nabla\sqrt{c}\nabla\mathbf{u}\nabla\sqrt{c}.\end{aligned}$$

Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla\sqrt{c}|^2 \,\mathrm{d}x \,\mathrm{d}y + \frac{1}{2}\int_{\Omega_l} c|D^2\ln c|^2 \,\mathrm{d}x \,\mathrm{d}y + \int_{\Omega_l} n|\nabla\sqrt{c}|^2 \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_{\Omega_l} \Delta|\nabla\sqrt{c}|^2 dxdy - \int_{\Omega_l} \nabla\cdot(\mathbf{u}|\nabla\sqrt{c}|^2) \,\mathrm{d}x \,\mathrm{d}y \\ &\quad - \frac{1}{2}\int_{\Omega_l} \nabla n\nabla c \,\mathrm{d}x \,\mathrm{d}y - 2\int_{\Omega_l} \nabla\sqrt{c}\nabla\mathbf{u}\nabla\sqrt{c} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_{\Gamma_T} \partial_y|\nabla\sqrt{c}|^2 \,\mathrm{d}S - \int_{\Gamma_B} \partial_y|\nabla\sqrt{c}|^2 \,\mathrm{d}S - \frac{1}{2}\int_{\Omega_l} \nabla n\nabla c \,\mathrm{d}x \,\mathrm{d}y - \frac{1}{2}\int_{\Omega_l} \frac{\nabla c\nabla\mathbf{u}\nabla c}{c} \,\mathrm{d}x \,\mathrm{d}y.\end{aligned} \tag{3.5}$$

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Noticing that $c = \sigma c_{air}$ on Γ_T, then $c_t\Big|_{\Gamma_T} = 0$, $\partial_x c\Big|_{\Gamma_T} = 0$, $\partial_{xx} c\Big|_{\Gamma_T} = 0$, $\partial_y c\Big|_{\Gamma_T} \geqslant 0$. Recalling the equation c satisfied, it gives

$$-\partial_{yy} c + u\partial_x c + v\partial_y c = -nc, \text{ on } \Gamma_T,$$

which implies that

$$\partial_{yy} c = \sigma n c_{air}, \text{ on } \Gamma_T.$$

Therefore, we have

$$\begin{aligned}\int_{\Gamma_T} \partial_y |\nabla\sqrt{c}|^2 \,\mathrm{d}S &= -\frac{1}{4}\int_{\Gamma_T} \frac{|\nabla c|^2 \partial_y c}{c^2}\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T} \frac{\nabla c \cdot \nabla \partial_y c}{c}\,\mathrm{d}S \\ &= -\frac{1}{4}\int_{\Gamma_T} \frac{|\partial_y c|^2 \partial_y c}{c^2}\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T} \frac{\partial_y c \partial_{yy} c}{c}\,\mathrm{d}S \\ &= -\frac{1}{4}\int_{\Gamma_T} \frac{|\partial_y c|^3}{c^2}\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T} n\partial_y c\,\mathrm{d}S. \end{aligned} \tag{3.6}$$

Noticing that $\partial_y c = 0$ on Γ_B, it implies that $\partial_{xy} c = 0$ on Γ_B. Thus, we have

$$\int_{\Gamma_B} \partial_y |\nabla\sqrt{c}|^2 \,\mathrm{d}S = -\frac{1}{4}\int_{\Gamma_B} \frac{|\nabla c|^2 \partial_y c}{c^2}\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_B} \frac{\partial_x c\partial_{xy} c + \partial_y c \partial_{yy} c}{c}\,\mathrm{d}S = 0. \tag{3.7}$$

Noting that $c = \sigma c_{air}$ on Γ_T, substituting (3.6), (3.7) into (3.5) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla\sqrt{c}|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\Omega_l} c|D^2 \ln c|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} n|\nabla\sqrt{c}|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{1}{4\sigma^2 c_{air}^2}\int_{\Gamma_T} |\partial_y c|^3 \,\mathrm{d}S \\ &= \frac{1}{2}\int_{\Gamma_T} n\partial_y c\,\mathrm{d}S - \frac{1}{2}\int_{\Omega_l} \nabla n \nabla c\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_{\Omega_l} \frac{\nabla c \nabla \mathbf{u} \nabla c}{c}\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \frac{1}{2}\int_{\Gamma_T} n\partial_y c\,\mathrm{d}S - \frac{1}{2}\int_{\Omega_l} \nabla n \nabla c\,\mathrm{d}x\,\mathrm{d}y + \eta\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y + \frac{c_{air}}{16\eta}\int_{\Omega_l} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{3.8}$$

for any small $\eta > 0$. By a direct calculation, we see that

$$\begin{aligned}&\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_l} |\nabla \ln c|^2 \nabla \ln c \nabla c\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\Gamma_T} |\nabla \ln c|^2 \nabla c \cdot \nu\,\mathrm{d}S - \int_{\Omega_l} c^{-1}\left(|\nabla c|^2 \Delta \ln c + 2\nabla c D^2 \ln c \nabla c\right)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \int_{\Gamma_T} \frac{|\nabla c|^2 \nabla c \cdot \nu}{c^2}\,\mathrm{d}S + \left(\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\Omega_l} c|\Delta \ln c|^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\ &\quad + 2\left(\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\Omega_l} c|D^2 \ln c|^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\end{aligned}$$

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$$\leqslant \int_{\Gamma_T} \frac{|\partial_y c|^3}{c^2}\,\mathrm{d}S + \sqrt{2}\left(\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}$$

$$+ 2\left(\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{\sigma^2 c_{air}^2} \int_{\Gamma_T} |\partial_y c|^3\,\mathrm{d}S + \frac{1}{2}\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y + \frac{(2+\sqrt{2})^2}{2}\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y,$$

that is

$$\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{2}{\sigma^2 c_{air}^2}\int_{\Gamma_T} |\partial_y c|^3\,\mathrm{d}S + (6+4\sqrt{2})\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y. \quad (3.9)$$

Taking $\eta = \frac{1}{4(6+4\sqrt{2})}$ in (3.8), and combining with (3.9), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{4}\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \frac{\sqrt{2}-1}{2\sigma^2 c_{air}^2}\int_{\Gamma_T} |\partial_y c|^3\,\mathrm{d}S$$

$$\leqslant \frac{1}{2}\int_{\Gamma_T} n\partial_y c\,\mathrm{d}S - \frac{1}{2}\int_{\Omega_l} \nabla n \nabla c\,\mathrm{d}x\,\mathrm{d}y + \frac{(3+2\sqrt{2})c_{air}}{2}\int_{\Omega_l} |\nabla \mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y. \quad (3.10)$$

Multiplying the first equation of (3.1) by $1+\ln n$, and integrating the resulting equation over Ω_l gives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} n\ln n\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} \frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y = -\chi\int_{\Gamma_T} n(1+\ln n)\partial_y c\,\mathrm{d}S$$

$$+ \chi\int_{\Omega_l} \nabla c \nabla n\,\mathrm{d}x\,\mathrm{d}y. \quad (3.11)$$

Multiplying the third equation of (3.1) by u, and integrating the resulting equation over Ω_l, using the boundary conditions and Poincaré inequality, we see that for any $1 < p < 2$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} |\nabla \mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y &= \sigma\int_{\Omega_l} n\mathbf{u}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \sigma\|\nabla\varphi\|_{L^\infty}\|\mathbf{u}\|_{L^{\frac{p}{p-1}}}\|n\|_{L^p} \\ &\leqslant C\sigma\|\nabla\varphi\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|n\|_{L^p} \\ &\leqslant \frac{1}{2}\int_{\Omega_l} |\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y + C\|n\|_{L^p}^2,\end{aligned}$$

since $\mathbf{u}\Big|_{\Gamma_B} = 0$, which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} |\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C_p\|n\|_{L^p}^2, \text{ for any } 1 < p < 2. \quad (3.12)$$

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Combining (3.10), (3.11) and (3.12), we arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_l}|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2\chi}\int_{\Omega_l}n\ln n\,\mathrm{d}x\,\mathrm{d}y+(3+2\sqrt{2})c_{air}\int_{\Omega_l}|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\right)\\&\quad+\frac{(3+2\sqrt{2})c_{air}}{2}\int_{\Omega_l}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\\&\quad+\frac{1}{2\chi}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{4}\int_{\Omega_l}c|D^2\ln c|^2\,\mathrm{d}x\,\mathrm{d}y\\&\quad+\int_{\Omega_l}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y+\frac{\sqrt{2}-1}{2\sigma^2c_{air}^2}\int_{\Gamma_T}|\partial_y c|^3\,\mathrm{d}S\\&\leqslant-\frac{1}{2}\int_{\Gamma_T}n\ln n\partial_y c\,\mathrm{d}S+C_p(3+2\sqrt{2})c_{air}\|n\|_{L^p}^2.\end{aligned}\tag{3.13}$$

Noticing that $\partial_y c\geqslant 0$ on Γ_T, then

$$\begin{aligned}-\frac{1}{2}\int_{\Gamma_T}n\ln n\partial_y c\,\mathrm{d}S&=-\frac{1}{2}\int_{\Gamma_T}n|\ln n|\partial_y c\,\mathrm{d}S+\int_{\Gamma_T\cap\{n<1\}}n|\ln n|\partial_y c\,\mathrm{d}S\\&\leqslant-\frac{1}{2}\int_{\Gamma_T}n|\ln n|\partial_y c\,\mathrm{d}S+C_1\int_{\Gamma_T}\partial_y c\,\mathrm{d}S\\&\leqslant-\frac{1}{2}\int_{\Gamma_T}n|\ln n|\partial_y c\,\mathrm{d}S+\frac{\sqrt{2}-1}{4c_{air}^2}\int_{\Gamma_T}|\partial_y c|^3\,\mathrm{d}S+C_2.\end{aligned}\tag{3.14}$$

From Gagliardo-Nirenberg interpolation inequality, and noticing that $4-\frac{4}{p}<2$ we infer that

$$\begin{aligned}C_p(3+2\sqrt{2})c_{air}\|n\|_{L^p}^2&=C_p(3+2\sqrt{2})c_{air}\|\sqrt{n}\|_{L^{2p}}^4\\&\leqslant C_3\|\nabla\sqrt{n}\|_{L^2}^{4-\frac{4}{p}}\|\sqrt{n}\|_{L^2}^{\frac{4}{p}}+C_4\|\sqrt{n}\|_{L^2}^4\\&\leqslant\frac{1}{2\chi}\|\nabla\sqrt{n}\|_{L^2}^2+C_5.\end{aligned}\tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_l}|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2\chi}\int_{\Omega_l}n\ln n\,\mathrm{d}x\,\mathrm{d}y+(3+2\sqrt{2})c_{air}\int_{\Omega_l}|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\right)\\&\quad+\frac{(3+2\sqrt{2})c_{air}}{2}\int_{\Omega_l}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\\&\quad+\frac{3}{8\chi}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{4}\int_{\Omega_l}c|D^2\ln c|^2\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_l}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y\\&\quad+\frac{\sqrt{2}-1}{4\sigma^2c_{air}^2}\int_{\Gamma_T}|\partial_y c|^3\,\mathrm{d}S+\frac{1}{2}\int_{\Gamma_T}n|\ln n|\partial_y c\,\mathrm{d}S\\&\leqslant C_6,\end{aligned}\tag{3.16}$$

which implies (3.3). □

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Lemma 3.2. *Assume that $\mathcal{T}:(\mathbf{u},n,\sigma)=(\mathbf{u},n)$. Then*

$$\sup_{0<t<T}\|\nabla c(\cdot,t)\|_{L^2}^2+\sup_{0<t<T-1}\int_t^{t+1}(\|D^2c\|_{L^2}^2+\|c_t\|_{L^2}^2)\,\mathrm{d}s\leqslant C, \tag{3.17}$$

$$\sup_{0<t<T}\|\nabla \mathbf{u}(\cdot,t)\|_{L^2}^2+\sup_{0<t<T-1}\int_t^{t+1}(\|D^2\mathbf{u}\|_{L^2}^2+\|\mathbf{u}_t\|_{L^2}^2)\,\mathrm{d}s\leqslant C, \tag{3.18}$$

where the constants C depend only on $n_0,c_0,\mathbf{u}_0$, χ, and c_{air}, and they are independent of T.

