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CONDITIONS FOR OSCILLATION OF FIRST ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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In this paper, some new sufficient conditions for oscillation of first order neutral delay differential equations with several variable coefficients are obtained. These sufficient conditions include and are in many cases weaker than those known.

1. INTRODUCTION

The oscillation theory of first order neutral delay differential equations (NDDEs for short) has been extensively developed during the past few years. We refer to the works of Grammatikopoulos et al [**1, 2, 3**] , Ladas and Sficas [**4**], Gopalsamy and Zhang [**5**], Jiang Ziwen [**6**] for some results related to the oscillations of NDDEs. Recently, there has been some interest in the oscillation theory of first order NDDEs when the NDDE has one or more variable coefficients (for example see Gopalsamy and Zhang [**5**], Jiang Ziwen [**6**]).

In this paper, we study the oscillation of first order NDDEs with several variable coefficients

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big(x(t)-\sum_{i=1}^{m}c_i x(t-r_i)\Big)+\sum_{j=1}^{n}p_j(t)x(t-s_j)=0 \tag{1.1}$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big(x(t)-\sum_{i=1}^{m}c_i(t) x(t-r_i)\Big)+\sum_{j=1}^{n}p_j(t)x(t-s_j)=0. \tag{1.2}$$

When $m = n = 1$, Gopalsamy and Zhang in [**5**], Jiang Ziwen in [**6**] obtained some sufficient conditions for oscillation of the first order NDDEs (1.1) and (1.2). But these sufficient conditions in [**5**] or [**6**] are strong. The purpose of this paper is to give some new sufficient conditions which are weaker than those in [**5**] and [**6**] for oscillation of the first order NDDEs (1.1) and (1.2). In order to achieve this aim, we first obtain some new sufficient conditions for oscillation of the first order NDDEs with several coefficients

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big(x(t)-\sum_{i=1}^{m}c_i x(t-r_i)\Big)+\sum_{j=1}^{n}p_j x(t-s_j)=0. \tag{1.3}$$

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Then we use these new sufficient conditions for oscillation of (1.3) and two Lemmas to derive some new sufficient conditions for oscillation of (1.1) and (1.2). All of these new sufficient conditions we obtain include and are in many cases weaker than those in the articles [**2, 3, 4, 5, 6**]; furthermore these new sufficient conditions can be verified when the coefficients of NDDEs are given. That is, these sufficient conditions have relevance to the coefficients of NDDEs only.

2. Lemmas and Theorems

In this section, we shall prove some lemmas and theorems which are the foundation of our main results. First, we consider the first order NDDEs

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big(x(t)-\sum_{i=1}^{m}c_i x(t-r_i)\Big)+\sum_{j=1}^{n}p_j x(t-s_j)=0 \tag{1.3}$$

where the coefficients satisfy

$$c_i>0,\, i=1,2,\cdots,m, \sum_{i=1}^{m}c_i<1 \quad \text{and} \quad 0<r_1\leqslant r_2\leqslant\cdots\leqslant r_m; \tag{2.1}$$

$$p_j>0,\, j=1,2,\cdots,n \quad \text{and} \quad 0<s_1\leqslant s_2\leqslant\cdots\leqslant s_n. \tag{2.2}$$

We have the following result:

Lemma 2.1. *Assume that*

$$\sum_{i=1}^{m}c_i\exp(vr_i)+\sum_{j=1}^{n}p_j\exp(vs_j)/v>1 \quad \textit{for all} \quad v>0. \tag{2.3}$$

Then all solutions of (1.3) are oscillatory.

Proof: The characteristic equation of (1.3) is

$$f(\lambda)=\lambda-\lambda\sum_{i=1}^{m}c_i\exp(-\lambda r_i)+\sum_{j=1}^{n}p_j\exp(-\lambda s_j)=0. \tag{2.4}$$

To prove this result, it suffices to prove that (2.4) has no real roots under the assumption (2.1) and (2.2) (see [**1**, Theorem]). We note that any real root of (2.4) cannot be positive under the assumption (2.1) and (2.2). Since $f(0)=\sum_{j=1}^{n}p_j>0$, $\lambda=0$ is not a root of (2.4). Thus any real root of (2.4) can only be negative. Let $\lambda=-v_0<0$ be a root of (2.4), then

$$1-\sum_{i=1}^{m}c_i\exp(v_0 r_i)-\sum_{j=1}^{n}p_j\exp(v_0 s_j)/v_0=0,\ v_0>0.$$

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which contradicts the condition (2.3). Hence (2.4) has no real roots under the assumption (2.1) and (2.2). The proof is complete. ∎

THEOREM 2.2. *Assume that*

$$e\sum_{j=1}^{n} p_j s_j \geqslant 1 - \sum_{i=1}^{m} c_i \exp\left(r_i \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)\right). \tag{2.5}$$

Then all solutions of (1.3) are oscillatory.

