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LARGE DEVIATIONS AND QUASI-STATIONARITY FOR DENSITY-DEPENDENT BIRTH-DEATH PROCESSES

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Abstract

Consider a density-dependent birth-death process X_N on a finite state space of size N. Let P_N be the law (on $D([0, T])$ where $T > 0$ is arbitrary) of the density process X_N/N and let Π_N be the unique stationary distribution (on [0,1]) of X_N/N, if it exists. Typically, these distributions converge weakly to a degenerate distribution as $N \to \infty$, so the probability of sets not containing the degenerate point will tend to 0; large deviations is concerned with obtaining the exponential decay rate of these probabilities. Friedlin-Wentzel theory is used to establish the large deviations behaviour (as $N \to \infty$) of P_N. In the one-dimensional case, a large deviations principle for the stationary distribution Π_N is obtained by elementary explicit computations. However, when the birth-death process has an absorbing state at 0 (so Π_N no longer exists), the same elementary computations are still applicable to the quasi-stationary distribution, and we show that the quasi-stationary distributions obey the same large deviations principle as in the recurrent case. In addition, we address some questions related to the estimated time to absorption and obtain a large deviations principle for the invariant distribution in higher dimensions by studying a quasi-potential.

1. Introduction

Let X_N be a continuous-time birth-death process on the state space $\{0, 1, \dots, N\}^d \subset \mathbb{Z}^d$, whose Q-matrix has the form

$$q_N(n, n + e_i) = N b_i^{(N)}(n/N), \quad q_N(n, n - e_i) = N d_i^{(N)}(n/N), \tag{1.1}$$

where $n = (n_1, n_2, \dots, n_d)$ and $b^{(N)} = (b_1^{(N)}, \dots, b_d^{(N)})$ and $d^{(N)} = (d_1^{(N)}, \dots, d_d^{(N)})$ are Lipschitz-continuous $(\mathbb{R}^+)^d$-valued functions on $[0, 1]^d$ such that $b_i^{(N)}(x) = 0$ if $x_i = 1$ and $d_i^{(N)}(x) = 0$ if $x_i = 0$. We would also like to allow the possibility of absorbing states: for example, 0 may be absorbing, in which case we would

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have $b^{(N)}(0) = 0$; alternatively, the process might be absorbed when it hits one of faces $x_i = 0$ of the cube $[0, 1]^d$, in which case we would have $b_i^{(N)}(x) = 0$ if $x_i = 0$. We assume that $b_i^{(N)}, d_i^{(N)} > 0$ otherwise. Suppose further that, as $N \to \infty$, $b^{(N)} - d^{(N)} \to F$ for some Lipschitz-continuous $\mathbb{R}^d$-valued function F on $[0, 1]^d$. In the case where $b^{(N)} \equiv b$ and $d^{(N)} \equiv d$, such a process is said to be density-dependent, an idea first introduced by Kurtz [6]. In the more general situation just described, such a process may be called asymptotically density-dependent (see [8]). Although the notion of asymptotic density-dependency can be applied equally well to a process on an infinite state space and many of the results presented here remain true in the case of infinite state spaces, it is more natural and in some technical respects simpler (especially in the context of quasi-stationary distributions) to discuss such processes in the framework of finite state spaces. We shall concentrate on birth-death processes with finite state spaces in this paper.

Denote by $Y_N(t) := X_N(t)/N$ the density process associated with X_N. Kurtz [6] established the following law of large numbers.

THEOREM 1.1. *Suppose that $Y_N(0) \to y_0$ as $N \to \infty$ and let $T > 0$ be an arbitrary fixed positive number. Then for any $\epsilon > 0$,*

$$\mathbb{P}\left(\sup_{t \le 0 \le T} |Y_N(t) - y(t)| > \epsilon\right) \to 0$$

as $N \to \infty$, where $y(\cdot)$ is the solution to the ODE

$$\dot{y}(t) = F(y(t)), \qquad y(0) = y_0. \tag{1.2}$$

In particular, if $P_{N,T}$ is the law of $\{Y_N(t) : t \in [0, T]\}$ (so $P_{N,T}$ is a probability measure on $D([0, T], [0, 1]^d)$), we have $P_{N,T} \Rightarrow \delta_{y(\cdot)}$. The original version of this theorem stated in [6] applies only to the density-dependent case (that is, $b^{(N)} \equiv b$ and $d^{(N)} \equiv d$), however, the proof extends readily to the general asymptotically density-dependent case (see [8]).

Once a law of large numbers like Theorem 1.1 has been established, it is natural to ask whether the measure P_N and the stationary distribution obey a large deviations principle, to investigate the relationship between the path-wise large deviations at the process level and the large deviations of the stationary measure.

2. Large deviations for the density process

The aim of this section is to present a large deviations theorem for the law P_N of the density process Y_N. The main result in this section is not really new, although some relatively minor technical details are different from what has been established before.

Certainly, Theorem 2.1 below will come as no surprise to anyone familiar with the large deviations theory of similar Markov processes.

First, a reminder.

DEFINITION. Let E be a Polish space, μ_n a sequence of probability measures on E and a_N a sequence of positive numbers with $a_N \uparrow \infty$. Let $I : E \to [0, \infty]$ be a lower semi-continuous function. Then (μ_N, a_N) is said to obey a large deviations principle with rate function I if

$$-\inf_G I(x) \leq \liminf_{N\to\infty} \frac{1}{a_N} \log \mu_n(G)$$

for any open $G \subset E$,

$$\limsup_{N\to\infty} \frac{1}{a_N} \log \mu_N(F) \leq -\inf_F I(x)$$

for any closed $F \subset E$ and for each $s \geq 0$, the level set $\{x : I(x) \leq s\}$ is compact.

The above definition is a precise way of expressing the rough idea that if B_x is a small neighbourhood of x, then $\mu_N(B_x) \approx exp\{-a_N I(x)\}$. Example 4.1 illustrates one way in which this idea can be applied in practice.