Proof. By Gagliardo-Nirenberg interpolation inequality, we get that

$$\|\nabla c\|_{L^4}^4\leqslant C_1\|D^2c\|_{L^2}^2\|c\|_{L^\infty}^2+C_2\|c\|_{L^\infty}^4\leqslant C_3\|D^2c\|_{L^2}^2+C_4, \tag{3.19}$$

$$\|n\|_{L^2}^2=\|\sqrt{n}\|_{L^4}^4\leqslant C_5\|\nabla\sqrt{n}\|_{L^2}^2\|\sqrt{n}\|_{L^2}^2+C_6\|\sqrt{n}\|_{L^2}^2\leqslant C_7\|\nabla\sqrt{n}\|_{L^2}^2+C_8, \tag{3.20}$$

$$\|u\|_{L^4}^4\leqslant C_9\|\nabla u\|_{L^2}^2\|u\|_{L^2}^2+C_{10}\|u\|_{L^2}^4\leqslant C_{11}\|\nabla u\|_{L^2}^2+C_{12}. \tag{3.21}$$

Applying ∇ to the second equation of (3.1), and multiplying the resulting equation by ∇c, noticing that $c_y=c_{xy}=0$ on Γ_B, and $c_x=c_{xx}=0$, $c_{yy}=nc$ on Γ_T, and using (3.19)–(3.21), we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l}|\nabla c|^2\,\mathrm{d}x\,\mathrm{d}y=\int_{\Omega_l}\nabla\Delta c\nabla c\,\mathrm{d}x\,\mathrm{d}y-\int_{\Omega_l}\nabla(nc)\nabla c\,\mathrm{d}x\,\mathrm{d}y-\int_{\Omega_l}\nabla(\mathbf{u}\nabla c)\nabla c\,\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{2}\int_{\Omega_l}\Delta|\nabla c|^2\,\mathrm{d}x\,\mathrm{d}y-\int_{\Omega_l}|D^2c|^2\,\mathrm{d}x\,\mathrm{d}y-\int_{\Gamma_T}ncc_y\,\mathrm{d}S+\int_{\Omega_l}(nc+\mathbf{u}\nabla c)\Delta c\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant\frac{1}{2}\int_{\Gamma_T}\partial_y|\nabla c|^2\,\mathrm{d}S-\frac{1}{2}\int_{\Gamma_B}\partial_y|\nabla c|^2\,\mathrm{d}S-\int_{\Omega_l}|D^2c|^2\,\mathrm{d}x\,\mathrm{d}y-\int_{\Gamma_T}ncc_y\,\mathrm{d}S\\
&\quad+\frac{1}{4}\int_{\Omega_l}|\Delta c|^2\,\mathrm{d}x\,\mathrm{d}y+2\int_{\Omega_l}|\nabla c|^2|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y+2\int_{\Omega_l}|nc|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant\int_{\Gamma_T}(c_xc_{xy}+c_yc_{yy})\,\mathrm{d}S-\int_{\Omega_l}|D^2c|^2\,\mathrm{d}x\,\mathrm{d}y-\int_{\Gamma_T}ncc_y\,\mathrm{d}S\\
&\quad+\frac{1}{2}\int_{\Omega_l}|D^2c|^2\,\mathrm{d}x\,\mathrm{d}y+2\left(\int_{\Omega_l}|\nabla c|^4\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{\Omega_l}|u|^4\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}+2\int_{\Omega_l}|nc|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant-\frac{1}{4}\int_{\Omega_l}|D^2c|^2\,\mathrm{d}x\,\mathrm{d}y+C_{13}\int_{\Omega_l}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y+C_{14}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+C_{15},
\end{aligned}$$

which implies

$$\sup_{0<t<T}\int_{\Omega_l}|\nabla c|^2\,\mathrm{d}x\,\mathrm{d}y+\sup_{0<t<T-1}\int_t^{t+1}\mathrm{d}s\int_{\Omega_l}|D^2c|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant C.$$

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Multiplying the second equation of (3.1) by c_t, then we obtain

$$\sup_{0<t<T-1}\int_t^{t+1} \mathrm{d}s \int_{\Omega_l} |c_t|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant C.$$

Then (3.17) is proved. Multiplying the third equation of (3.1) by $\mathbf{u}_t$, and integrating it over Ω_l yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} |\mathbf{u}_t|^2 \,\mathrm{d}x\,\mathrm{d}y = -\int_{\Omega_l} \mathbf{u}\cdot\nabla\mathbf{u}\;\mathbf{u}_t \,\mathrm{d}x\,\mathrm{d}y + \sigma\int_{\Omega_l} n\nabla\varphi\mathbf{u}_t \,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant \frac{1}{2}\int_{\Omega_l} |\mathbf{u}_t|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} (|\mathbf{u}|^2|\nabla\mathbf{u}|^2 + n^2|\nabla\varphi|^2) \,\mathrm{d}x\,\mathrm{d}y.$$

Noticing that

$$-\Delta\mathbf{u} + \nabla\pi = -\mathbf{u}_t - \mathbf{u}\cdot\nabla\mathbf{u} + \sigma n\nabla\varphi,$$

then by L^2 theory of Stokes operator [2], we have

$$\|D^2\mathbf{u}\|_{L^2}^2 + \|\nabla\pi\|_{L^2}^2 \leqslant C_{17}(\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\nabla\mathbf{u}\|_{L^2}^2 + \|n\nabla\varphi\|_{L^2}^2).$$

Combining the above two inequalities, and using (3.3) we arrive at

$$2C_{17}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y + C_{17}\int_{\Omega_l} |\mathbf{u}_t|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} |\nabla^2\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} |\nabla\pi|^2 \,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C_{18}\int_{\Omega_l} (|\mathbf{u}|^2|\nabla\mathbf{u}|^2 + n^2) \,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C_{18}\|\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2 + C_{18}\|n\|_{L^2}^2$$
$$\leqslant C_{19}\left(\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{L^2}^2\right)\left(\|\nabla\mathbf{u}\|_{L^2}\|\Delta\mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{L^2}^2\right) + C_{20}\|\nabla\sqrt{n}\|_{L^2}^2 + C_{21}$$
$$\leqslant \frac{1}{2}\int_{\Omega_l} |\nabla^2\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y + C_{22}\|\nabla\mathbf{u}\|_{L^2}^4 + C_{20}\int_{\Omega_l} |\nabla\sqrt{n}|^2 \,\mathrm{d}x\,\mathrm{d}y + C_{23},$$

that is

$$2C_{17}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\Omega_l} |D^2\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y + C_{17}\int_{\Omega_l} |\mathbf{u}_t|^2 \,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C_{22}\|\nabla\mathbf{u}\|_{L^2}^4 + C_{20}\int_{\Omega_l} |\nabla\sqrt{n}|^2 \,\mathrm{d}x\,\mathrm{d}y + C_{23}. \tag{3.22}$$

Recalling (3.3), we see that

$$\sup_{0<t<T-1}\int_t^{t+1}\int_{\Omega_l} |\nabla\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \leqslant \hat{C},$$

by mean value theorem of integrals, for any $t\in(0,T-1)$ there exists $t_0\in(t,t+1)$ such that

$$\int_{\Omega_l} |\nabla\mathbf{u}(x,t_0)|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant \hat{C}.$$

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From (3.22), and by a direct calculation, we derive that for any t_0,

$$\|\nabla \mathbf{u}(\cdot,t)\|_{L^2}^2 \leqslant \left(\|\nabla \mathbf{u}(\cdot,t_0)\|_{L^2}^2 + \frac{1}{2C_{17}}\int_{t_0}^{t}(C_{20}\|\nabla\sqrt{n}(\cdot,s)\|_{L^2}^2 + C_{23})\,\mathrm{d}s\right)$$
$$\times \exp\left\{\frac{C_{22}}{2C_{17}}\int_{t_0}^{t}\|\nabla \mathbf{u}(\cdot,s)\|_{L^2}^2\,\mathrm{d}s\right\}.$$

Combining the above two inequalities, we get that

$$\sup_{0<t<T}\|\nabla \mathbf{u}(\cdot,t)\|_{L^2}^2 \leqslant C_{24}.$$

Using this inequality, and integrating (3.22) from t to $t+1$ yields

$$\sup_{0<t<T-1}\int_t^{t+1}(\|D^2\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}_t\|_{L^2}^2)\,\mathrm{d}s \leqslant C_{25}.$$

Then (3.18) is proved. □

LEMMA 3.3. *Assume that $\mathcal{T}(\mathbf{u},n,\sigma) = (\mathbf{u},n)$. Then for any $r>0$*

$$\sup_{0<t<T}\int_{\Omega_l}|n(\cdot,t)|^{r+1}\,\mathrm{d}x\,\mathrm{d}y + \sup_{0<t<T-1}\int_t^{t+1}\int_{\Omega_l}|\nabla n^{\frac{r+1}{2}}|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau$$
$$+ \sup_{0<t<T-1}\int_t^{t+1}\int_{\Gamma_T} n^{r+1}c_y\,\mathrm{d}S\,\mathrm{d}\tau \leqslant C, \tag{3.23}$$

where C depends only on $n_0, c_0, \mathbf{u}_0$, χ, r and c_{air}, and it is independent of T.

Proof. Multiplying the first equation of (3.1) by n^r for $r>0$, and integrating the resulting equation over Ω yields

$$\frac{1}{r+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l}n^{r+1}\,\mathrm{d}x\,\mathrm{d}y + \frac{4r}{(r+1)^2}\int_{\Omega_l}|\nabla n^{\frac{r+1}{2}}|^2\,\mathrm{d}x\,\mathrm{d}y + \chi\int_{\Gamma_T}n^{r+1}c_y\,\mathrm{d}S$$
$$= \frac{2r\chi}{r+1}\int_{\Omega_l}n^{\frac{r+1}{2}}\nabla c\nabla n^{\frac{r+1}{2}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant \frac{2r\chi}{r+1}\|\nabla n^{\frac{r+1}{2}}\|_{L^2}\|n^{\frac{r+1}{2}}\|_{L^4}\|\nabla c\|_{L^4}$$
$$\leqslant \frac{2Cr\chi}{r+1}\|\nabla n^{\frac{r+1}{2}}\|_{L^2}\left(\|n^{\frac{r+1}{2}}\|_{L^2}^{\frac12}\|\nabla n^{\frac{r+1}{2}}\|_{L^2}^{\frac12} + \|n\|_{L^1}^{\frac{r+1}{2}}\right)\left(\|\nabla c\|_{L^2}^{\frac12}\|\Delta c\|_{L^2}^{\frac12} + \|\nabla c\|_{L^2}\right)$$
$$\leqslant \frac{r}{(r+1)^2}\|\nabla n^{\frac{r+1}{2}}\|_{L^2}^2 + C\|n\|_{L^{r+1}}^{r+1}(\|\Delta c\|_{L^2}^2+1) + C\|\Delta c\|_{L^2}^2 + C.$$

By Gagliardo-Nirenberg interpolation inequality, we get that

$$\|n\|_{L^{r+2}}^{r+2} = \|n^{\frac{r+1}{2}}\|_{L^{\frac{2(r+2)}{r+1}}}^{\frac{2(r+2)}{r+1}} \leqslant C_1\|\nabla n^{\frac{r+1}{2}}\|_{L^2}^2\|n^{\frac{r+1}{2}}\|_{L^{\frac{2}{r+1}}}^{\frac{2}{r+1}}$$
$$+ C_2\|n\|_{L^1}^{r+2} \leqslant C_3\|\nabla n^{\frac{r+1}{2}}\|_{L^2}^2 + C_4. \tag{3.24}$$

Combining the above two inequalities, there exists a small constant $\eta > 0$ such that

$$\frac{1}{r+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} n^{r+1}\,\mathrm{d}x\,\mathrm{d}y + \frac{2r}{(r+1)^2}\int_{\Omega_l}|\nabla n^{\frac{r+1}{2}}|^2\,\mathrm{d}x\,\mathrm{d}y$$
$$+\chi\int_{\Gamma_T} n^{r+1}c_y\,\mathrm{d}S + \eta\int_{\Omega_l} n^{r+2}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C\|n\|_{L^{r+1}}^{r+1}(\|\Delta c\|_{L^2}^2+1) + C\|\Delta c\|_{L^2}^2 + C.$$

Using lemma 2.2 and (3.17), we complete the proof. □

LEMMA 3.4. *Assume that $\mathcal{T}(\mathbf{u}, n, \sigma) = (\mathbf{u}, n)$. Then for any $r > 0$*

$$\sup_{0<t<T}\int_{\Omega_l}|\nabla c|^r\,\mathrm{d}x\,\mathrm{d}y + \sup_{0<t<T-1}\int_t^{t+1}\int_{\Omega_l}|\nabla c|^{r-2}|D^2c|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \leqslant C_r, \quad (3.25)$$

where C_r depends only on $n_0, c_0, \mathbf{u}_0$, χ, r and c_{air}, and it is independent of T.