PROOF: We define the function f so that

$$f(v) = \sum_{j=1}^{n} p_j \exp(vs_j)/v + \sum_{i=1}^{m} c_i \exp(vr_i) - 1, \ v > 0.$$

Consider the value of f at v for $0 < v \leqslant \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)$, then

$$f(v) > \sum_{j=1}^{n} p_j/v + \sum_{i=1}^{m} c_i - 1.$$

Hence $f(v) > 0$ for all v satisfying $0 < v \leqslant \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)$.

Let us now consider the value of f at v for $v > \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)$ and note that $p_j \exp(vs_j)/v$ has a global minimum at $1/s_j$ and the minimum value is ep_js_j for $j = 1, 2, \cdots, n$. Then

$$f(v) > e\sum_{j=1}^{n} p_j s_j + \sum_{i=1}^{m} c_i \exp\left(r_i \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)\right) - 1.$$

But by our assumption (2.5), $e\sum_{j=1}^{n} p_j s_j + \sum_{i=1}^{m} c_i \exp\left(r_i \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)\right) - 1 \geqslant 0$ showing that $f(v) > 0$ for all $v > \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)$.

Thus we have shown that (2.3) holds. It follows from Lemma 2.1 that all solutions of (1.3) are oscillatory and the proof is complete. ∎

THEOREM 2.3. *Assume that*

(i) $\sum_{j=1}^{n} p_j s_j \geqslant 1 - \sum_{i=1}^{m} c_i$, *or*

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(ii) $\sum_{j=1}^{n} p_j s_j < 1 - \sum_{i=1}^{m} c_i$ *and*

$$e \sum_{j=1}^{n} p_j s_j \geqslant 1 - \sum_{i=1}^{m} c_i \exp\left(r_i \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} p_j s_j \right) \right). \tag{2.6}$$

Then all solutions of (1.3) are oscillatory.

PROOF: (i) Since

$$e \sum_{j=1}^{n} p_j s_j \geqslant e\left(1 - \sum_{i=1}^{m} c_i \right) > 1 - \sum_{i=1}^{m} c_i \exp\left(r_i \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i \right) \right),$$

by Theorem 2.2, all solutions of (1.3) are oscillatory.

(ii) We define the function f as in the proof of Theorem 2.2. We first consider the value of f at v for $v > \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} p_j s_j \right)$ and note that $p_j \exp(v s_j)/v$ has a global minimum at $1/s_j$ and the minimum value is $e p_j s_j$ for $j = 1, 2, \cdots, n$. Then

$$\begin{aligned} f(v) &= \sum_{j=1}^{n} p_j \exp(v s_j)/v + \sum_{i=1}^{m} c_i \exp(v r_i) - 1 \\ &> e \sum_{j=1}^{n} p_j s_j + \sum_{i=1}^{m} c_i \exp\left(r_i \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} p_j s_j \right) \right) - 1. \end{aligned}$$

By our assumption (2.6),

$$e \sum_{j=1}^{n} p_j s_j + \sum_{i=1}^{m} c_i \exp\left(r_i \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} p_j s_j \right) \right) - 1 > 0$$

showing that $f(v) > 0$ for all $v > \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} p_j s_j \right)$.

Next we consider the value of f at v for v satisfying

$$0 < v \leqslant \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} p_j s_j \right).$$

We have

$$f(v) = \sum_{j=1}^{n} p_j \exp(v s_j)/v + \sum_{i=1}^{m} c_i \exp(v r_i) - 1 > \sum_{j=1}^{n} p_j (1 + v s_j)/v + \sum_{i=1}^{m} c_i - 1.$$

Hence $f(v) > 0$ for all v satisfying $0 < v \leqslant \sum_{j=1}^{n} p_j \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} p_j s_j\right)$.

Thus we have shown that (2.3) holds. By Lemma 2.1, all solutions of (1.3) are oscillatory and the proof is complete. ∎

In general, the conditions (2.5) and (2.6) are difficult to verify. In the following, we give some sufficient conditions which are easier to verify than the conditions (2.5) and (2.6).

THEOREM 2.4. *Assume that there exists a nonnegative integer N satisfying*

$$(2.7) \qquad e\sum_{j=1}^{n} p_j s_j \geqslant 1 - \sum_{i=1}^{m} c_i \sum_{k=0}^{N} \left(\left(r_i \sum_{j=1}^{n} p_j\right) \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)\right)^k \Big/ (k!).$$

Then all solutions of (1.3) are oscillatory.