From the Q-matrix defined by (1.1), we see that the density process Y_N is a Markov chain on the state space $S_N = \{0, N^{-1}, 2N^{-1}, \dots 1\}^d$ whose generator is

$$A_N f(x) = \int (f(x+u) - f(x)) Q^{(N)}(x, du), \quad x \in S_N \tag{2.1}$$

where $Q^{(N)}(x, du) = N \sum_{i=1}^{d} b_i^{(N)}(x)\delta_{e_i/N} + d_i^{(N)}(x)\delta_{-e_i/N}$. Consider the "logarithmic moment generating function" $G^{(N)}(x, z)$, defined as follows:

$$G^{(N)}(x, z) := \frac{d}{dt}\mathbb{E}^x[e^{z^T Y_N(t)-x}]\Big|_{t=0} = \int (e^{z^T u} - 1) Q^{(N)}(x, du). \tag{2.2}$$

Let $Q_N(x, du) = \sum_i b_i^{(N)}(x)\delta_{+e_i} + d_i^{(N)}(x)\delta_{-e_i}$ and

$$G_N(x, z) = \int (e^{z^T u} - 1) Q_N(x, du). \tag{2.3}$$

Next, let

$$H^N(x, u) = \sup_z \{z^T u - G^{(N)}(x, z)\},$$

$$H_N(x, u) = \sup_z \{z^T u - G_N(x, z)\},$$

be the Legendre transforms of $z \mapsto G^{(N)}(x, z)$ and $z \mapsto G_N(x, z)$ respectively, for fixed x. Observe that

$$G^{(N)}(x, z) = NG_N(x, z/N) \quad \text{and} \quad H^{(N)}(x, u) = NH_N(x, u), \tag{2.4}$$

which exactly reflects the scaling property exhibited by (1.1). (We effectively have $Y_N(t) = N^{-1}X(Nt)$ where X is the process whose Q-matrix is defined *via* the kernel $Q_N(x, du)$ in a manner analogous to (2.1).)

It is easy to show that, in the case of a birth-death process,

$$H_N(x, u) = \sum_{i=1}^{d}\left(u_i \log \frac{u_i + \sqrt{u_i^2 + 4b_i^{(N)}(x)d_i^{(N)}(x)}}{2b_i{}^{(N)}(x)}\right.$$
$$\left. - \sqrt{u_i^2 + 4b_i^{(N)}(x)d_i^{(N)}(x)} + b_i^{(N)}(x) + d_i^{(N)}(x)\right). \tag{2.5}$$

For the purposes of this section, we shall make the simplifying assumption that

$$b^{(N)} \to b \quad \text{and} \quad d^{(N)} \to d \tag{2.6}$$

uniformly in $x \in [0, 1]^d$ for some Lipschitz $(\mathbb{R}^+)^d$-valued functions b and d which are positive except on the boundaries as described in Section 1. (Thus, in the context of Theorem 1.1, $F \equiv b - d$.) These assumptions on $b^{(N)}$ and $d^{(N)}$ imply that $G_N(x, z) \to G(x, z)$ and $H_N(x, u) \to H(x, u)$ as $N \to \infty$ for fixed x, u, where $G(x, z) = \sum_i b_i(x)(e^{z_i} - 1) + d_i(x)(e^{-z_i} - 1)$ and

$$H(x, u) = \sum_{i}\left(u_i \log \frac{u_i + \sqrt{u_i^2 + 4b_i(x)d_i(x)}}{2b_i(x)}\right.$$
$$\left. - \sqrt{u_i^2 + 4b_i(x)d_i(x)} + b_i(x) + d_i(x)\right). \tag{2.7}$$

(Moreover, for fixed u, this convergence is uniform for $x \in [\epsilon, 1 - \epsilon]^d$ for any $\epsilon > 0$.) Define, for any arbitrary fixed $0 < R < T$,

$$S_{R,T}(\phi) := \int_R^T H(\phi(t), \dot{\phi}(t))\, dt \tag{2.8}$$

if $\phi \in C([0, T], \mathbb{R}^d)$ is absolutely continuous, otherwise set $S_{R,T}(\phi) = \infty$. Let $P^x_{N,T}$ be the law of $\{Y_N(t) : t \in [0, T]\}$ with $Y_N(0) = x$. (Since $P^x_{N,T}$ is a probability measure on $D([0, T])$, we need to think of $S_{R,T}$ as a function on $D([0, T])$.) It can be shown by standard arguments that $S_{R,T}$ is lower semi-continuous and has compact level sets in the uniform topology on $C([0, T], \mathbb{R}^d)$ (see, for example, [10], Theorem 5.1).

We have the following theorem, originally due to Wentzel [10].

THEOREM 2.1. *Suppose that the density-dependent process X_N satisfies the assumption* (2.6). *Then for any fixed $T > 0$ and each fixed $x \in (0, 1)$, $(P^x_{N,T}, N)$ obeys the large deviations principle with rate function $S \equiv S_{0,T}$ defined by* (2.8).

PROOF. Let $\rho_{0,T}(\phi, \psi) = \sup_{t\in[0,T]} |\phi(t) - \psi(t)|$. For any $s \geq 0$, let $\Phi_x(s) = \{\phi : S(\phi) \geq s\}$. Following [10], we prove the following two inequalities: for any $\delta, \gamma, s_0 > 0$, there exists N_0 such that for all $N \geq N_0$ and any function ϕ with $\phi(0) = x$ and $S(\phi) \leq s_0$,

$$P^x_{N,T}(\rho_{0,T}(Y_N, \phi) < \delta) \geq \exp\{-N(S(\phi) + \gamma)\}, \tag{2.9a}$$

$$P^x_{N,T}(\rho_{0,T}(Y_N, \Phi_x(s)) > \delta) \leq \exp\{-N(s - \gamma)\}. \tag{2.9b}$$

Consider first the case that $b^{(N)} \equiv b$ and $d^{(N)} \equiv d$ (so that $G_n \equiv G$ and $H_N \equiv H$). In this case, the proofs of these two inequalities are essentially identical to the proofs of, respectively, Theorem 6.1 and Theorem 6.2 in [10]. (Indeed, the present situation is almost identical to that of the example on pp. 228–229 of [10].) However, because of our assumption that $b_i(x) = 0$ for $x_i = 1$ and possibly for $x_i = 0$, some of the assumptions of [10] are not quite satisfied and a few points in the proof require a slightly different treatment. We shall concentrate on the main differences and omit the other details which can be found in [10].

The main assumption of [10] which is violated is that it is not true that

$$\Delta H(\delta) := \sup_{|y-y'|<\delta} \frac{H(y', u) - H(y, u)}{1 + H(y, u)} \to 0$$

as $\delta \downarrow 0$. However it is true that, for any $\epsilon > 0$,

$$\sup_{|y-y'|<\delta;\ y,y'\in[\epsilon,1-\epsilon]^d} \frac{H(y', u) - H(y, u)}{1 + H(y, u)} \to 0 \tag{2.10}$$

as $\delta \downarrow 0$. In addition, for any compact set $U \subset \mathbb{R}^d$, there is a constant M such that for all $x \in [\epsilon, 1 - \epsilon]^d$ and for all $u \in U$

$$|\nabla_u H| \leq M.$$

(Reference [10] also assumes that the above holds with $\epsilon = 0$.)