Proof. Using the boundary value conditions, we see that

$$\mathbf{u}\nabla c = uc_x + vc_y = 0, \quad \text{on } \Gamma_T,$$

$$\partial_y|\nabla c|^r = r(c_x^2+c_y^2)^{\frac{r}{2}-1}(c_xc_{xy}+c_yc_{yy}) = rc_y^{r-1}c_{yy} = rncc_y^{r-1}, \quad \text{on } \Gamma_T,$$

and

$$\partial_y|\nabla c|^r = 0, \quad \text{on } \Gamma_B.$$

Applying ∇ to the second equation of (3.1), and multiplying the resulting equation by $|\nabla c|^{r-2}\nabla c$ yields

$$\begin{aligned}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l}|\nabla c|^r\,\mathrm{d}x\,\mathrm{d}y &= \int_{\Omega_l}\nabla\Delta c|\nabla c|^{r-2}\nabla c\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega_l}\nabla(nc)|\nabla c|^{r-2}\nabla c\,\mathrm{d}x\,\mathrm{d}y\\
&\quad - \int_{\Omega_l}\nabla(\mathbf{u}\nabla c)|\nabla c|^{r-2}\nabla c\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{r}\int_{\Omega_l}\Delta|\nabla c|^r\,\mathrm{d}x\,\mathrm{d}y - (r-1)\int_{\Omega_l}|\nabla c|^{r-2}|D^2c|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad - \int_{\partial\Omega_l} nc|\nabla c|^{r-2}\nabla c\cdot\nu\,\mathrm{d}S\\
&\quad + \int_{\Omega_l}(\mathbf{u}\nabla c + nc)\left(|\nabla c|^{r-2}\Delta c + (r-2)|\nabla c|^{r-4}\nabla cD^2c\nabla c\right)\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant \frac{1}{r}\int_{\Gamma_T}\partial_y|\nabla c|^r\,\mathrm{d}S - \frac{1}{r}\int_{\Gamma_B}\partial_y|\nabla c|^r\,\mathrm{d}S - (r-1)\end{aligned}$$

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$$
\begin{aligned}
&\times \int_{\Omega_l} |\nabla c|^{r-2}|D^2c|^2 \,\mathrm{d}x\,\mathrm{d}y - \int_{\Gamma_T} nc|c_y|^{r-2}c_y \,\mathrm{d}S \\
&+ \frac{r-1}{2}\int_{\Omega_l} |\nabla c|^{r-2}|D^2c|^2 \,\mathrm{d}x\,\mathrm{d}y + (r-1)\int_{\Omega_l} |\nabla c|^{r}|\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&+ (r-1)\|c\|_{L^\infty}\int_{\Omega_l} |\nabla c|^{r-2}|n|^2 \,\mathrm{d}x\,\mathrm{d}y \\
= &-\frac{r-1}{2}\int_{\Omega_l} |\nabla c|^{r-2}|D^2c|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&+ (r-1)\int_{\Omega_l} |\nabla c|^{r}|\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y + \|c\|_{L^\infty}(r-1)\int_{\Omega_l} |\nabla c|^{r-2}|n|^2 \,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{3.26}
$$

and noticing that

$$
\begin{aligned}
\|\nabla c\|_{L^{r+2}}^{r+2} &= \||\nabla c|^{\frac{r}{2}}\|_{L^{\frac{2(r+2)}{r}}}^{\frac{2(r+2)}{r}} \leqslant C_1\|\nabla|\nabla c|^{\frac{r}{2}}\|_{L^2}^2\||\nabla c|^{\frac{r}{2}}\|_{L^{\frac{4}{r}}}^{\frac{4}{r}} \\
&+ C_2\|\nabla c\|_{L^2}^{r+2} \leqslant C_3\||\nabla c|^{\frac{r-2}{2}}D^2c\|_{L^2}^2 + C_4,
\end{aligned} \tag{3.27}
$$

then there exists a constant $\eta > 0$ such that

$$
\begin{aligned}
&\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla c|^{r} \,\mathrm{d}x\,\mathrm{d}y + \frac{r-1}{4}\int_{\Omega_l} |\nabla c|^{r-2}|D^2c|^2 \,\mathrm{d}x\,\mathrm{d}y + \eta\int_{\Omega_l} |\nabla c|^{r+2} \,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \frac{\eta}{2}\int_{\Omega_l} |\nabla c|^{r+2} \,\mathrm{d}x\,\mathrm{d}y + C\int_{\Omega_l} |\mathbf{u}|^{r+2} \,\mathrm{d}x\,\mathrm{d}y + C\int_{\Omega_l} |n|^{\frac{r+2}{2}} \,\mathrm{d}x\,\mathrm{d}y + C.
\end{aligned}
$$

Recalling (3.18) and (3.23), we infer that

$$
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla c|^{r} \,\mathrm{d}x\,\mathrm{d}y + \frac{r-1}{4}\int_{\Omega_l} |\nabla c|^{r-2}|D^2c|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{\eta}{2}\int_{\Omega_l} |\nabla c|^{r+2} \,\mathrm{d}x\,\mathrm{d}y \leqslant C_r.
$$

Thus (3.25) is proved. □

LEMMA 3.5. *Assume that $\mathcal{T}(\mathbf{u}, n, \sigma) = (\mathbf{u}, n)$. Then for any $r > 0$*

$$
\sup_{0<t<T}(\|n\|_{L^\infty} + \|\nabla c\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty}) \leqslant C, \tag{3.28}
$$

and for any $\beta \in \left(\frac{1}{2}, 1\right)$,

$$
\sup_{0<t<T}\|A^\beta \mathbf{u}\|_{L^\infty} \leqslant \tilde{C}, \tag{3.29}
$$

where $A = P\Delta$, P is Helmholtz projection, C, $\tilde{C}$ depend only on $n_0, c_0, \mathbf{u}_0$, χ, c_{air} and β, and they are independent of T.

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Proof. Multiplying the first equation of (3.1) by u^r for $r > 1$, and integrating the resulting equation over Ω yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} n^{r+1}\,\mathrm{d}x\,\mathrm{d}y + \frac{4r}{r+1}\int_{\Omega_l}|\nabla n^{\frac{r+1}{2}}|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \chi(r+1)\int_{\Gamma_T} n^{r+1}c_y\,\mathrm{d}S + \int_{\Omega_l} n^{r+1}\,\mathrm{d}x\,\mathrm{d}y \\
&= 2r\chi\int_{\Omega_l} n^{\frac{r+1}{2}}\nabla c\nabla n^{\frac{r+1}{2}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} n^{r+1}\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant 2r\chi\|\nabla n^{\frac{r+1}{2}}\|_{L^2}\|n^{\frac{r+1}{2}}\|_{L^4}\|\nabla c\|_{L^4} + \|n^{\frac{r+1}{2}}\|_{L^2}^2 \\
&\leqslant 2Cr\chi\|\nabla n^{\frac{r+1}{2}}\|_{L^2}\left(\|n^{\frac{r+1}{2}}\|_{L^1}^{\frac{1}{4}}\|\nabla n^{\frac{r+1}{2}}\|_{L^2}^{\frac{3}{4}} + \|n\|_{L^{\frac{r+1}{2}}}^{\frac{r+1}{2}}\right) \\
&\quad + C\left(\|\nabla n^{\frac{r+1}{2}}\|_{L^2}^{\frac{1}{2}}\|n^{\frac{r+1}{2}}\|_{L^1}^{\frac{1}{2}} + \|n^{\frac{r+1}{2}}\|_{L^1}^2\right) \\
&\leqslant \frac{2r}{r+1}\|\nabla n^{\frac{r+1}{2}}\|_{L^2}^2 + Cr(r+1)^7\|n\|_{L^{\frac{r+1}{2}}}^{r+1},
\end{aligned}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} n^{r+1}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} n^{r+1}\,\mathrm{d}x\,\mathrm{d}y \leqslant C(r+1)^8\|n\|_{L^{\frac{r+1}{2}}}^{r+1}.$$

Taking $r_j = 2r_{j-1} = 2^j r_0,\ r_0 = 1,\ M_j = \max\left\{1, \|n_0\|_{L^\infty}, \sup_{t\in(0,T)}\|n\|_{L^{r_j}}\right\}$, then we have

$$\begin{aligned}
M_j &\leqslant C^{\frac{1}{r_j}} r_j^{\frac{8}{r_j}} M_{j-1} = C^{\frac{1}{r_0 2^j}} r_0^{\frac{8}{r_0 2^j}} 2^{\frac{8j}{r_0 2^j}} M_{j-1} \\
&\leqslant C^{\sum_{k=1}^j \frac{1}{r_0 2^k}} r_0^{\sum_{k=1}^j \frac{8}{r_0 2^k}} 2^{\sum_{k=1}^j \frac{8k}{r_0 2^k}} M_0 \leqslant \tilde{C},
\end{aligned}$$

where $\tilde{C}$ is independent of j, letting $j \to \infty$, we obtain the L^∞ estimate of n. Recalling (3.26), using (3.18) and (3.23),for any $r \geqslant 3$, we see that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l}|\nabla c|^r\,\mathrm{d}x\,\mathrm{d}y + \frac{r(r-1)}{2}\int_{\Omega_l}|\nabla c|^{r-2}|D^2c|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l}|\nabla c|^r\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant r(r-1)\int_{\Omega_l}|\nabla c|^r|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y + Cr(r-1)\int_{\Omega_l}|\nabla c|^{r-2}|n|^2\,\mathrm{d}x\,\mathrm{d}y + + \int_{\Omega_l}|\nabla c|^r\,\mathrm{d}x\,\mathrm{d}y \\
&= r(r-1)\||\nabla c|^{\frac{r}{2}}\|_{L^4}^2\|\mathbf{u}\|_{L^4}^2 + Cr(r-1)\||\nabla c|^{\frac{r}{2}}\|_{L^{\frac{4(r-2)}{r}}}^{\frac{2(r-2)}{r}}\|n\|_{L^4}^2 + \||\nabla c|^{\frac{r}{2}}\|_{L^2}^2 \\
&\leqslant C_1 r(r-1)\left(\||\nabla c|^{\frac{r}{2}}\|_{L^1}^{\frac{1}{2}}\|\nabla|\nabla c|^{\frac{r}{2}}\|_{L^2}^{\frac{3}{2}} + \||\nabla c|^{\frac{r}{2}}\|_{L^1}^2\right) \\
&\quad + C_2 r(r-1)\left(\||\nabla c|^{\frac{r}{2}}\|_{L^1}^{\frac{1}{2}}\|\nabla|\nabla c|^{\frac{r}{2}}\|_{L^2}^{\frac{3r-8}{2r}} + \||\nabla c|^{\frac{r}{2}}\|_{L^1}^{\frac{2(r-2)}{r}}\right)
\end{aligned}$$

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$$+ C_3 \left(\| |\nabla c|^{\frac{r}{2}} \|_{L^1} \| \nabla |\nabla c|^{\frac{r}{2}} \|_{L^2} + \| |\nabla c|^{\frac{r}{2}} \|_{L^1}^2 \right)$$

$$\leqslant \frac{r(r-1)}{4} \int_{\Omega_l} |\nabla c|^{r-2} |D^2 c|^2 \, \mathrm{d}x \, \mathrm{d}y + C_4 r(r-1) \|\nabla c\|_{L^{\frac{r}{2}}}^r + C_5 r(r-1) \|\nabla c\|_{L^{\frac{r}{2}}}^{\frac{r^2}{r+8}}$$

$$+ C_6 r(r-1) \|\nabla c\|_{L^{\frac{r}{2}}}^{r-2},$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_l} |\nabla c|^r \, \mathrm{d}x \, \mathrm{d}y + \int_{\Omega_l} |\nabla c|^r \, \mathrm{d}x \, \mathrm{d}y \leqslant C_4 r(r-1) \|\nabla c\|_{L^{\frac{r}{2}}}^r$$

$$+ C_5 r(r-1) \|\nabla c\|_{L^{\frac{r}{2}}}^{\frac{r^2}{r+8}} + C_6 r(r-1) \|\nabla c\|_{L^{\frac{r}{2}}}^{r-2}.$$

Then, similar to above, we obtain the L^∞ estimates of ∇c, and we complete the proof.