PROOF: Since

$$\sum_{i=1}^{m} c_i \sum_{k=0}^{N} \left(\left(r_i \sum_{j=1}^{n} p_j\right) \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)\right)^k \Big/ (k!) < \sum_{i=1}^{m} c_i \exp\left(\left(r_i \sum_{j=1}^{n} p_j\right) \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)\right),$$

(2.5) holds. It follows from Theorem 2.2 that all solutions of (1.3) are oscillatory and the proof is complete. ∎

THEOREM 2.5. *Assume that $\sum_{j=1}^{n} p_j s_j < 1 - \sum_{i=1}^{m} c_i$, and that there exists a nonnegative integer N satisfying*

$$(2.8) \qquad e\sum_{j=1}^{n} p_j s_j \geqslant 1 - \sum_{i=1}^{m} c_i \sum_{k=0}^{N} \left(\left(r_i \sum_{j=1}^{n} p_j\right) \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} p_j s_j\right)\right)^k \Big/ (k!).$$

Then all solutions of (1.3) are oscillatory.

PROOF: Since

$$\sum_{k=0}^{N} \left(\left(r_i \sum_{j=1}^{n} p_j\right) \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} p_j s_j\right)\right)^k \Big/ (k!) < \exp\left(\left(r_i \sum_{j=1}^{n} p_j\right) \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} p_j s_j\right)\right),$$

(2.6) holds. It follows from Theorem 2.3 that all solutions of (1.3) are oscillatory and the proof is complete. ∎

REMARK. The sufficient conditions of Theorem 2.2.–2.5. for oscillation of first order NDDEs include and are in many cases weaker than those of [**2**, **3**, **4**, **5**, **6**], so the results of Theorem 2.2.–2.5. develop the results of [**2**, **3**, **4**, **5**, **6**]. See Example 1 and Example 2 in Section 4 of this paper.

Second, we consider the first order NDDEs

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(x(t)-\sum_{i=1}^{m}c_i x(t-r_i)\right)+\sum_{j=1}^{n}p_j(t)x(t-s_j)=0, \tag{1.1}$$

where the coefficients satisfy

(i) $c_i>0,\ i=1,2,\cdots,m,\ \sum\limits_{i=1}^{m}c_i<1;\ 0<r_1\leqslant r_2\leqslant\cdots\leqslant r_m;\ 0<s_1\leqslant s_2\leqslant\cdots\leqslant s_n$.

(ii) $p_j(t)$ is continuous and $p_j(t)\geqslant q_j\geqslant 0,\ j=1,2,\cdots,n,\ t\in R,\ \sum\limits_{j=1}^{n}q_j>0$.

Then we have the following result:

LEMMA 2.6. *Assume that $y(t)$ is a nonoscillatory solution of (1.1). Then* $\lim\limits_{t\to\infty}y(t)=0$.

The proof of Lemma 2.6 is similar to the proof of Lemma 2.7 , so we omit it.

Finally, we consider the first order NDDEs

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(x(t)-\sum_{i=1}^{m}c_i(t)x(t-r_i)\right)+\sum_{j=1}^{n}p_j(t)x(t-s_j)=0 \tag{1.2}$$

where the coefficients satisfy

(i) $0<r_1\leqslant r_2\leqslant\cdots\leqslant r_m;\ 0<s_1\leqslant s_2\leqslant\cdots\leqslant s_m$;

(ii) $p_j(t)$ is continuous and $p_j(t)\geqslant q_j\geqslant 0,\ j=1,2,\cdots,n,\ t\in R,\ \sum\limits_{j=1}^{n}q_j>0$.

(iii) $c_i(t)$ is continuous and $0\leqslant u_i\leqslant c_i(t),\ t\in R,\ \lim\limits_{t\to\infty}c_i(t)=c_i,\ i=1,2,\cdots,m$ and $0<\sum\limits_{i=1}^{m}u_i\leqslant\sum\limits_{i=1}^{m}c_i<1$

Then we have the following result:

LEMMA 2.7. *Assume that $y(t)$ is a nonoscillatory solution of (1.2). Then* $\lim\limits_{t\to\infty}y(t)=0$.