For arbitrary $\epsilon > 0$, define a truncated process $\bar{Y}_N$ as follows:

$$\bar{Y}_N(t) = \begin{cases} 1 - 1/N & \text{if } Y_N(t) = 1 \\ 1/N & \text{if } Y_N(t) = 0 \\ Y_N(t) & \text{otherwise.} \end{cases}$$

Thus $\rho_{0,T}(Y_N, \bar{Y}_N) \leq 1/N$. Since

$$\begin{aligned} P^x_{N,T}(\rho_{0,T}(Y_N, \phi) < \delta) &\geq P^x_{N,T}(\rho_{0,T}(\bar{Y}_N, \phi) + \rho_{0,T}(Y_N, \bar{Y}_N) < \delta) \\ &= P^x_{N,T}(\rho_{0,T}(\bar{Y}_N, \phi) < \delta - \rho_{0,T}(Y_N, \bar{Y}_N)) \\ &\geq P^x_{N,T}(\rho_{0,T}(\bar{Y}_N, \Phi_x(s)) < \delta - 1/N) \end{aligned}$$

and similarly

$$P^x_{N,T}(\rho_{0,T}(Y_N, \Phi_x(s)) > \delta) \leq P^x_{N,T}(\rho_{0,T}(\bar{Y}_N, \Phi_x(s)) > \delta - 1/N),$$

we see that it is sufficient to prove the inequalities (2.9) with $\bar{Y}_N$ in place of Y_N.

We prove first the inequality (2.9a). It is clearly sufficient to prove this for all ϕ such that $\phi(t) \in [\epsilon, 1-\epsilon]^d$ for arbitrary $\epsilon > 0$. Then defining

$$\Delta H^{(N)}(t) := \sup_{|y-\phi(t)|<\delta_N} [H^{(N)}(y, \dot{\phi}(t)) - H^{(N)}(\phi(t), \dot{\phi}(t))],$$

we can then use (2.4) together with (2.10) to get an estimate for $\Delta H^{(N)}(t)$, exactly as in the proof of Theorem 6.1 in [10]. Next, a key step in the proof involves making a Girsanov type change of measure, which is done using Theorem 3.1 of [10]. For this, we merely need to define $z(t, x) := \nabla_u H(x, \dot{\phi}(t))$ and

$$D(t) := \sup_{y\in[\epsilon,1-\epsilon]^d} \sum_i \frac{\partial^2 G(x, z(t, y))}{\partial z_i^2},$$

$$ZDZ(t) := \sup_{y\in[\epsilon,1-\epsilon]^d} \sum_i \frac{\partial^2 G(x, z(t, y))}{\partial z_i^2} z_i(t, y)^2,$$

and the rest of the argument is now the same as in [10].

For the upper bound (2.9b), the only modification needed is in the definition of ΔH_1 in Theorem 4.1 in [10]:

$$\Delta H_1 := \sup_{|y-y'|<2\delta', y, y'\in[\epsilon,1-\epsilon]} \frac{H(y', u) - H(y, u)}{A + H(y, u)}.$$

Finally, the asymptotically density-dependent case, where $b^{(N)} \to b$ and $d^{(N)} \to d$, is an easy extension for not only do $G_N \to G$ and $H_N \to H$, but also

$$\frac{\partial^2 G_N(x, z)}{\partial z_i^2} \to \frac{\partial^2 G(x, z)}{\partial z_i^2} \quad \text{and} \quad \nabla_u H_N(x, u) \to \nabla_u H(x, u).$$

Moreover, these convergences take place uniformly in u for u in each compact set and uniformly in x at least for $x \in [\epsilon, 1-\epsilon]^d$. The argument of Wentzel then goes through without difficulty.

REMARKS. (i) It should be emphasised that the large deviations estimates of Theorem 2.1—unlike those of Theorems 6.1 and 6.2 in [10]—do not hold uniformly for $x \in [0.1]$. However, they do hold uniformly for $x \in [\epsilon, 1-\epsilon]$ for any $\epsilon > 0$.

(ii) Of course, the result of Theorem 2.1 holds in much greater generality than in the context of birth-death processes: the result has been formulated for a very general class of (not necessarily time-homogeneous) Markov processes in [10]—in particular, it holds for any density-dependent Markov chain. Similar results have been obtained in the context of processes arising from queueing theory: see for example, [9], [3] and [2]. However, in the case of birth-death processes, the function H and the rate function have particularly simple forms and it is not easy to extend the results in some of the subsequent sections to general density-dependent Markov processes.

3. Large deviations of the stationary distribution in one dimension

Consider first a one-dimensional density-dependent birth-death process; the higher-dimensional case is deferred until Section 5. Suppose that the process X_N introduced in Section 1 has a unique stationary distribution—in particular, both end-points of $[0, 1]$ are reflecting barriers and we must have $b^{(N)}(0) > 0$ and $d^{(N)}(1) > 0$.

Let Π_N be the stationary distribution of the density process Y_N on $\{0, 1/N, 2/N, \dots, 1\}$:

$$\Pi_N(j/N) = \Pi_N(0) \prod_{k=0}^{j-1} \frac{b^{(N)}(k/N)}{d^{(N)}((k+1)/N)}, \tag{3.1}$$

where

$$\Pi_N(0)^{-1} = \sum_{j=0}^{N} \prod_{k=1}^{j-1} \frac{b^{(N)}(k/N)}{d^{(N)}((k+1)/N)}. \tag{3.2}$$

Suppose there exists a function r on $[0, 1]$ satisfying the following conditions:

$$\int_{0+} |\log r(u)|\, du < \infty, \tag{3.3a}$$

$$\frac{1}{N} \sum_{k=1}^{N-1} \left(\log \frac{d^{(N)}((k+1)/N)}{b^{(N)}(k/N)} - \log r(k/N) \right) \to 0 \quad \text{as} \quad N \to \infty. \tag{3.3b}$$

Note that a necessary condition for (3.3b) to hold is that

$$\log \frac{d^{(N)}(x)}{b^{(N)}(x)} \to \log r(x)$$

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as $N \to \infty$, for almost all $x \in [0, 1]$. However, this alone is not sufficient—we require that the convergence takes place in a sufficiently uniform manner, although (3.3b) is weaker than uniform convergence. Also, note that the conditions (3.3a,b) together with our assumptions on $b^{(N)}$ and $d^{(N)}$ imply that $\int_0^x |\log r(u)|\,du < \infty$ for any $x < 1$ but it is possible that $\int_0^1 |\log r(u)|\,du = \infty$.