By semigroup theory of Stokes operator, we see that

$$\|\mathbf{u}\|_{L^\infty} \leqslant \mathrm{e}^{-\delta t} \|\mathbf{u_0}\|_{L^\infty} + C_7 \int_0^t \mathrm{e}^{-\delta(t-s)} (t-s)^{-\frac{3}{4}} \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^{\frac{4}{3}}} \, \mathrm{d}s$$

$$+ C_8 \int_0^t \mathrm{e}^{-\delta(t-s)} (t-s)^{-\frac{1}{2}} \|n\|_{L^2}$$

$$\leqslant \mathrm{e}^{-\delta t} \|\mathbf{u_0}\|_{L^\infty} + C_7 \sup_{t<T} \|\mathbf{u}(\cdot,t)\|_{L^4} \|\nabla \mathbf{u}(\cdot,t)\|_{L^2} \int_0^t \mathrm{e}^{-\delta s} s^{-\frac{3}{4}} \, \mathrm{d}s$$

$$+ C_8 \sup_{t<T} \|n(\cdot,t)\|_{L^2} \int_0^t \mathrm{e}^{-\delta s} s^{-\frac{1}{2}} \, \mathrm{d}s$$

$$\leqslant C_9.$$

Similarly, for any $\beta \in \left(\frac{1}{2}, 1\right)$, we also have

$$\|A^\beta \mathbf{u}\|_{L^\infty} \leqslant \mathrm{e}^{-\delta t} \|A^\beta \mathbf{u_0}\|_{L^\infty} + C_{10} \int_0^t \mathrm{e}^{-\delta(t-s)} (t-s)^{-\beta} \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^\infty} \, \mathrm{d}s$$

$$+ C_{11} \int_0^t \mathrm{e}^{-\delta(t-s)} (t-s)^{-\beta} \|n\|_{L^\infty}$$

$$\leqslant \mathrm{e}^{-\delta t} \|A^\beta \mathbf{u_0}\|_{L^\infty} + C_{10} \sup_{t<T} \|\mathbf{u}(\cdot,t)\|_{L^\infty} \|\nabla \mathbf{u}(\cdot,t)\|_{L^\infty} \int_0^t \mathrm{e}^{-\delta s} s^{-\beta} \, \mathrm{d}s$$

$$+ + C_{11} \sup_{t<T} \|n(\cdot,t)\|_{L^\infty} \int_0^t \mathrm{e}^{-\delta s} s^{-\beta} \, \mathrm{d}s$$

$$\leqslant \mathrm{e}^{-\delta t} \|A^\beta \mathbf{u_0}\|_{L^\infty} + C_{12} \sup_{t<T} \|\mathbf{u}(\cdot,t)\|_{L^\infty} (\|\mathbf{u}(\cdot,t)\|_{L^\infty}^{\frac{2\beta-1}{2\beta}} \|A^\beta \mathbf{u}(\cdot,t)\|_{L^\infty}^{\frac{1}{2\beta}}$$

$$+ \|\mathbf{u}(\cdot,t)\|_{L^\infty}) + C_{13}$$

$$\leqslant C_{14} + C_{15} \sup_{t<T} \|A^\beta \mathbf{u}(\cdot,t)\|_{L^\infty}^{\frac{1}{2\beta}},$$

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which implies that

$$\sup_{t<T} \|A^\beta \mathbf{u}(\cdot,t)\|_{L^\infty} \leqslant C_{14} + C_{15} \sup_{t<T} \|A^\beta \mathbf{u}(\cdot,t)\|_{L^\infty}^{\frac{1}{2\beta}}.$$

Noticing that $\frac{1}{2\beta} < 1$, then

$$\sup_{t<T} \|A^\beta \mathbf{u}(\cdot,t)\|_{L^\infty} \leqslant C. \qquad \square$$

LEMMA 3.6. *Let $\mathcal{T} : (\mathbf{u}, n, \sigma) = (\mathbf{u}, n)$. Then for any $p > 1$,*

$$\sup_{0<t<T-1} \left(\|(\mathbf{u}, n, c)\|_{W_p^{2,1}(Q_1^l(t))} + \|\pi\|_{W_p^{1,0}(Q_1^l(t))} \right) \leqslant C_p, \tag{3.30}$$

where $Q_1^l(t) = \Omega_l \times (t, t+1)$, C_p is independent of T, it depends only on $n_0, c_0, \mathbf{u}_0$, χ, c_{air} and p.

Proof. Noticing that

$$c_t - \Delta c + \mathbf{u} \cdot \nabla c + c = c - nc,$$

and (3.28), using the L^p theory of linear parabolic equations, it is easy to obtain that

$$\sup_{0<t<T-1} \|c\|_{W_p^{2,1}(Q_1^l(t))}^p \leqslant C \sup_{0<t<T-1} \|c - nc\|_{L^p(Q_1^l(t))}^p + C\|c_0\|_{W^{2,p}}^p \leqslant C_1. \tag{3.31}$$

For $\mathbf{u}$, we see that

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla \pi + \mathbf{u} = \mathbf{u} + \sigma n \nabla \varphi,$$

then

$$\begin{aligned} &\sup_{0<t<T-1} \left(\|\mathbf{u}\|_{W_p^{2,1}(Q_1^l(t))}^p + \|\pi\|_{W_p^{1,0}(Q_1^l(t))}^p \right) \\ &\quad \leqslant C \sup_{0<t<T-1} \|\mathbf{u} + \sigma n \nabla \varphi\|_{L^p(Q_1^l(t))}^p + C\|\mathbf{u}_0\|_{W^{2,p}}^p \leqslant C_2. \end{aligned} \tag{3.32}$$

Recalling that

$$n_t + \mathbf{u} \cdot \nabla n - \Delta n + n = n - \chi \nabla \cdot (n \nabla c).$$

Then for any $q > 1$,

$$\begin{aligned} &\int_t^{t+1} (\|n\|_{W^{2,q}}^q + \|n_t\|_{L^q}^q)\, \mathrm{d}s \leqslant C\|n_0\|_{W^{2,q}}^q + C \int_t^{t+1} \|\nabla \cdot (n \nabla c)\|_{L^q}^q \, \mathrm{d}s \\ &\quad + C \int_t^{t+1} \|n\|_{L^q}^q \, \mathrm{d}s \\ &\leqslant C\|n_0\|_{W^{2,q}}^q + C_2 \|\nabla c\|_{L^\infty(Q_T^l)}^q \int_t^{t+1} \|\nabla n\|_{L^q}^q \, \mathrm{d}s \\ &\quad + C\|n\|_{L^\infty(Q_T^l)}^q \int_t^{t+1} \|\Delta c\|_{L^q}^q \, \mathrm{d}s + +C \int_t^{t+1} \|n\|_{L^q}^q \, \mathrm{d}s \\ &\leqslant C\|n_0\|_{W^{2,q}}^q + \frac{1}{2} \int_t^{t+1} \|\nabla n\|_{W^{1,q}}^q \, \mathrm{d}s + \tilde{C}, \end{aligned}$$

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which implies that

$$\sup_{0<t<T-1} \int_t^{t+1} (\|n\|_{W^{2,q}}^q + \|n_t\|_{L^q}^q)\, \mathrm{d}s \leqslant C_3.$$

Thus, (3.30) is proved. □

Proof of theorem 1.1. Noticing that $W_p^{2,1}(Q_T^l) \hookrightarrow C^{\beta,\frac{\beta}{2}}(\overline{Q}_T^l)$, for any $\beta < 2 - \frac{4}{p}$, then by lemma 3.6, we derive that

$$\|(n, c, \mathbf{u})\|_{C^{\beta,\frac{\beta}{2}}(Q_T^l)} \leqslant C.$$

Using Leray-Schauder fixed point theorem, the problem (1.1), (1.2), and (1.4) admits a classical solution $(n, c, \mathbf{u}, \pi)$ in Q_T^l. From the above lemmas, we see that all these estimates are independent of T, which implies that the solution $(n, c, \mathbf{u}, \pi)$ is a global classical solution, and we conclude that

$$\|(n, c, \mathbf{u})\|_{C^{\beta,\frac{\beta}{2}}(Q^l)} \leqslant C. \tag{3.33}$$

Combining (3.33) and lemma 3.6, and using the classical theory of linear parabolic equations, we have

$$\|(c, \mathbf{u})\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q^l)} + \|\nabla\pi\|_{C^{\alpha,\frac{\alpha}{2}}(Q^l)} \leqslant C. \tag{3.34}$$

Using (3.34), we further have

$$\|n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q^l)} \leqslant C. \tag{3.35}$$

The global existence in theorem 1.1 is proved.

The proof of uniqueness is standard, for the completeness of the paper, in what follows, we still give the proof. Suppose the contrary. Let $(n_1, c_1, \mathbf{u_1}, \pi_1)$, $(n_2, c_2, \mathbf{u_2}, \pi_2)$ be two solutions of (1.1), (1.2), and (1.4). Denote $\tilde{n} = n_1 - n_2$, $\tilde{c} = c_1 - c_2$, $\tilde{\mathbf{u}} = \mathbf{u_1} - \mathbf{u_2} = (\tilde{u}, \tilde{v})$, $\tilde{\pi} = \pi_1 - \pi_2$. Then

$$\begin{cases} \tilde{n}_t + \tilde{\mathbf{u}} \cdot \nabla n_1 + \mathbf{u}_2 \cdot \nabla \tilde{n} = \Delta \tilde{n} - \chi \nabla \cdot (\tilde{n} \nabla c_1 + n_2 \nabla \tilde{c}), & \text{in } Q, \\ \tilde{c}_t - \Delta \tilde{c} + \tilde{\mathbf{u}} \cdot \nabla c_1 + \mathbf{u}_2 \cdot \nabla \tilde{c} = -\tilde{n} c_1 - n_2 \tilde{c}, & \text{in } Q, \\ \tilde{\mathbf{u}}_t + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_1 + \mathbf{u}_2 \cdot \nabla \tilde{\mathbf{u}} = \Delta \tilde{\mathbf{u}} - \nabla \tilde{\pi} + \tilde{n} \nabla \varphi, & \text{in } Q, \\ \nabla \cdot \tilde{\mathbf{u}} = 0, & \text{in } Q, \\ \tilde{n}_y = 0, \ \tilde{c} = 0, \ \tilde{v} = 0, \ \tilde{u}_y = 0, & (x,y) \in \Gamma_T, \\ \tilde{n}_y = \tilde{c}_y = 0, \ \tilde{u} = \tilde{v} = 0, & (x,y) \in \Gamma_B, \\ \tilde{n}(x,y,0) = 0, \ \tilde{c}(x,y,0) = 0, \ \tilde{\mathbf{u}}(x,y,0) = 0, & x \in \Omega, \end{cases} \tag{3.36}$$

Multiplying the first equation of (3.36) by $\tilde{n}$, and integrating it over Ω yields

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\tilde{n}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega |\nabla\tilde{n}|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_\Omega \tilde{n}\tilde{\mathbf{u}} \cdot \nabla n_1 \,\mathrm{d}x\,\mathrm{d}y - \chi \int_{\Gamma_T} (\tilde{n}\partial_y c_1 + n_2 \partial_y \tilde{c})\tilde{n} \,\mathrm{d}S + \chi \int_\Omega (\tilde{n}\nabla c_1 + n_2 \nabla\tilde{c})\nabla\tilde{n} \,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

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$$
\begin{aligned}
&\leqslant \frac{1}{4}\int_{\Omega}|\nabla\tilde{n}|^2\,\mathrm{d}x\,\mathrm{d}y + C\int_{\Omega}(|\tilde{\mathbf{u}}|^2+|\tilde{n}|^2+|\nabla\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad -\chi\int_{\Gamma_T}|\tilde{n}|^2\partial_y c_1\,\mathrm{d}S - \chi\int_{\Gamma_T} n_2\partial_y\tilde{c}\tilde{n}\,\mathrm{d}S\\
&\leqslant \frac{1}{4}\int_{\Omega}|\nabla\tilde{n}|^2\,\mathrm{d}x\,\mathrm{d}y + C\int_{\Omega}(|\tilde{\mathbf{u}}|^2+|\tilde{n}|^2+|\nabla\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}y + C\int_{\Gamma_T}(|\tilde{n}|^2+|\partial_y\tilde{c}|^2)\,\mathrm{d}S,
\end{aligned}
$$

that is

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|\tilde{n}|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{3}{4}\int_{\Omega}|\nabla\tilde{n}|^2\,\mathrm{d}x\,\mathrm{d}y &\leqslant C\int_{\Omega}(|\tilde{\mathbf{u}}|^2+|\tilde{n}|^2+|\nabla\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + C\int_{\Gamma_T}(|\tilde{n}|^2+|\partial_y\tilde{c}|^2)\,\mathrm{d}S. \qquad (3.37)
\end{aligned}
$$

Multiplying the second equation of (3.36) by $\tilde{c}$, and integrating it over Ω yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega}|\nabla\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&= -\int_{\Omega}\tilde{c}\tilde{\mathbf{u}}\cdot\nabla c_1\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega}\tilde{c}\mathbf{u}_2\cdot\nabla\tilde{c}\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega}\tilde{c}(\tilde{n}c_1+n_2\tilde{c})\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_{\Omega}\nabla\tilde{c}\cdot\tilde{\mathbf{u}}c_1\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega}\tilde{c}(\tilde{n}c_1+n_2\tilde{c})\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant \frac{1}{4}\int_{\Omega}|\nabla\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y + C\int_{\Omega}(|\tilde{\mathbf{u}}|^2+|\tilde{n}|^2+|\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

namely,

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{3}{4}\int_{\Omega}|\nabla\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C\int_{\Omega}(|\tilde{\mathbf{u}}|^2+|\tilde{n}|^2+|\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}y. \qquad (3.38)
$$

Applying ∇ to the second equation of (3.36), and multiplying the resulting equation by $\nabla\tilde{c}$, and integrating it over Ω yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|\nabla\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega}\nabla\Delta\tilde{c}\nabla\tilde{c}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad - \int_{\Omega}\nabla(\tilde{\mathbf{u}}\cdot\nabla c_1+\mathbf{u}_2\cdot\nabla\tilde{c})\nabla\tilde{c}\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega}\nabla\tilde{c}\nabla(\tilde{n}c_1+n_2\tilde{c})\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{2}\int_{\Omega}\Delta|\nabla\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega}|D^2\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega}(\tilde{\mathbf{u}}\cdot\nabla c_1+\mathbf{u}_2\cdot\nabla\tilde{c})\Delta\tilde{c}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad - \int_{\Gamma_T}(\tilde{n}c_1+n_2\tilde{c})\partial_y\tilde{c}\,\mathrm{d}S + \int_{\Omega}(\tilde{n}c_1+n_2\tilde{c})\Delta\tilde{c}\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{2}\int_{\Gamma_T}\partial_y|\nabla\tilde{c}|^2\,\mathrm{d}S - \frac{1}{2}\int_{\Gamma_B}\partial_y|\nabla\tilde{c}|^2\,\mathrm{d}S - \int_{\Omega}|D^2\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

$$+\int_\Omega (\tilde{\mathbf{u}}\cdot\nabla c_1 + \mathbf{u}_2\cdot\nabla\tilde{c})\Delta\tilde{c}\,\mathrm{d}x\,\mathrm{d}y$$
$$-\int_{\Gamma_T}(\tilde{n}c_1 + n_2\tilde{c})\partial_y\tilde{c}\,\mathrm{d}S + \int_\Omega(\tilde{n}c_1 + n_2\tilde{c})\Delta\tilde{c}\,\mathrm{d}x\,\mathrm{d}y,$$

noticing that $\tilde{c}_y = \tilde{c}_{xy} = 0$ on Γ_B, $\tilde{c} = \tilde{c}_x = \tilde{c}_{xx} = 0$ on Γ_T, and $\tilde{c}_{yy} = \tilde{n}c_1 + n_2\tilde{c}$ on Γ_T, it implies that

$$\frac{1}{2}\partial_y|\nabla\tilde{c}|^2 = \tilde{c}_x\tilde{c}_{xy} + \tilde{c}_y\tilde{c}_{yy} = 0, \quad \text{on } \Gamma_B,$$

$$\frac{1}{2}\partial_y|\nabla\tilde{c}|^2 = \tilde{c}_x\tilde{c}_{xy} + \tilde{c}_y\tilde{c}_{yy} = \tilde{c}_y(\tilde{n}c_1 + n_2\tilde{c}), \quad \text{on } \Gamma_T.$$

Then

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega|D^2\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_\Omega(\tilde{\mathbf{u}}\cdot\nabla c_1 + \mathbf{u}_2\cdot\nabla\tilde{c})\Delta\tilde{c}\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega(\tilde{n}c_1 + n_2\tilde{c})\Delta\tilde{c}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant \frac{1}{4}\int_\Omega|D^2\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y + C\int_\Omega\left(|\tilde{\mathbf{u}}|^2 + |\nabla\tilde{c}|^2 + |\tilde{c}|^2 + |\tilde{n}|^2\right)\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

That is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{3}{4}\int_\Omega|D^2\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C\int_\Omega\left(|\tilde{\mathbf{u}}|^2 + |\nabla\tilde{c}|^2 + |\tilde{c}|^2 + |\tilde{n}|^2\right)\mathrm{d}x\,\mathrm{d}y. \tag{3.39}$$

Similarly, we also have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\tilde{\mathbf{u}}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega|\nabla\tilde{\mathbf{u}}|^2\,\mathrm{d}x\,\mathrm{d}y = -\int_\Omega\tilde{\mathbf{u}}\cdot\nabla\mathbf{u}_1\tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega\tilde{n}\nabla\varphi\tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant C\int_\Omega|\tilde{\mathbf{u}}|^2\,\mathrm{d}x\,\mathrm{d}y + C\int_\Omega|\tilde{n}|^2\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{3.40}$$

Combining (3.37)–(3.40) yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\tilde{n}|^2 + |\tilde{\mathbf{u}}|^2 + |\tilde{c}|^2 + |\nabla\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \frac{3}{4}\int_\Omega\left(|\nabla\tilde{n}|^2 + |\nabla\tilde{c}|^2 + |D^2\tilde{c}|^2 + |\nabla\tilde{\mathbf{u}}|^2\right)\mathrm{d}x\,\mathrm{d}y\\
&\leqslant C_1\int_\Omega\left(|\tilde{\mathbf{u}}|^2 + |\nabla\tilde{c}|^2 + |\tilde{c}|^2 + |\tilde{n}|^2\right)\mathrm{d}x\,\mathrm{d}y + C_2\int_{\Gamma_T}(|\tilde{n}|^2 + |\partial_y\tilde{c}|^2)\,\mathrm{d}S.
\end{aligned} \tag{3.41}$$

Using Sobolev trace embedding inequality, we see that

$$C_2\int_{\Gamma_T}(|\tilde{n}|^2 + |\partial_y\tilde{c}|^2)\,\mathrm{d}S \leqslant \frac{1}{4}\int_\Omega\left(|\nabla\tilde{n}|^2 + |D^2\tilde{c}|^2\right)\mathrm{d}x\,\mathrm{d}y + C_3\int_\Omega\left(\tilde{n}|^2 + |\nabla\tilde{c}|^2\right)\mathrm{d}x\,\mathrm{d}y.$$

Substituting it into (3.42) gives

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\tilde{n}|^2+|\tilde{\mathbf{u}}|^2+|\tilde{c}|^2+|\nabla\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2}\int_\Omega \big(|\nabla\tilde{n}|^2\\
&\quad+|\nabla\tilde{c}|^2+|D^2\tilde{c}|^2+|\nabla\tilde{\mathbf{u}}|^2\big)\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant C_4\int_\Omega \big(|\tilde{\mathbf{u}}|^2+|\nabla\tilde{c}|^2+|\tilde{c}|^2+|\tilde{n}|^2\big)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}\tag{3.42}$$

which implies that

$$\int_\Omega (|\tilde{n}|^2+|\tilde{\mathbf{u}}|^2+|\tilde{c}|^2+|\nabla\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}y\equiv 0,\ \text{for any } t>0,$$

and the uniqueness is proved. □

4. Global classical solution: Zero Flux-Dirichlet(Robin)-Navier slip boundary conditions

In this section, we consider the global classical solution of the problem (1.1), (1.3) and (1.4).

Consider the following linear problem

$$\begin{cases}
n_t+\mathbf{u}\cdot\nabla n=\Delta n-\chi\nabla\cdot(n_+\nabla c), & \text{in } Q,\\
c_t-\Delta c+\mathbf{u}\cdot\nabla c=-\tilde{n}_+c, & \text{in } Q,\\
\mathbf{u}_t+\tilde{\mathbf{u}}\cdot\nabla\mathbf{u}=\Delta\mathbf{u}-\nabla\pi+\sigma\tilde{n}\nabla\varphi, & \text{in } Q,\\
\nabla\cdot\mathbf{u}=0, & \text{in } Q,\\
n_y-\chi n_+c_y=0,\ \tau c_y=-c+\sigma c_{air},\ v=0,\ u_y=0, & (x,y)\in\Gamma_T,\\
n_y=c_y=0,\ u=v=0, & (x,y)\in\Gamma_B,\\
n(x,y,0)=\sigma n_0(x,y)\geqslant 0,\ c(x,y,0) & \\
\quad=\sigma c_0(x,y)\geqslant 0,\ \mathbf{u}(x,y,0)=\sigma\mathbf{u}_0(x,y), & x\in\Omega.
\end{cases}\tag{4.1}$$

Similar to §2, for any $T>1$ and for any given $\tilde{\mathbf{u}}\in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T^l)\cap W_2^{1,0}(Q_T^l), \tilde{n}\in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T^l)$ with $\nabla\cdot\tilde{\mathbf{u}}=0$ and $\tilde{\mathbf{u}}\cdot\nu|_{\Gamma_T}=0$. The above problem admits a classical solution in Q_T^l. By comparison lemma, we also have $0\leqslant c\leqslant\sigma c_{air}$. Next, we show that $n\geqslant 0$. Let $n_-=\min\{0,n\}$. Multiplying the first equation of (4.1) by n_1, and integrating it over $\Omega_l\times(0,t)$ yields

$$\begin{aligned}
&\int_{\Omega_l}|n_-(x,y,t)|^2\,\mathrm{d}x\,\mathrm{d}y\\
&=-\int_0^t\int_{\Omega_l}\mathbf{u}\cdot\nabla n n_-\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau+\int_0^t\int_{\Omega_l}n_-\Delta n\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\\
&\quad-\chi\int_0^t\mathrm{d}\tau\int_{\Omega_l}n_-\nabla\cdot(n_+\nabla c)\,\mathrm{d}x\,\mathrm{d}y\\
&=-\frac{1}{2}\int_0^t\int_{\partial J(\tau)}|n_-|^2\mathbf{u}\cdot\nu\,\mathrm{d}S\,\mathrm{d}\tau+\int_0^t\int_{\partial J(\tau)}n_-\left(\frac{\partial n}{\partial\nu}-\chi n_+\frac{\partial c}{\partial\nu}\right)\mathrm{d}S\,\mathrm{d}\tau
\end{aligned}$$

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$$
\begin{aligned}
&- \int_0^t \mathrm{d}\tau \int_{\Omega_l} |\nabla n_-|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&= - \int_0^t \int_{\Omega_l} |\nabla n_-|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \leqslant 0,
\end{aligned}
$$

where $J(t) = \{x \in \Omega_l; n(x,t) \leqslant 0\}$, which implies that $\int_{\Omega_l} |n_-(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}y = 0$, that is $n \geqslant 0$.

We define the mapping

$$
\begin{aligned}
\mathcal{T} : C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \cap W_2^{1,0}(Q^l_T) \times C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \times [0,1] \\
\to C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \cap W_2^{1,0}(Q^l_T) \times C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T),
\end{aligned}
$$

$$
\mathcal{T} : (\tilde{\mathbf{u}}, \tilde{n}, \sigma) \to (\mathbf{u}, n).
$$

From the above analysis, we see that $\mathbf{u}, n \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}^l \times (0,T]) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T)$, and noticing that $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}^l_T) \cap C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \hookrightarrow C^{\alpha,\frac{\alpha}{2}}(\overline{Q}^l_T) \cap W_2^{1,0}(Q^l_T)$, then the operator $\mathcal{T}$ is completely continuous. It is easy to verify that

$$
\mathcal{T} : (\tilde{\mathbf{u}}, \tilde{n}, 0) \equiv \mathbf{0}.
$$

Next, we use Leray-Schauder's fixed point theorem, to show the existence of classical solutions. For this purpose, some a prior energy estimates are necessary.

LEMMA 4.1. *Let $\mathcal{T} : (\mathbf{u}, n, \sigma) = (\mathbf{u}, n)$. Then we have*

$$
n \geqslant 0, \quad 0 \leqslant c \leqslant \sigma c_{air}, \tag{4.2}
$$

and

$$
\|n(\cdot,t)\|_{L^1} = \sigma \|n_0\|_{L^1}. \tag{4.3}
$$

Moreover, when $\tau = 0$, for appropriately small $\chi > 0$, we have

$$
\begin{aligned}
&\sup_{t \leqslant T} \int_{\Omega_l} \left(|\nabla \sqrt{c}|^2 + n \ln n + |\mathbf{u}|^2\right) \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \sup_{t \leqslant T-1} \int_t^{t+1} \mathrm{d}s \int_{\Omega_l} \left(|\nabla \mathbf{u}|^2 + \frac{|\nabla n|^2}{n} + c|D^2 \ln c|^2 + n|\nabla \sqrt{c}|^2\right) \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \sup_{t \leqslant T-1} \int_t^{t+1} \mathrm{d}s \int_{\Gamma_T} |\partial_y c|^3 \,\mathrm{d}S \leqslant C_1.
\end{aligned} \tag{4.4}
$$

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When $\tau = 1$,

$$\sup_{t\leqslant T}\left(\int_{\Omega_l}\left(|\nabla\sqrt{c}|^2 + n\ln n + |\mathbf{u}|^2 + c\ln c\right)\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_T}(c-\sigma c_{air}\ln c)\,\mathrm{d}S\right)$$
$$+ \sup_{t\leqslant T-1}\int_t^{t+1}\mathrm{d}s\int_{\Omega_l}\left(\frac{|\nabla n|^2}{n} + c|D^2\ln c|^2 + n|\nabla\sqrt{c}|^2 + |\nabla\mathbf{u}|^2\right)\mathrm{d}x\,\mathrm{d}y$$
$$+ \sup_{t\leqslant T-1}\int_t^{t+1}\mathrm{d}s\int_{\Gamma_T}\left(\frac{|\nabla c|^2 c_y}{c^2} + \frac{|c_x|^2}{c} + \sigma c_{air}\frac{|c_x|^2}{c^2} + (c-\sigma c_{air}\ln c)\right)\mathrm{d}S \leqslant C_2. \tag{4.5}$$

Here C_1, C_2 *depend only on* $n_0, c_0, \mathbf{u}_0$, χ, *and* c_{air}, *and they are independent of* T.