Define a function J (the "free energy function") on $[0, 1]$ by

$$J(x) = \int_0^x \log r(u)\,du \tag{3,4}$$

and let J_0 be its global minimum in $[0, 1]$ (the "Gibbs canonical free energy"):

$$J_0 = \min_{y\in[0,1]} \int_0^y \log r(u)\,du. \tag{3.5}$$

THEOREM 3.1. *Suppose the conditions* (3.3) *are satisfied. Then the family* (Π_N, N) *obeys a large deviations principle with rate function* $I : [0, 1] \to [0, \infty]$ *given by* $I(x) = J(x) - J_0$.

PROOF. It is sufficient to prove that for any $\gamma > 0$ and any $x \in [0, 1]$, there exists $\epsilon > 0$ such that

$$\liminf_{N\to\infty} \frac{1}{N} \log \Pi_N((x-\epsilon, x+\epsilon)) \geq -(I(x)+\gamma), \tag{3.6a}$$

$$\limsup_{N\to\infty} \frac{1}{N} \log \Pi_N((x-\epsilon, x+\epsilon)) \leq -(I(x)-\gamma). \tag{3.6b}$$

The proof relies on the following elementary fact: if $a_1, a_2, \ldots, a_n$ are positive numbers, then

$$\max_i a_i \leq \sum_{k=1}^n a_k \leq n \max_i a_i$$

and hence

$$\max_i \log a_i \leq \log \sum_{k=1}^n a_k \leq \log n + \max_i \log a_i. \tag{3.7}$$

The assumptions (3.3) imply that, given any x and any $\gamma > 0$, we can choose ϵ so that

$$\left| \int_0^z \log r(u)\,du - \int_0^x \log r(u)\,du \right| \leq \frac{\gamma}{2}, \tag{3.8a}$$

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} \log \frac{d^{(n)}((k+1)/N)}{b^{(N)}(k/N)} - \frac{1}{N} \sum_{k=1}^{N-1} \log r(k/N) \right| \leq \frac{\gamma}{2} \tag{3.8b}$$

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for all $z \in (x-\epsilon, x+\epsilon)$ and for all sufficiently large N. (The slightly different ranges of summation in (3.8b) is to avoid problems caused by the fact that $r(0) = 0$; it is inconsequential as far as the subsequent arguments are concerned.)

We first show that, as $N \to \infty$,

$$\frac{1}{N} \log \Pi_N(0) \to J_0. \tag{3.9}$$

From (3.2) and the lower bound in (3.7)

$$\begin{aligned} -\frac{1}{N} \log \Pi_N(0) &\geq \max_{1 \leq j \leq N} \left(\frac{1}{N} \sum_{k=0}^{j-1} \log \frac{b^{(N)}(k/N)}{d^{(N)}((k+1)/N)} \right) \\ &\geq \max_{1 \leq j \leq N} \left(\frac{1}{N} \sum_{k=1}^{j-1} - \log r(k/N) - \gamma/2 \right) \\ &\to \max_{0 \leq z \leq 1} \int_0^z - \log r(u)\, du - \gamma/2 = -J_0 - \gamma/2, \end{aligned}$$

where we have used (3.8b) to obtain the second inequality. Similarly, using the upper bound in (3.7) we obtain

$$\lim_{N \to \infty} \frac{1}{N} \log \Pi_N(0) \geq J_0 - \gamma/2$$

and since γ is arbitrary, we have established (3.9).

Consider now the lower bound (3.6a). We have

$$\Pi_N(x-\epsilon, x+\epsilon) = \Pi_N(0) \sum_{j=[(x-\epsilon)N]}^{[(x+\epsilon)N]} \prod_{k=0}^{j-1} \frac{b^{(N)}(k/N)}{d^{(N)}((k+1)/N)}.$$

(Here, $[\cdot]$ denotes the integer part.) Hence, using the lower bound in (3.7),

$$\begin{aligned} &\frac{1}{N} \log \Pi_N((x-\epsilon, x+\epsilon)) \\ &\quad \geq \frac{1}{N} \log \Pi_N(0) + \max_{[(x-\epsilon)N] \leq j \leq [(x+\epsilon)N]} \frac{1}{N} \sum_{k=0}^{j-1} \log \frac{b^{(N)}(k/N)}{d^{(N)}((k+1)/N)} \\ &\quad \geq \frac{1}{N} \log \Pi_N(0) + \max_{[(x-\epsilon)N] \leq j \leq [(x+\epsilon)N]} \frac{1}{N} \sum_{k=0}^{j-1} - \log r(k/N) - \gamma/2 \\ &\quad \to J_0 + \max_{z \in (x-\epsilon, x+\epsilon)} \int_o^z - \log r(u)\, du - \gamma/2 \\ &\quad \geq J_0 - \int_0^x \log r(u) du - \gamma, \end{aligned}$$

where we have used (3.8) in the last two steps. This establishes (3.6a). The upper bound (3.6b) is proved similarly, but now applying the upper bound in (3.7) as the first step in the above calculation.

EXAMPLE 3.1. Consider the Ehrenfest chain: $b^{(N)}(x) = b(x) = \lambda(1-x)$ and $d^{(N)}(x) = d(x) = \mu x$ where $\lambda, \mu > 0$. Then Π_N is the binomial distribution

$$\Pi_N(j/N) = \binom{N}{j} p^j (j-p)^{N-j},$$

where $p = \lambda/(\lambda+\mu)$. In the context of Theorem 3.1, we have $r(x) = d(x)/b(x)$ and

$$\begin{aligned} J(x) &= \int_0^x \log r(u)\,du = \int_0^x \log \frac{\mu u}{\lambda(1-u)}\,du \\ &= x\log \mu x + (1-x)\log \lambda(1-x) - \log \lambda. \end{aligned}$$

The minimum value of $J(x)$ is attained at $x = p$, giving the minimum value as $J_0 = \log(1-p)$. Putting all this together, we find that

$$I(x) = J(x) - J_0 = x\log x + (1-x)\log(1-x) - x\log p - (1-x)\log(1-p).$$

Observe that this last expression is the relative entropy

$$\int \log \frac{d\nu_x}{d\nu_p}\,d\nu_x,$$

where $\nu_x = (1-x)\delta_0 + x\delta_1$, which is exactly the rate function given by Sanov's theorem if we were to write the binomial distribution as a sum of independent Bernoulli measures.