Proof. From the above analysis, it is easy to obtain (4.2). And (4.3) is derived from a direct integration as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} n(x,y,t)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Multiplying the first equation of (4.1) by $1+\ln n$, and integrating the resulting equation over Ω_l gives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} n\ln n\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y = \chi\int_{\Omega_l}\nabla c\nabla n\,\mathrm{d}x\,\mathrm{d}y. \tag{4.6}$$

For u it is completely similar to the proof of (3.12), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l}|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C_p\|n\|^2_{L^p},\ \text{for any } 1<p<2. \tag{4.7}$$

From (3.5) and (3.7), we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l}|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\Omega_l}c|D^2\ln c|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{\Gamma_T}\partial_y|\nabla\sqrt{c}|^2\,\mathrm{d}S - \frac{1}{2}\int_{\Omega_l}\nabla n\nabla c\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_{\Omega_l}\frac{\nabla c\nabla\mathbf{u}\nabla c}{c}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.8}$$

(i) When $\tau = 0$, completely similar to the proof of (3.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l}|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{4}\int_{\Omega_l}c|D^2\ln c|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \frac{\sqrt{2}-1}{2\sigma^2c_{air}^2}\int_{\Gamma_T}|\partial_y c|^3\,\mathrm{d}S$$
$$\leqslant \frac{1}{2}\int_{\Gamma_T}nc_y\,\mathrm{d}S - \frac{1}{2}\int_{\Omega_l}\nabla n\nabla c\,\mathrm{d}x\,\mathrm{d}y + \frac{(3+2\sqrt{2})c_{air}}{2}\int_{\Omega_l}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y.$$

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Combining with (4.6), (4.7), and using (3.15), we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_l}|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2\chi}\int_{\Omega_l}n\ln n\,\mathrm{d}x\,\mathrm{d}y+(3+2\sqrt{2})c_{air}\int_{\Omega_l}|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\right)\\
&\quad+\frac{(3+2\sqrt{2})c_{air}}{2}\int_{\Omega_l}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\frac{1}{2\chi}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{4}\int_{\Omega_l}c|D^2\ln c|^2\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_l}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\frac{\sqrt{2}-1}{2\sigma^2c_{air}^2}\int_{\Gamma_T}|\partial_y c|^3\,\mathrm{d}S\\
&\leqslant\frac{1}{2}\int_{\Gamma_T}nc_y\,\mathrm{d}S+C_p(3+2\sqrt{2})c_{air}\|n\|_{L^p}^2\\
&\leqslant\frac{\sqrt{2}-1}{4\sigma^2c_{air}^2}\int_{\Gamma_T}|c_y|^3\,\mathrm{d}S+\frac{\sigma c_{air}}{\sqrt{2\sqrt{2}-2}}\int_{\Gamma_T}n^{\frac{3}{2}}\,\mathrm{d}S+\frac{1}{8\chi}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+C.
\end{aligned}$$

By lemma 2.1, we have

$$\begin{aligned}
\|n\|_{L^{\frac{3}{2}}(\Gamma_T)}^{\frac{3}{2}}&=\|\sqrt{n}\|_{L^3(\Gamma_T)}^3\leqslant C_1\|\nabla\sqrt{n}\|_{L^2}^2\|\sqrt{n}\|_{L^2}+C_2\|n\|_{L^1}\\
&\leqslant C_3\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+C_4.
\end{aligned}$$

Combining the above two inequalities, then when χ is appropriately small, such that $\frac{C_3\sigma c_{air}}{\sqrt{2\sqrt{2}-2}}\leqslant\frac{1}{8\chi}$, we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_l}|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2\chi}\int_{\Omega_l}n\ln n\,\mathrm{d}x\,\mathrm{d}y+(3+2\sqrt{2})c_{air}\int_{\Omega_l}|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\right)\\
&\quad+\frac{(3+2\sqrt{2})c_{air}}{2}\int_{\Omega_l}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\frac{1}{4\chi}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{4}\int_{\Omega_l}c|D^2\ln c|^2\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_l}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\frac{\sqrt{2}-1}{4\sigma^2c_{air}^2}\int_{\Gamma_T}|\partial_y c|^3\,\mathrm{d}S\leqslant C.
\end{aligned}\tag{4.9}$$

Noticing that

$$\|n\ln n\|_{L^1}\leqslant\|n\|_{L^2}^2+\|n\|_{L^1}\leqslant C\|\nabla\sqrt{n}\|_{L^2}^2+C,$$

and

$$|\nabla\sqrt{c}|^2\leqslant\frac{|\nabla c|^4}{c^3}+c,$$

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combining (3.9) and Poincaré inequality, we see that

$$\int_{\Omega_l} \left(|\nabla\sqrt{c}|^2 + n\ln n + |\mathbf{u}|^2 + c\ln c\right) \mathrm{d}x\,\mathrm{d}y \leqslant C\int_{\Omega_l}\left(\frac{|\nabla n|^2}{n} + c|D^2\ln c|^2 + |\nabla\mathbf{u}|^2\right)\mathrm{d}x\,\mathrm{d}y + C\int_{\Gamma_T}|c_y|^3\,\mathrm{d}S. \tag{4.10}$$

Combining (4.9), (4.11), and (4.4) is derived.

(ii) When $\tau = 1$, noticing that $c_y = -c + \sigma c_{air}$ on Γ_T, we take the derivative in the tangential direction and get that $c_{xy} = -c_x$ on Γ_T, then

$$\begin{aligned}\int_{\Gamma_T}\partial_y|\nabla\sqrt{c}|^2\,\mathrm{d}S &= -\frac{1}{4}\int_{\Gamma_T}\frac{|\nabla c|^2\partial_y c}{c^2}\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T}\frac{\nabla c\cdot\nabla\partial_y c}{c}\,\mathrm{d}S\\ &= -\frac{1}{4}\int_{\Gamma_T}\frac{|\nabla c|^2 c_y}{c^2}\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T}\frac{c_x c_{xy}}{c}\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T}\frac{c_y c_{yy}}{c}\,\mathrm{d}S\\ &= -\frac{1}{4}\int_{\Gamma_T}\frac{|\nabla c|^2 c_y}{c^2}\,\mathrm{d}S - \frac{1}{2}\int_{\Gamma_T}\frac{|c_x|^2}{c}\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T}\frac{(\sigma c_{air} - c)c_{yy}}{c}\,\mathrm{d}S.\end{aligned} \tag{4.11}$$

Noticing that $c_t + uc_x = c_{xx} + c_{yy} - nc$ on Γ_T, multiplying this equation by $\frac{1}{2}\frac{(\sigma c_{air} - c)}{c}$, and integrating it on Γ_T yields

$$\begin{aligned}&\frac{1}{2}\int_{\Gamma_T}\frac{(\sigma c_{air} - c)c_{yy}}{c}\,\mathrm{d}S - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_T}(\sigma c_{air}\ln c - c)\,\mathrm{d}S\\ &= \frac{1}{2}\int_{\Gamma_T}\frac{(\sigma c_{air} - c)}{c}uc_x\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T}(\sigma c_{air} - c)n\,\mathrm{d}S - \frac{1}{2}\int_{\Gamma_T}\frac{(\sigma c_{air} - c)}{c}c_{xx}\,\mathrm{d}S\\ &= \frac{1}{2}\int_{\Gamma_T}\frac{(\sigma c_{air} - c)}{c}uc_x\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T}(\sigma c_{air} - c)n\,\mathrm{d}S\\ &\quad - \frac{1}{2}\int_0^l\frac{(\sigma c_{air} - c(x,1,t))}{c(x,1,t)}c_{xx}(x,1,t)\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{2}\int_{\Gamma_T}\frac{(\sigma c_{air} - c)}{c}uc_x\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T}(\sigma c_{air} - c)n\,\mathrm{d}S - \frac{\sigma c_{air}}{2}\int_{\Gamma_T}\frac{|c_x|^2}{c^2}\,\mathrm{d}S.\end{aligned} \tag{4.12}$$

Combining (4.8), (4.4), (4.11) and (4.12), we arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l}|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_T}(c - \sigma c_{air}\ln c)\,\mathrm{d}S\\ &\quad + \frac{1}{2}\int_{\Omega_l}c|D^2\ln c|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y\\ &\quad + \frac{1}{4}\int_{\Gamma_T}\frac{|\nabla c|^2 c_y}{c^2}\,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T}\frac{|c_x|^2}{c}\,\mathrm{d}S + \frac{\sigma c_{air}}{2}\int_{\Gamma_T}\frac{|c_x|^2}{c^2}\,\mathrm{d}S\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{2}\int_{\Gamma_T} \frac{(\sigma c_{air}-c)}{c} u c_x \,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T} (\sigma c_{air}-c) n \,\mathrm{d}S \\
&\quad - \frac{1}{2}\int_{\Omega_l} \nabla n \nabla c \,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_{\Omega_l} \frac{\nabla c \nabla \mathbf{u} \nabla c}{c} \,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \frac{\sigma c_{air}}{4}\int_{\Gamma_T} \frac{|c_x|^2}{c^2}\,\mathrm{d}S + \sigma c_{air}\int_{\Gamma_T} u^2 \,\mathrm{d}S + \frac{1}{4}\int_{\Gamma_T} \frac{|c_x|^2}{c}\,\mathrm{d}S \\
&\quad + \int_{\Gamma_T} c u^2 \,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_T} (\sigma c_{air}-c) n \,\mathrm{d}S \\
&\quad - \frac{1}{2}\int_{\Omega_l} \nabla n \nabla c \,\mathrm{d}x\,\mathrm{d}y + \eta \int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{16\eta}\int_{\Omega_l} c|\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \frac{\sigma c_{air}}{4}\int_{\Gamma_T} \frac{|c_x|^2}{c^2}\,\mathrm{d}S + 2\sigma c_{air}\int_{\Gamma_T} u^2 \,\mathrm{d}S + \frac{1}{4}\int_{\Gamma_T} \frac{|c_x|^2}{c}\,\mathrm{d}S \\
&\quad + \frac{\sigma c_{air}}{2}\int_{\Gamma_T} n \,\mathrm{d}S - \frac{1}{2}\int_{\Omega_l} \nabla n \nabla c \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \eta \int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{16\eta}\int_{\Omega_l} c|\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla\sqrt{c}|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_T} (c - \sigma c_{air}\ln c)\,\mathrm{d}S + \frac{1}{2}\int_{\Omega_l} c|D^2 \ln c|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\Omega_l} n|\nabla\sqrt{c}|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{4}\int_{\Gamma_T} \frac{|\nabla c|^2 c_y}{c^2}\,\mathrm{d}S + \frac{1}{4}\int_{\Gamma_T} \frac{|c_x|^2}{c}\,\mathrm{d}S + \frac{\sigma c_{air}}{4}\int_{\Gamma_T} \frac{|c_x|^2}{c^2}\,\mathrm{d}S \\
&\leqslant 2\sigma c_{air}\int_{\Gamma_T} u^2 \,\mathrm{d}S + \frac{\sigma c_{air}}{2}\int_{\Gamma_T} n \,\mathrm{d}S - \frac{1}{2}\int_{\Omega_l} \nabla n \nabla c \,\mathrm{d}x\,\mathrm{d}y + \eta\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{16\eta}\int_{\Omega_l} c|\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y. \qquad (4.13)
\end{aligned}
$$

Noticing that

$$
u(x,1,t) = u(x,0,t) + \int_0^1 u_y(x,y,t)\,\mathrm{d}y = \int_0^1 u_y(x,y,t)\,\mathrm{d}y,
$$

from Hölder's inequality, we infer that

$$
|u(x,1,t)|^2 \leqslant \int_0^1 |u_y(x,y,t)|^2 \,\mathrm{d}y,
$$

then

$$
2\sigma c_{air}\int_{\Gamma_T} u^2 \,\mathrm{d}S \leqslant 2\sigma c_{air}\int_0^l \mathrm{d}x \int_0^1 |u_y(x,y,t)|^2 \,\mathrm{d}y \leqslant 2\sigma c_{air}\int_{\Omega_l} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y. \qquad (4.14)
$$

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By a direct calculation, we see that

$$\begin{aligned}
&\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_l} |\nabla \ln c|^2 \nabla \ln c \nabla c\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\Gamma_T} |\nabla \ln c|^2 \nabla c\cdot\nu\,\mathrm{d}S - \int_{\Omega_l} c^{-1}\left(|\nabla c|^2 \Delta \ln c + 2\nabla c D^2 \ln c \nabla c\right)\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \int_{\Gamma_T} \frac{|\nabla c|^2 \nabla c\cdot\nu}{c^2}\,\mathrm{d}S + \left(\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\Omega_l} c|\Delta \ln c|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\quad + 2\left(\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\leqslant \int_{\Gamma_T} \frac{|\nabla c|^2 c_y}{c^2}\,\mathrm{d}S \\
&\quad + \sqrt{2}\left(\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\quad + 2\left(\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\leqslant \int_{\Gamma_T} \frac{|\nabla c|^2 c_y}{c^2}\,\mathrm{d}S + \frac{1}{2}\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y + \frac{(2+\sqrt{2})^2}{2}\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

that is

$$\int_{\Omega_l} \frac{|\nabla c|^4}{c^3}\,\mathrm{d}x\,\mathrm{d}y \leqslant 2\int_{\Gamma_T} \frac{|\nabla c|^2 c_y}{c^2}\,\mathrm{d}S + (6+4\sqrt{2})\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y. \tag{4.15}$$

Taking $\eta = \frac{1}{4(6+4\sqrt{2})}$ in (4.13), and combining with (4.2), (4.14) and (4.15), we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_T} (c - \sigma c_{air}\ln c)\,\mathrm{d}S + \frac{1}{4}\int_{\Omega_l} c|D^2 \ln c|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\Omega_l} n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{8}\int_{\Gamma_T} \frac{|\nabla c|^2 c_y}{c^2}\,\mathrm{d}S + \frac{1}{4}\int_{\Gamma_T} \frac{|c_x|^2}{c}\,\mathrm{d}S + \frac{\sigma c_{air}}{4}\int_{\Gamma_T} \frac{|c_x|^2}{c^2}\,\mathrm{d}S \\
&\leqslant \frac{\sigma c_{air}}{2}\int_{\Gamma_T} n\,\mathrm{d}S - \frac{1}{2}\int_{\Omega_l} \nabla n \nabla c\,\mathrm{d}x\,\mathrm{d}y + \frac{(7+2\sqrt{2})\sigma c_{air}}{2}\int_{\Omega_l} |\nabla \mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{4.16}$$