A similar procedure can be carried out for the one-species parasite model studied in [8], but the computations are messier.

Theorem 3.1 implies that Π_n converges weakly—at least along a subsequence—to a measure concentrated at those points a where $I(a) = 0$, that is at the points a where the function J defined at (3.4) attains its global minimum J_0 on $[0, 1]$.

If the global minimum is attained at $a \in (0, 1)$, we have $J'(a) = 0$, which is equivalent to $r(a) = 1$, which in turn is the same as $F(a) = 0$. In addition, since such points a occur at a (local) minimum, we must have $\log r(a-) < 0$ and $\log r(a+) > 0$, which is equivalent to $F(a-) > 0$ and $F(a+) < 0$; if F and r are differentiable, this is merely saying that $J''(a) < 0$ and $F'(a) < 0$. (A similar conclusion can be drawn if the global minimum is 0 at one of the endpoints $a = 0, 1$.) In other words, *the support of any limit point Π_∞ of the sequence $\{\Pi_N\}$ is a subset of the stable* *equilibrium points of the ODE* (1.2). In particular, if I attains its global minimum at a unique point $a^* \in [0, 1]$, then $\Pi_N \Rightarrow \delta_{a^*}$ as $N \to \infty$. For example, for the Ehrenfest chain in Example 3.1, I attains its global minimum at $x = p$ and so $\Pi_N \Rightarrow \delta_p$ as $N \to \infty$, which is easily checked directly using the normal approximation to the binomial distribution.

The ODE (1.2) may be written in the form

$$\dot{y}(t) = -U'(y(t)),$$

where $U(x) = -\int_0^x F(u)\,du$. Note that U and J have local minima (and local maxima) at the same points. However, the global minima of the two functions need not coincide. Suppose that U and J have more than one local minima but only one global minimum. Intuitively, one might expect that the limiting distribution Π_∞ should be concentrated at the global minimum of the potential function U. But this is not necessarily the case, since it is the global minimum of J that determines where Π_n converges to. The following is a rather extreme example of this distinction.

EXAMPLE 3.2. Let $\delta > 0$, $0 < \epsilon < K$, and $0 < a < 1/2$. Suppose that b and d are continuous functions satisfying

$$d(u) = \begin{cases} \epsilon & u \in (\delta, a-\delta) \\ K & u \in (a+\delta, 2a-\delta) \\ K/2 & u \in (2a+\delta, a+1/2-\delta) \\ K & u \in (a+1/2+\delta, 1] \end{cases}$$

and

$$b(u) = \begin{cases} K & u \in [0, a-\delta) \\ \epsilon & u \in (a+\delta, 2a-\delta) \\ K & u \in (2a+\delta, a+1/2-\delta) \\ K/2 & u \in (a+1/2+\delta, 1-\delta), \end{cases}$$

with reflecting boundary conditions $d(0) = b(1) = 0$. Following the standard notation we have established above, we have $J(x) = \int_0^x \log(d(u)/b(u))\,du$ and $U(x) = \int_0^x d(u) - b(u)\,du$. Suppose further that $b(a) = d(a)$, $b(2a) = d(2a)$ and $b(a+1/2) = d(a+1/2)$, that U and J are C^2 functions with $U(0) = J(0) = U(2a) = J(2a) = U(1) = J(1) = 0$. Thus U and J have local minima at a and $a + 1/2$ (and a local maximum at $2a$). We have $U(a) \approx a(\epsilon - K)$, $U(a + 1/2) \approx -K(1/2 - a)/2$, $J(a) \approx a(\log\epsilon - \log K)$ and $J(a + 1/2) \approx -(1/2 - a)\log 2$. We can clearly choose a and ϵ small enough and K large enough so that the global minimum of U occurs at $a + 1/2$ while the global minimum of J occurs at a. Note that we can make a as small as we please. Even though for very small a the well centred at a is much smaller than the well centred at $a + 1/2$, Theorem 3.1 tells us that nevertheless $\Pi_N \Rightarrow \delta_a$.

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The point that free energy is not the same as potential is well-documented in the physics literature—for example, see [5]. Stable equilibrium points of (1.2) which do not correspond to the global minimum of J (and I) are known as meta-stable points.

The one-dimensional case is the only case where an explicit product-form expression for the stationary distribution is known, without making additional assumptions. Of course, in the special cases where the stationary distribution in higher dimensions also has a product-form solution along the lines of (3.1), the proof of Theorem 3.1 can be readily modified to give a similar large deviations result. In Section 5, we present a more general large deviations result which does not rely on the existence of an explicit product-form formula.

4. Large deviations of quasi-stationary distributions in one dimension

We continue to work in one dimension. Suppose that $b^{(N)}(0) = 0$, so that 0 is an absorbing state and the set $\{1, 2, \dots, N\}$ is an irreducible transient class. Moreover, our other assumptions on $b^{(N)}$ and $d^{(N)}$ imply that the process Y_N will almost surely be eventually absorbed. The stationary distribution no longer exists; however, we may consider the quasi-stationary distributions and investigate their large deviations behaviour as $N \to \infty$. Note that since $b^{(N)}(0) = d^{(N)}(0) = 0$, we have $F(0) = 0$, so that 0 is an equilibrium point of the ODE (1.2). The most interesting case is that in which 0 is an unstable equilibrium and the ODE (1.2) has positive stable equilibria, for then Theorem 1.1 says the process Y_N should stay close to a trajectory of (1.2) which is attracted to a stable equilibrium, but of course Y_N must escape to 0. Therefore, it is interesting to enquire about the large deviation behaviour of the law of Y_N, conditioned on not having been absorbed.

To find the quasi-stationary distribution, we first need to obtain a μ-invariant measure. To this end, we make use of some results of [7], in particular Theorems 4.1 and 4.4 and Example 1 of that paper. Observe firstly that the birth-death process is reversible with respect to the subinvariant measure

$$v_j^{(N)} = v_1^{(N)} \prod_{k=1}^{j-1} \frac{b^{(N)}(k/N)}{d^{(N)}((k+1)/N)}, \qquad j = 1, 2. \dots, N. \tag{4.1}$$

Here and in the sequel, we shall not always distinguish between measures and vectors associated with the transient class $\{1, 2, \dots, N\}$ of X_N and the same quantities associated with the transient class $\{1/N, 2/N, \dots, 1\}$ of Y_N. Moreover, for notational convenience, we shall write $b_j^{(N)}$ and $d_j^{(N)}$ for $b^{(N)}(j/N)$ and $d^{(N)}(j/N)$.

Given $\mu > 0$, the μ-reverse of the Q-matrix of a birth-death process, with respect to the μ-invariant measure $m^{(N)}$ on $\{1, 2, \dots, N\}$ (or $\{1/N, 2/N, \dots, 1\}$), again describes a birth-death process (see [7]). The state space of this μ-reverse birth-death process is $\{1, 2, \ldots, N\}$ (or $\{1/N, 2/N, \ldots, 1\}$). Denote by $\tilde{b}^{(N)}$ and $\tilde{d}^{(N)}$ the birth and death rates respectively of the μ-reverse process. Example 1 of [7] derives the following recurrence equation for $\tilde{b}^{(N)}$ and $\tilde{d}^{(N)}$:

$$\tilde{b}_{j+1}^{(N)} = b_{j+1}^{(N)} + d_{j+1}^{(N)} - \mu - \frac{b_j^{(N)} d_{j+1}^{(N)}}{\tilde{b}_j^{(N)}}, \tag{4.2a}$$

$$\tilde{d}_{j+1}^{(N)} = b_{j+1}^{(N)} + d_{j+1}^{(N)} - \tilde{b}_{j+1}^{(N)} - \mu = \frac{b_j^{(N)} d_{j+1}^{(N)}}{\tilde{b}_j^{(N)}}, \tag{4.2b}$$

for $j = 1, 2, \ldots, N-1$. The requirements that $\tilde{d}_1^N = 0$ gives the initial equation

$$\tilde{b}_1^{(N)} = b_1^{(N)} + d_1^{(N)} - \mu. \tag{4.3}$$

The μ-invariant measure can then be expressed in terms of the μ-reverse birth rates:

$$m_j^{(N)} = m_1^{(N)} \prod_{k=1}^{j-1} \tilde{b}_k^{(N)} / d_{k+1}^{(N)}, \qquad j = 1, 2, \ldots, N. \tag{4.4}$$

However, in order that the μ-invariant measure $m^{(N)}$ be the unique (stationary conditional) quasi-stationary distribution (uniqueness follows from the fact that the state space is finite), we must choose $-\mu$ to be the maximal Perron-Frobenius eigenvalue of the Q-matrix, so that μ is the decay parameter. For a birth-death process, this must be given by $\mu = m_1^{(N)} d_1^{(N)}$. To see this, fix a state $i \neq 0$ and let

$$m_j^{(N)}(t) = \mathbb{P}(X_N(t) = j) | X_N(0) = i,\ X_N(t) \neq 0) = \frac{P_{ij}(t)}{1 - P_{i0}(t)},$$

where $P_{ij}(t)$ is the transition function. Writing $m^{(N)}(t) = (m_1^{(N)}(t), \ldots, m_N^{(N)}(t))$ as a row vector, the Kolmogorov forward equation shows that

$$\frac{dm^{(N)}}{dt} = m^{(N)} Q + (m^{(N)} \cdot q) m^{(N)},$$

where q is the vector with entries $q_k = Q_{k0}$. The limiting conditional quasi-stationary distribution is an equilibrium point of the above ODE. Now simply observe that for a birth-death process, $(m^{(N)} \cdot q) = m_1^{(N)} d_1^{(N)}$.

Since $d_0^{(N)} = 0$, we have $d_1^{(N)} = d^{(N)}(1/N) \to 0$ as $N \to \infty$ and hence $m_1^{(N)} d_1^{(N)} \to 0$ ($m_1^{(N)} < 1$). If $\mu = 0$ in (4.2), the solution to the recurrence relation (4.2a) is simply $\tilde{b}^{(N)} \equiv b^{(N)}$. Since $\mu \approx 0$ for large N, we should have

$$\tilde{b}_j^{(N)} \sim b_j^{(N)}, \quad \tilde{d}_j^{(N)} \sim d_j^{(N)}. \tag{4.5}$$

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Indeed, the finiteness of the state space $\{1, 2, \dots, N\}$ implies the boundary condition $\tilde{b}_N^{(N)} = 0$. Therefore, starting from $\tilde{b}_N^{(N)} = 0$ (rather than (4.3)), we have by induction

$$\begin{aligned} \tilde{b}_{N-1}^{(N)} &= \frac{b_{N-1}^{(N)} d_N^{(N)}}{d_N^{(N)} - m_1^{(N)} d_1^{(N)}} \sim b_{N-1}^{(N)}, \quad \Rightarrow \tilde{b}_{N-1}^{(N)} - b_{N-1}^{(N)} \to 0, \\ \tilde{b}_{N-2}^{(N)} &= \frac{b_{N-2}^{(N)} d_{N-1}^{(N)}}{b_{N-1}^{(N)} - \tilde{b}_{N-1}^{(N)} + d_{N-1}^{(N)} - m_1^{(N)} d_1^{(N)}} \sim b_{N-2}^{(N)}, \quad \Rightarrow \tilde{b}_{N-2}^{(N)} - b_{N-2}^{(N)} \to 0, \quad \text{etc.} \end{aligned} \tag{4.6}$$

Theorem 4.1 of [7] gives that the μ-invariant vector associated with $m^{(N)}$ is given by

$$x_j^{(N)} = m_j^{(N)} / v_j^{(N)} = x_1 \prod_{k=1}^{j-1} \tilde{b}_k^{(N)} / b_k^{(N)}. \tag{4.7}$$

The doubly limiting conditional quasi-stationary distribution is given by

$$\lim_{t\to\infty} \lim_{s\to\infty} \mathbb{P}(X_N(t) = j \mid X_N(t+s) \neq 0) = m_j^{(N)} x_j^{(N)}.$$

In view of (4.4), (4.5) and (4.7), we are led to the following theorem.

THEOREM 4.1. *Suppose the conditions* (3.3) *hold. Then each of the two families of quasi-stationary distributions* $(m^{(N)}, N)$ *and* $(m^{(N)} x^{(N)}, N)$ *obey a large deviations principle with the rate function* I *given by Theorem* 3.1.