Similar to (3.15), we have

$$\begin{aligned}
C_p(7+2\sqrt{2})c_{air}\|n\|_{L^p}^2 &= C_p(7+2\sqrt{2})c_{air}\|\sqrt{n}\|_{L^{2p}}^4 \\
&\leqslant C_1\|\nabla\sqrt{n}\|_{L^2}^{4-\frac{4}{p}}\|\sqrt{n}\|_{L^2}^{\frac{4}{p}} + C_2\|\sqrt{n}\|_{L^2}^4 \\
&\leqslant \frac{1}{2\chi}\|\nabla\sqrt{n}\|_{L^2}^2 + C_3.
\end{aligned} \tag{4.17}$$

Combining (4.6), (4.7) , (4.16), (4.17), and using Sobolev interpolation inequality, we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_l}|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2\chi}\int_{\Omega_l}n\ln n\,\mathrm{d}x\,\mathrm{d}y+(7+2\sqrt{2})c_{air}\int_{\Omega_l}|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\right.\\
&\quad\left.+\frac{1}{2}\int_{\Gamma_T}(c-\sigma c_{air}\ln c)\,\mathrm{d}S\right)\\
&\quad+\frac{1}{2\chi}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{4}\int_{\Omega_l}c|D^2\ln c|^2\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_l}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\frac{1}{8}\int_{\Gamma_T}\frac{|\nabla c|^2c_y}{c^2}\,\mathrm{d}S+\frac{1}{4}\int_{\Gamma_T}\frac{|c_x|^2}{c}\,\mathrm{d}S\\
&\quad+\frac{\sigma c_{air}}{4}\int_{\Gamma_T}\frac{|c_x|^2}{c^2}\,\mathrm{d}S+\frac{(7+2\sqrt{2})c_{air}}{2}\int_{\Omega_l}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant\frac{\sigma c_{air}}{2}\int_{\Gamma_T}n\,\mathrm{d}S+C_p(7+2\sqrt{2})c_{air}\|n\|^2_{L^p}\\
&\leqslant\frac{1}{8\chi}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_l}n\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{8\chi}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+C,
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_l}|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2\chi}\int_{\Omega_l}n\ln n\,\mathrm{d}x\,\mathrm{d}y+(7+2\sqrt{2})c_{air}\int_{\Omega_l}|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\right.\\
&\quad\left.+\frac{1}{2}\int_{\Gamma_T}(c-\sigma c_{air}\ln c)\,\mathrm{d}S\right)\\
&\quad+\frac{1}{4\chi}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{4}\int_{\Omega_l}c|D^2\ln c|^2\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_l}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\frac{1}{8}\int_{\Gamma_T}\frac{|\nabla c|^2c_y}{c^2}\,\mathrm{d}S+\frac{1}{4}\int_{\Gamma_T}\frac{|c_x|^2}{c}\,\mathrm{d}S\\
&\quad+\frac{\sigma c_{air}}{4}\int_{\Gamma_T}\frac{|c_x|^2}{c^2}\,\mathrm{d}S+\frac{(7+2\sqrt{2})c_{air}}{2}\int_{\Omega_l}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant C.
\end{aligned}
\tag{4.18}
$$

Multiplying the second equation of (4.1) by $1+\ln c$, and integrating it over Ω_l yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l}c\ln c\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_l}\frac{|\nabla c|^2}{c}\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_l}nc(1+\ln c)\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{\Gamma_T}c_y(1+\ln c)\,\mathrm{d}S=\int_{\Gamma_T}(\sigma c_{air}-c)(1+\ln c)\,\mathrm{d}S\\
&=-\int_{\Gamma_T}(c\ln c+c-\sigma c_{air}\ln c)\,\mathrm{d}S+\sigma c_{air}l\\
&=-\int_{\Gamma_T}(c-\sigma c_{air}\ln c)\,\mathrm{d}S+C.
\end{aligned}
\tag{4.19}
$$

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Combining (4.18) and (4.19), we arrive that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_l}\left(|\nabla\sqrt{c}|^2+\frac{1}{2\chi}n\ln n+(7+2\sqrt{2})c_{air}|\mathbf{u}|^2+c\ln c\right)\mathrm{d}x\,\mathrm{d}y\right.\\
&\quad\left.+\frac{1}{2}\int_{\Gamma_T}(c-\sigma c_{air}\ln c)\,\mathrm{d}S\right)\\
&\quad+\frac{1}{4\chi}\int_{\Omega_l}\frac{|\nabla n|^2}{n}\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{4}\int_{\Omega_l}c|D^2\ln c|^2\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_l}n|\nabla\sqrt{c}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\frac{1}{8}\int_{\Gamma_T}\frac{|\nabla c|^2c_y}{c^2}\,\mathrm{d}S+\frac{1}{4}\int_{\Gamma_T}\frac{|c_x|^2}{c}\,\mathrm{d}S\\
&\quad+\frac{\sigma c_{air}}{4}\int_{\Gamma_T}\frac{|c_x|^2}{c^2}\,\mathrm{d}S+\frac{(7+2\sqrt{2})c_{air}}{2}\int_{\Omega_l}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\int_{\Gamma_T}(c-\sigma c_{air}\ln c)\,\mathrm{d}S\leqslant C.
\end{aligned}
\tag{4.20}
$$

Noticing that

$$
\|n\ln n\|_{L^1}\leqslant\|n\|_{L^2}^2+\|n\|_{L^1}\leqslant C\|\nabla\sqrt{n}\|_{L^2}^2+C,
$$

and

$$
|\nabla\sqrt{c}|^2\leqslant\frac{|\nabla c|^4}{c^3}+c,
$$

combining (4.15) and Poincaré inequality, we see that

$$
\begin{aligned}
&\int_{\Omega_l}\left(|\nabla\sqrt{c}|^2+n\ln n+|\mathbf{u}|^2+c\ln c\right)\mathrm{d}x\,\mathrm{d}y\\
&\leqslant C\int_{\Omega_l}\left(\frac{|\nabla n|^2}{n}+c|D^2\ln c|^2+|\nabla\mathbf{u}|^2\right)\mathrm{d}x\,\mathrm{d}y+C\int_{\Gamma_T}\frac{|\nabla c|^2c_y}{c^2}\,\mathrm{d}S.
\end{aligned}
\tag{4.21}
$$

Combining (4.20) and (4.21), we finally conclude (4.5). □

LEMMA 4.2. *Assume that $\mathcal{T}:(\mathbf{u},n,\sigma)=(\mathbf{u},n)$. Then for $\tau=0$ with small $\chi>0$, or $\tau=1$, we have*

$$
\sup_{0<t<T}\|\nabla c(\cdot,t)\|_{L^2}^2+\sup_{0<t<T-1}\int_t^{t+1}(\|D^2c\|_{L^2}^2+\|c_t\|_{L^2}^2)\,\mathrm{d}s\leqslant C,
\tag{4.22}
$$

$$
\sup_{0<t<T}\|\nabla\mathbf{u}(\cdot,t)\|_{L^2}^2+\sup_{0<t<T-1}\int_t^{t+1}(\|D^2\mathbf{u}\|_{L^2}^2+\|\mathbf{u}_t\|_{L^2}^2)\,\mathrm{d}s\leqslant C,
\tag{4.23}
$$

where the constants C depend only on $n_0,c_0,\mathbf{u}_0$, χ, and c_{air}, and they are independent of T.

Proof. When $\tau=0$, the proof is completely similar to lemma 3.2. We omit it. In what follows, we consider the case $\tau=1$. Applying ∇ to the second equation of

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(3.1), and multiplying the resulting equation by ∇c, noticing that $c_{xy} = -c_x$ on Γ_T, and using (3.19)–(3.21) yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla c|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_l} |D^2 c|^2 \,\mathrm{d}x\,\mathrm{d}y = -\int_{\Omega_l} \nabla(\mathbf{u}\nabla c)\nabla c \,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\Omega_l} \nabla(cn)\nabla c \,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_T} (c_x c_{xy} + c_y c_{yy}) \,\mathrm{d}S \\
&\leqslant \|\nabla \mathbf{u}\|_{L^2} \|\nabla c\|_{L^4}^2 + \|\mathbf{u}\|_{L^4} \|D^2 c\|_{L^2} \|\nabla c\|_{L^4} \\
&\quad + 2\|c\|_{L^\infty} \|\nabla \sqrt{n}\|_{L^2} \|\sqrt{n}\nabla c\|_{L^2} - \|\sqrt{n}\nabla c\|_{L^2}^2 \\
&\quad - \int_{\Gamma_T} |c_x|^2 \,\mathrm{d}S + \int_{\Gamma_T} (\sigma c_{air} - c) c_{yy} \,\mathrm{d}S \\
&\leqslant \frac{1}{2}\int_{\Omega_l} |D^2 c|^2 \,\mathrm{d}x\,\mathrm{d}y + \tilde{C}_1 \int_{\Omega_l} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \tilde{C}_2 \int_{\Omega_l} \frac{|\nabla n|^2}{n} \,\mathrm{d}x\,\mathrm{d}y - \int_{\Gamma_T} |c_x|^2 \,\mathrm{d}S + \int_{\Gamma_T} (\sigma c_{air} - c) c_{yy} \,\mathrm{d}S.
\end{aligned} \tag{4.24}$$

Noticing that $c_t + uc_x = c_{xx} + c_{yy} - nc$ on Γ_T, multiplying this equation by $(\sigma c_{air} - c)$, integrating it on Γ_T, and using (4.14) yields

$$\begin{aligned}
&\int_{\Gamma_T} (\sigma c_{air} - c) c_{yy} \,\mathrm{d}S - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_T} (\sigma c_{air} c - \frac{|c|^2}{2}) \,\mathrm{d}S \\
&= \int_{\Gamma_T} (\sigma c_{air} - c) u c_x \,\mathrm{d}S + \int_{\Gamma_T} (\sigma c_{air} - c) cn \,\mathrm{d}S - \int_{\Gamma_T} (\sigma c_{air} - c) c_{xx} \,\mathrm{d}S \\
&\leqslant \frac{1}{2}\int_{\Gamma_T} |c_x|^2 \,\mathrm{d}S + \frac{|\sigma c_{air}|^2}{2}\int_{\Gamma_T} |u|^2 \,\mathrm{d}S + \frac{\sigma c_{air}}{2}\int_{\Gamma_T} n \,\mathrm{d}S \\
&\quad - \frac{1}{2}\int_0^l (\sigma c_{air} - c(x,1,t)) c_{xx}(x,1,t) \,\mathrm{d}x \\
&\leqslant \frac{1}{2}\int_{\Gamma_T} |c_x|^2 \,\mathrm{d}S + \frac{|\sigma c_{air}|^2}{2}\int_{\Gamma_T} |\nabla \mathbf{u}|^2 \,\mathrm{d}S + C\int_{\Gamma_T} \frac{|\nabla n|^2}{n} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int_{\Gamma_T} |c_x|^2 \,\mathrm{d}S + \tilde{C}_5.
\end{aligned} \tag{4.25}$$

Combining (4.24) with (4.25) gives

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l} |\nabla c|^2 \,\mathrm{d}x\,\mathrm{d}y - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_T} (\sigma c_{air} c - \frac{|c|^2}{2}) \,\mathrm{d}S + \frac{1}{2}\int_{\Omega_l} |D^2 c|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_T} |c_x|^2 \,\mathrm{d}S \\
&\leqslant \tilde{C}_3 \int_{\Omega_l} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}y + \tilde{C}_4 \int_{\Omega_l} \frac{|\nabla n|^2}{n} \,\mathrm{d}x\,\mathrm{d}y + \tilde{C}_5.
\end{aligned} \tag{4.26}$$

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Noticing that c is bounded uniformly, using (4.5), we arrive at

$$\sup_{0<t<T}\int_{\Omega_l}|\nabla c|^2\,\mathrm{d}x\,\mathrm{d}y+\sup_{0<t<T-1}\int_t^{t+1}\int_{\Omega_l}|D^2c|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+\sup_{0<t<T-1}\int_t^{t+1}\int_{\Gamma_T}|c_x|^2\,\mathrm{d}S\,\mathrm{d}s\leqslant C.$$

Multiplying the second equation of (4.8) by c_t, then we obtain

$$\sup_{0<t<T-1}\int_t^{t+1}\int_{\Omega_l}|c_t|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\leqslant C.$$

Then (4.22) is proved. The proof of (4.23) is also completely similar to (3.18), we omit it. □

Then similar to the proof of lemma 3.3 by deleting the term $\chi\int_{\Gamma_T}n^{r+1}c_y\,\mathrm{d}S$, and lemma 3.5, we also have

Lemma 4.3. *Assume that* $\mathcal{T}(\mathbf{u},n,\sigma)=(\mathbf{u},n)$. *For* $\tau=0$ *with small* $\chi>0$, *or* $\tau=1$, *we have*

$$\sup_{0<t<T}\int_{\Omega_l}|n(\cdot,t)|^{r+1}\,\mathrm{d}x\,\mathrm{d}y+\sup_{0<t<T-1}\int_t^{t+1}\int_{\Omega_l}|\nabla n^{\frac{r+1}{2}}|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\leqslant C_r,\quad \text{for any } r>0, \tag{4.27}$$

and

$$\sup_{0<t<T}\|\mathbf{u}\|_{L^\infty}\leqslant C, \tag{4.28}$$

where C_r, C *depend on* $n_0,c_0,\mathbf{u}_0$, χ, r *and* c_{air}, *and* C_r *also depends on* r, *and all of them are independent of* T.