PROOF. Assume (3.3) holds and consider first the limiting conditional distribution $m^{(N)}$ given by (4.4). Comparing (4.4) with (3.1), we see that the only difference is that $b^{(N)}$ in (3.1) has been replaced by $\tilde{b}^{(N)}$ in (4.4). Therefore, if we can show that (3.3) holds with $\tilde{b}^{(N)}$ in the place of $b^{(N)}$, the proof of Theorem 3.1 will go through exactly the same as before. (The fact that the product in (3.1) starts from $k = 0$ whereas the starting point is $k = 1$ in (4.4) will not affect the relevant asymptotics.) In order to check that (3.3) still hold with $\tilde{b}^{(N)}$ in the place of $b^{(N)}$, we simply need to check that

$$\frac{1}{N} \sum_{k=1}^{N-1} \log \left(\frac{\tilde{b}_k^{(N)}}{b_k^{(N)}} \right) \to 0 \tag{4.8}$$

as $N \to \infty$, which is true because, since $d^{(N)}$ is bounded away from 0 in a neighbourhood of 1, (4.6) shows that $\tilde{b}_k^{(N)} / b_k^{(N)} \to 0$ as $N \to \infty$ uniformly in k for $k \geq K_0$, for some fixed K_0.

The same sort of argument applies to the doubly limiting conditional distribution $m^{(N)} x^{(N)}$, as (4.8) shows that $N^{-1} \log x^{(N)} \to 0$.

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EXAMPLE 4.1. Suppose $b^{(N)}(x) = b(x) = \lambda x(1-x)$ and $d^{(N)}(x) = d(x) = \mu x$ where $\lambda, \mu > 0$. (Such a model might describe a population inhabiting a finite habitat and individuals can only give birth if there is some unoccupied habitat for the offspring; here, N is the number of sites in the habitat and $x \in [0, 1]$ is the proportion of habitat already occupied and the birth rate is proportional to the unoccupied area.) With $r \equiv d/b$, (3.4) gives

$$J(x) = x \log \frac{\mu}{\lambda} + (1-x)\log(1-x) + x. \tag{4.9}$$

The most interesting case is where $\mu < \lambda$, for then $1 - \mu/\lambda$ is the unique stable equilibrium of (1.2) in [0,1] and the absorbing state 0 is an unstable equilibrium. The function J attains its global minimum on [0, 1] uniquely at $x = 1 - \mu/\lambda$ and the global minimum is given by

$$J_0 = 1 - \frac{\mu}{\lambda} + \log \frac{\mu}{\lambda}.$$

According to Theorem 4.1, the quasi-stationary distribution has large deviations rate function I given by

$$I(x) = (1-x)\left(\log(1-x) - \log \frac{\mu}{\lambda} - 1\right) + \frac{\mu}{\lambda}. \tag{4.10}$$

Since I and J attain their global minimum at a unique point $x = 1 - \mu/\lambda$, Theorem 4.1 implies that $m^{(N)} \Rightarrow \delta_{1-\mu/\lambda}$.

We can use the large deviations result to obtain an approximate confidence interval for the population density, conditional on non-extinction. For an α-confidence interval, we seek an interval of the form $(1 - \mu/\lambda - a, 1 - \mu/\lambda + a)$ whose $m^{(N)}$-measure is α. Since the rate function I is convex and has a unique minimum at $1 - \mu/\lambda$, the large deviations principle tells us that $m^{(N)}(1 - \mu/\lambda + a, 1) \approx e^{-NI(1-\mu/\lambda+a)}$ (for large N). Similarly, $m^{(N)}(0, 1 - \mu/\lambda - a) \approx e^{-NI(1-\mu/\lambda-a)}$. Therefore, an approximate α-confidence interval can be found by solving

$$e^{-NI(1-\mu/\lambda-a)} + e^{-NI(1-\mu/\lambda+a)} = 1 - \alpha, \tag{4.11}$$

for $0 < a < \max(\mu/\lambda, 1 - \mu/\lambda)$. It is not hard to see that (4.11) has a unique solution, at least for large enough N. For example, suppose $\mu/\lambda = 1/2$ and take $N = 100, \alpha = 0.95$. Solving (4.11) numerically, we find $a \approx 0.1926$ and so an approximate 95% confidence interval for the population density given non-extinction is $(0.307, 0.693)$. Repeating this for $N = 500$, we find $a \approx 0.0859$, giving an approximate 95% confidence interval $(0.414, 0.586)$.

On the other hand, if $\mu > \lambda$, then the absorbing state 0 is the unique stable equilibrium of (1.2) in [0, 1]. The global minimum on [0, 1] of the function J at

(4.9) is attained at $x = 0$ and the global minimum is $J_0 = 0$. Thus $I(x) = J(x)$ and $m^{(N)} \Rightarrow \delta_0$. In this case, there is no "genuine" quasi-stationary distribution in the sense of a long-term equilibrium conditional on non-absorption because the process reaches the absorbing state 0 very quickly.

5. The quasi-potential and applications

We again make the assumption (2.6) and consider for the moment only the one-dimensional case. Thus in particular, $F \equiv b-d$ in (1.2) and (3.3) holds with $r \equiv d/b$.

Let y^* be any stable equilibrium of the ODE (1.2) with domain of attraction D. The quasi-potential relative to y^* is the function V_{y^*} defined by

$$V_{y^*}(x) = \inf\{S_{T_1,T_2}(\phi) : -\infty < T_1 < T_2 < \infty, \phi(T_1) = y^*, \phi(T_2) = x\}, \quad (5.1)$$

where S_{T_1,T_2} is the action functional given by (2.7) and (2.8). Note that $x \mapsto V_{y^*}(x)$ is continuous, non-negative and $V_{y^*}(y^*) = 0$.

The quasi-potential plays a central role in the study of exit times and large deviations of the invariant distribution. Our first aim, however, is to obtain an explicit formula for V. This is a relatively straight-forward exercise in the calculus of variations. Similar calculations can be found in [9]. The Euler-Lagrange equation

$$\frac{d}{dt}\left[H - \dot{\phi}\frac{\partial H}{\partial \dot{\phi}}\right] = 0$$

together with a transversality condition (the times T_1 and T_2 are not fixed) gives

$$H - \dot{\phi}\frac{\partial H}{\partial \dot{\phi}} = 0.$$

Putting the expression (2.7) for H into the above identity results in

$$\dot{\phi}^2 = (b(\phi) - d(\phi))^2, \qquad \phi(T_1) = y^*, \phi(T_2) = x. \quad (5.2)$$

For ϕ satisfying (5.2), we have

$$V_{y^*}(x) = \int_{T_1}^{T_2} \dot{\phi} \log \frac{\dot{\phi} + \sqrt{\dot{\phi}^2 + 4b(\phi)d(\phi)}}{2b(\phi)}\, dt. \quad (5.3)$$

Since y^* is a stable fixed point of (1.2) (with $F \equiv b - d$), $b(z) > d(z)$ for $z \in D \cap (-\infty, y^*)$ and $b(z) < d(z)$ for $z \in D \cap (y^*, \infty)$.