Lemma 4.4. *Assume that* $\mathcal{T}(\mathbf{u},n,\sigma)=(\mathbf{u},n)$. *When* $\tau=0$ *with small* $\chi>0$, *or* $\tau=1$, *for any* $r>0$,

$$\sup_{0<t<T}\int_{\Omega_l}|\nabla c|^r\,\mathrm{d}x\,\mathrm{d}y\leqslant C_r, \tag{4.29}$$

where C_r *depends only on* $n_0,c_0,\mathbf{u}_0$, χ, r *and* c_{air}, *and it is independent of* T.

Proof. When $\tau=0$, the proof is completely similar to (3.25). In what follows, we only consider the case $\tau=1$. Let

$$\tilde{c}=\mathrm{e}^{\frac{y^2}{2}}(c-\sigma c_{air}).$$

It is easy to see that $\tilde{c}(x,y,t)$ is periodic on x with period l, $-\sigma c_{air}\,\mathrm{e}^{\frac{l^2}{2}}\leqslant\tilde{c}\leqslant 0$, and

$$\begin{cases}\dfrac{\partial\tilde{c}}{\partial t}+\mathbf{u}\nabla\tilde{c}-yv\tilde{c}=\Delta\tilde{c}+(y^2-1)\tilde{c}-y\tilde{c}_y-n\tilde{c}-\sigma c_{air}n\,\mathrm{e}^{\frac{y^2}{2}},\\ \dfrac{\partial\tilde{c}}{\partial\nu}\Big|_{\Gamma_T\cup\Gamma_B}=0.\end{cases} \tag{4.30}$$

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From a direct calculation, we derive that

$$\partial_y|\nabla\tilde{c}|^r\,\big|_{\Gamma_T\cup\Gamma_B} = r|\nabla\tilde{c}|^{r-2}(\tilde{c}_x\tilde{c}_{xy}+\tilde{c}_y\tilde{c}_{yy})\big|_{\Gamma_T\cup\Gamma_B} = 0.$$

Applying ∇ to the first equation of (4.30), multiplying the resulting equation by $|\nabla\tilde{c}|^{r-2}\nabla\tilde{c}$, and using (3.27), (4.22), (4.27), (4.28), it yields

$$\begin{aligned}
&\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_l}|\nabla\tilde{c}|^r\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{\Omega_l}\nabla\Delta\tilde{c}|\nabla\tilde{c}|^{r-2}\nabla\tilde{c}\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_l}\nabla(yv\tilde{c}-\mathbf{u}\nabla\tilde{c}\\
&\quad+(y^2-1)\tilde{c}-y\tilde{c}_y-n\tilde{c}-\sigma c_{air}n\,\mathrm{e}^{\frac{y^2}{2}})|\nabla\tilde{c}|^{r-2}\nabla\tilde{c}\,\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{r}\int_{\Omega_l}\Delta|\nabla\tilde{c}|^r\,\mathrm{d}x\,\mathrm{d}y-(r-1)\int_{\Omega_l}|\nabla\tilde{c}|^{r-2}|D^2\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad-\int_{\Omega_l}(yv\tilde{c}-\mathbf{u}\nabla\tilde{c}+(y^2-1)\tilde{c}-y\tilde{c}_y-n\tilde{c}-\sigma c_{air}n\,\mathrm{e}^{\frac{y^2}{2}})\big(|\nabla\tilde{c}|^{r-2}\Delta\tilde{c}\\
&\quad+(r-2)|\nabla\tilde{c}|^{r-4}\nabla\tilde{c}D^2\tilde{c}\nabla\tilde{c}\big)\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant-\frac{r-1}{4}\int_{\Omega_l}|\nabla\tilde{c}|^{r-2}|D^2\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y+C\int_{\Omega_l}(yv\tilde{c}-\mathbf{u}\nabla\tilde{c}\\
&\quad+(y^2-1)\tilde{c}-y\tilde{c}_y-n\tilde{c}-\sigma c_{air}n\,\mathrm{e}^{\frac{y^2}{2}})^2|\nabla\tilde{c}|^{r-2}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant-\frac{r-1}{4}\int_{\Omega_l}|\nabla\tilde{c}|^{r-2}|D^2\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y+\eta\int_{\Omega_l}|\nabla\tilde{c}|^{r+2}\,\mathrm{d}x\,\mathrm{d}y+C_\eta\int_{\Omega_l}(n^{\frac{r+2}{2}}+1)\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant-\frac{r-1}{2}\int_{\Omega_l}|\nabla\tilde{c}|^{r-2}|D^2\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}y+C_r,
\end{aligned}$$

then

$$\sup_{0<t<T}\|\nabla\tilde{c}\|^r_{L^r}+\sup_{0<t<T-1}\int_t^{t+1}\int_{\Omega_l}|\nabla\tilde{c}|^{r-2}|D^2\tilde{c}|^2\,\mathrm{d}xmy\,\mathrm{d}s\leqslant\tilde{C}_r,\quad\text{for any}\quad r>2, \tag{4.31}$$

which implies (4.29). □

Then similar to the proof of lemmas 3.5 and 3.6, we conclude that

LEMMA 4.5. *Assume that $\mathcal{T}(\mathbf{u},n,\sigma)=(\mathbf{u},n)$. For $\tau=0$ with small $\chi>0$, or $\tau=1$,*

$$\sup_{0<t<T}(\|n\|_{L^\infty}+\|\nabla c\|_{L^\infty}+\|A^\beta\mathbf{u}\|_{L^\infty})\leqslant C,\quad \textit{for any } \beta\in\left(\frac{1}{2},1\right), \tag{4.32}$$

$$\sup_{0<t<T-1}\Big(\|(\mathbf{u},n,c)\|_{W_p^{2,1}(Q_1^l(t))}+\|\pi\|_{W_p^{1,0}(Q_1^l(t))}\Big)\leqslant C_p,\quad \textit{for any } p>1, \tag{4.33}$$

where C, C_p depend only on $n_0,c_0,\mathbf{u}_0$, χ, c_{air}, β and p, and they are independent of T.

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Next, we show theorem 1.2.

Proof of theorem 1.2. Using lemma 4.5, completely similar to the proof theorem 1.1, we complete the proof of global existence in theorem 1.2.

Next, we show the uniqueness. Suppose the contrary. Let $(n_1, c_1, \mathbf{u_1}, \pi_1)$, $(n_2, c_2, \mathbf{u_2}, \pi_2)$ be two solutions of (1.1), (1.3), and (1.4). Denote $\tilde{n} = n_1 - n_2$, $\tilde{c} = c_1 - c_2$, $\tilde{\mathbf{u}} = \mathbf{u_1} - \mathbf{u_2} = (\tilde{u}, \tilde{v})$, $\tilde{\pi} = \pi_1 - \pi_2$. Then

$$\begin{cases} \tilde{n}_t + \tilde{\mathbf{u}} \cdot \nabla n_1 + \mathbf{u}_2 \cdot \nabla \tilde{n} = \Delta \tilde{n} - \chi \nabla \cdot (\tilde{n} \nabla c_1 + n_2 \nabla \tilde{c}), & \text{in } Q, \\ \tilde{c}_t - \Delta \tilde{c} + \tilde{\mathbf{u}} \cdot \nabla c_1 + \mathbf{u}_2 \cdot \nabla \tilde{c} = -\tilde{n} c_1 - n_2 \tilde{c}, & \text{in } Q, \\ \tilde{\mathbf{u}}_t + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_1 + \mathbf{u}_2 \cdot \nabla \tilde{\mathbf{u}} = \Delta \tilde{\mathbf{u}} - \nabla \tilde{\pi} + \tilde{n} \nabla \varphi, & \text{in } Q, \\ \nabla \cdot \tilde{\mathbf{u}} = 0, & \text{in } Q, \\ \tilde{n}_y - \chi(\tilde{n} \partial_y c_1 + n_2 \partial_y \tilde{c}) = 0, \ \tau \tilde{c}_y = -\tilde{c}, \ \tilde{v} = 0, \ \tilde{u}_y = 0, & (x, y) \in \Gamma_T, \\ \tilde{n}_y = \tilde{c}_y = 0, \ \tilde{u} = \tilde{v} = 0, & (x, y) \in \Gamma_B, \\ \tilde{n}(x, y, 0) = 0, \ \tilde{c}(x, y, 0) = 0, \ \tilde{\mathbf{u}}(x, y, 0) = 0, & x \in \Omega, \end{cases} \tag{4.34}$$

Multiplying the first equation of (4.34) by $\tilde{n}$, and integrating it over Ω yields

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\tilde{n}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega |\nabla \tilde{n}|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_\Omega \tilde{n} \tilde{\mathbf{u}} \cdot \nabla n_1 \,\mathrm{d}x\,\mathrm{d}y + \chi \int_\Omega (\tilde{n} \nabla c_1 + n_2 \nabla \tilde{c}) \nabla \tilde{n} \,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \frac{1}{4} \int_\Omega |\nabla \tilde{n}|^2 \,\mathrm{d}x\,\mathrm{d}y + C_1 \int_\Omega (|\tilde{\mathbf{u}}|^2 + |\tilde{n}|^2 + |\nabla \tilde{c}|^2) \,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

that is

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\tilde{n}|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{3}{4} \int_\Omega |\nabla \tilde{n}|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant C_1 \int_\Omega (|\tilde{\mathbf{u}}|^2 + |\tilde{n}|^2 + |\nabla \tilde{c}|^2) \,\mathrm{d}x\,\mathrm{d}y. \tag{4.35}$$

Multiplying the second equation of (3.36) by $\tilde{c}$, and integrating it over Ω yields

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\tilde{c}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega |\nabla \tilde{c}|^2 \,\mathrm{d}x\,\mathrm{d}y + \tau \int_{\Gamma_T} |\tilde{c}_y|^2 \,\mathrm{d}S \\ &= -\int_\Omega \tilde{c} \tilde{\mathbf{u}} \cdot \nabla c_1 \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega \tilde{c} \mathbf{u}_2 \cdot \nabla \tilde{c} \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega \tilde{c}(\tilde{n} c_1 + n_2 \tilde{c}) \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega \nabla \tilde{c} \cdot \tilde{\mathbf{u}} c_1 \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega \tilde{c}(\tilde{n} c_1 + n_2 \tilde{c}) \,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \frac{1}{4} \int_\Omega |\nabla \tilde{c}|^2 \,\mathrm{d}x\,\mathrm{d}y + C_2 \int_\Omega (|\tilde{\mathbf{u}}|^2 + |\tilde{n}|^2 + |\tilde{c}|^2) \,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

namely,

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\tilde{c}|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{3}{4} \int_\Omega |\nabla \tilde{c}|^2 \,\mathrm{d}x\,\mathrm{d}y + \tau \int_{\Gamma_T} |\tilde{c}_y|^2 \,\mathrm{d}S \leqslant C_2 \int_\Omega (|\tilde{\mathbf{u}}|^2 + |\tilde{n}|^2 + |\tilde{c}|^2) \,\mathrm{d}x\,\mathrm{d}y. \tag{4.36}$$

Similarly, we also have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\tilde{\mathbf{u}}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega |\nabla\tilde{\mathbf{u}}|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C_3\int_\Omega (|\tilde{\mathbf{u}}|^2 + |\tilde{n}|^2)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.37}$$

Combining (4.35)–(4.37) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}|\tilde{n}|^2 + \frac{1}{2}|\tilde{\mathbf{u}}|^2 + C_1|\tilde{c}|^2\right)\mathrm{d}x\,\mathrm{d}y + \frac{3}{4}\int_\Omega \left(|\nabla\tilde{n}|^2 + \frac{2C_1}{3}|\nabla\tilde{c}|^2 + |\nabla\tilde{\mathbf{u}}|^2\right)\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C_4\int_\Omega \left(|\tilde{\mathbf{u}}|^2 + |\tilde{c}|^2 + |\tilde{n}|^2\right)\mathrm{d}x\,\mathrm{d}y,$$

which implies that

$$\int_\Omega (|\tilde{n}|^2 + |\tilde{\mathbf{u}}|^2 + |\tilde{c}|^2 + |\nabla\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}y \equiv 0,\ \text{for any } t > 0,$$

and the uniqueness is proved. □

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