Consider first the case that $x \in D$. If $x < y^*$ then $\dot{\phi}(t) < 0$ for $t \in (T_1, T_2)$ and (5.2) shows that $\dot{\phi} = d(\phi) - b(\phi)$. Similarly, if $x > y^*$ and $x \in D$ then $\dot{\phi}(t) > 0$ for

$t \in (T_1, T_2)$ and again $\dot{\phi} = d(\phi) - b(\phi)$. Hence, from (5.3) we have (bearing in mind that $r \equiv d/b$),

$$V_{y^*}(x) = \int_{y^*}^{x} \log \frac{d(z)}{b(z)}\, dz.$$

The above is also true for $x \in \partial D$ since V_{y^*} is continuous. These results can be summarized as the following theorem:

THEOREM 5.1. *Let y^* be a stable fixed point of* (1.2) *and let V_{y^*} be the quasi-potential with respect to y^* defined by* (5.1). *Then for x in closure of the domain of attraction of y^*,*

$$V_{y^*}(x) = I(x) - I(y^*),$$

where I is the large deviations rate function given by Theorem 3.1. *Moreover, the infimum in* (5.1) *is achieved by the time-reversal of the trajectory of* (1.2) *which goes from x to y^*: that is, for $u \in (-\infty, 0]$, $\phi(u) = \psi(-u)$ where $\psi(u)$ satisfies* (1.2) *with $\psi(0) = x$. Up to a constant time-shift, this action-minimizing trajectory is unique.*

Although in our case it is essentially just a rather pretentious way of presenting some elementary computations, Theorem 5.1 is nevertheless a result of rather wide scope which holds in many other similar situations—for example, see [1].

We can now use the quasi-potential to study the problem of exit from an interval $D = (\alpha, \beta)$ by Y_N; in particular we wish to take D to be the domain of attraction of a stable fixed point y^* of (1.2). The following result is due to Freidlin and Wentzel [4]

THEOREM 5.2. *Let y^* be a stable fixed point of* (1.2) *and let $D = (\alpha, \beta)$ be an open interval containing y^*. Let $\tau_N = \inf\{t : Y_N(t) \notin D\}$. Then for any $x \in D$ and any $\epsilon > 0$,*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}^x[\tau_N] = \min(V_{y^*}(\alpha), V_{y^*}(\beta)).$$

If there is a unique $y_0 \in \{\alpha, \beta\}$ such that $V_{y^*}(y_0) = \min(V_{y^*}(\alpha), V_{y^*}(\beta))$ then

$$\lim_{N\to\infty} \mathbb{P}^x(|Y_N(\tau_N) - y_0| > \epsilon) = 0.$$

Theorem 5.2 is essentially a restatement of Theorems 2.1 and 4.1 in Chapter 4 of [4], specialised to the present situation. Although the corresponding results in [4] deal with diffusion perturbations of (1.2), the proofs can be readily extended to our situation (see Chapter 5, Section 4 of [4]) because for the most part they do not rely on any specific properties of the diffusions or the associated rate function but only on the underlying large deviations structure of the processes.

EXAMPLE 4.1 REVISITED. We can obtain an estimate for the expected time to extinction in this example. We have seen that when $\mu < \lambda$, the unique stable point is

$y^* = 1 - \mu/\lambda$ and since $I(y^*) = 0$, $V_{y^*}(x) = I(x)$ where I is given by (4.10). We have

$$V_{y^*}(0) = \frac{\mu}{\lambda} - \log\frac{\mu}{\lambda} - 1,$$
$$V_{y*}(1) = \frac{\mu}{\lambda}.$$

Therefore, provided $\mu/\lambda > e^{-1}$, $V_{y^*}(0) < V_{y^*}(1)$ so applying Theorem 5.2 (with $D = (0, 1)$), we find that

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}^x[\tau_N] = \frac{\mu}{\lambda} - \log\frac{\mu}{\lambda} - 1$$

and for any $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}^x(Y_N(\tau_N) > \epsilon) = 0;$$

in other words, τ_N is the time of extinction with high probability. If the coefficient μ of this death rate is too small (specifically if $\mu/\lambda < e^{-1}$, $V_{y^*}(0) > V_{y^*}(1)$ and τ_N is likely to be the first hitting time of 1 rather than 0. In any case, we have the following lower bound for the extinction time $\tau_N^{(0)}$:

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}^x[\tau_N^{(0)}] \geq \min\left(\frac{\mu}{\lambda} - \log\frac{\mu}{\lambda} - 1, \frac{\mu}{\lambda}\right).$$

Observe that if the ODE (1.2) has a unique stable fixed point y^* whose domain of attraction is $(0, 1)$, then Theorem 5.1 shows that $I(x) = V_{y^*}(x)$. This result extends to higher dimensions and can be used to obtain the large deviations behaviour of the invariant distribution even if an explicit formula is not known to exist. The following result is the same as Theorem 4.3 in Chapter 4 of [4] for diffusions; as with Theorem 5.2, the proof can be easily adapted from the diffusion case.

THEOREM 5.3. *Suppose that* (1.2) *has a unique stable equilibrium y^* whose domain of attraction is* $(0, 1)^d$. *Suppose that the density process Y_N has birth and death rates $b^{(N)}$ and $d^{(N)}$ which satisfy* (2.6) *and the invariant distribution Π_N exists. Then for any connected domain $D \subset (0.1)^d$,*

$$\lim_{N\to\infty} \frac{1}{N} \log \Pi_N(D) = -\inf_{x\in D} V_{y^*}(x),$$

where V_{y^*} is the quasi-potential relative to y^*.

Theorem 5.3 implies that (Π_N, N) obeys a large deviations principle with rate function V_{y^*}. The calculus of variations used at the beginning of this section is also applicable to higher dimensions, and we find that

$$V_{y^*}(x) = \sum_{i=1}^{d} \int_{-\infty}^{0} \dot{\phi}_i \log\frac{d_i(\phi)}{b_i(\phi)}\, dt,$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_d)$ is the time-reversal of the trajectory of (1.2) going from x to y^* and b_i, d_i are the individual components of the functions b and d. Unfortunately, this calculation does not yield an explicit formula for $V_{y^*}(x)$ in terms of x, as in the one-dimensional case. Finally, an interesting question is whether Theorem 5.3 also applies to the quasi-stationary distribution.

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