PROOF: The negative of a solution of (1.2) is again a solution of (1.2) and a nonoscillatory solution is an eventually positive or negative solution, so without loss of generality, we may assume that $y(t)$ is an eventually positive solution of (1.2). Then

$$y(t)>0,\ y(t-r_i)>0,\ i=1,2,\cdots,m;\ y(t-s_j)>0,\ j=1,2,\cdots,n$$

for all $t \geqslant T_1$ for some $T_1 > 0$. Let

$$z(t) = y(t) - \sum_{i=1}^{m} c_i(t) y(t - r_i). \tag{2.9}$$

Then

$$\frac{\mathrm{d}}{\mathrm{dt}} z(t) = -\sum_{j=1}^{n} p_j(t) y(t - s_j) < 0,\ t \geqslant T_1. \tag{2.10}$$

We see that $z(t)$ is a strictly monotone decreasing function for $t \geqslant T_1$, and so that $\lim_{t\to\infty} z(t)$ exists. If $\lim_{t\to\infty} z(t) = -\infty$, then $z(t) < 0$ for all $t \geqslant T_2$ for some $T_2 \geqslant T_1$. Hence

$$y(t) < \sum_{i=1}^{m} c_i(t) y(t - r_i),\ t \geqslant T_2. \tag{2.11}$$

Note that from conditions (iii) of (1.2),

$$u_i \leqslant c_i(t) < c_i + \left(1 - \sum_{i=1}^{m} c_i\right) \Big/ (2m),\ i = 1, 2, \cdots, m$$

for all $t \geqslant T_3$ for some $T_3 \geqslant T_2$. It follows from (2.11) that

$$y(t) < \sum_{i=1}^{m} \left(c_i + \left(1 - \sum_{i=1}^{m} c_i\right) \Big/ (2m)\right) y(t - r_i),\ t \geqslant T_3.$$

Let

$$y(T_3 - \tau) = \max_{1 \leqslant i \leqslant m} \{y(T_3 - r_i)\},$$

then

$$y(T_3) < \sum_{i=1}^{m} \left(c_i + \left(1 - \sum_{i=1}^{m} c_i\right) \Big/ (2m)\right) y(T_3 - \tau) = \left(1 + \sum_{i=1}^{m} c_i\right) y(T_3 - \tau)/2,\ \tau \geqslant r_1 > 0.$$

By iteration we have

$$y(T_3 + n\tau) < \left(\left(1 + \sum_{i=1}^{m} c_i\right) \Big/ 2\right)^n y(T_3),$$

which implies that $\lim_{n\to\infty} y(T_3 + n\tau) = 0$. Note that from (2.9), $\lim_{n\to\infty} z(T_3 + n\tau) = 0$. This contradicts the assumption that $\lim_{t\to\infty} z(t) = -\infty$. Hence

$$\lim_{t\to\infty} z(t) = \alpha \in R.$$

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From (2.10), we have

$$z(T_3) - \alpha = \int_{T_3}^{\infty} \sum_{j=1}^{n} p_j(s) y(s - s_j)\, ds \geqslant \sum_{j=1}^{n} q_j \int_{T_3}^{\infty} y(s - s_j)\, ds > 0.$$

Thus $y \in L^1[T_3 + s_n, \infty)$, and so that $z \in L^1[T_3, \infty)$. Hence α must equal 0. Note that

$$0 = \lim_{t\to\infty} z(t) = \liminf_{t\to\infty} z(t) = \liminf_{t\to\infty} y(t) - \left(\sum_{i=1}^{m} c_i\right) \limsup_{t\to\infty} y(t),$$

$$0 = \lim_{t\to\infty} z(t) = \limsup_{t\to\infty} z(t) = \limsup_{t\to\infty} y(t) - \left(\sum_{i=1}^{m} c_i\right) \liminf_{t\to\infty} y(t),$$

and then we have

$$\liminf_{t\to\infty} y(t) = \left(\sum_{i=1}^{m} c_i\right) \limsup_{t\to\infty} y(t) = \left(\sum_{i=1}^{m} c_i\right)^2 \liminf_{t\to\infty} y(t),$$

which implies that

$$\liminf_{t\to\infty} y(t) = \limsup_{t\to\infty} y(t) = 0 \quad \text{or} \quad \infty.$$

Hence we have

$$\lim_{t\to\infty} y(t) = 0 \quad \text{or} \quad \infty.$$

Note that $y \in L^1[T_3 + s_n, \infty)$, and hence

$$\lim_{t\to\infty} y(t) = 0.$$

The proof is complete. ∎

3. Main results

In this section, we use the results in Section 2 to study sufficient conditions for oscillation of first order NDDEs (1.1) and (1.2). We obtain the following main results of this paper.

When $q_k = 0$ where $1 \leqslant k \leqslant n$ or $u_k = 0$ where $1 \leqslant k \leqslant m$ in the NDDEs (1.1) and (1.2), we list it (or them) in the following conditions and proofs of Theorem 3.1. and Theorem 3.2. for convenience.

Theorem 3.1. *Assume that one of the following five conditions holds.*

(a) $e\sum_{j=1}^{n} q_j s_j \geqslant 1 - \sum_{i=1}^{m} c_i \exp\left(r_i \sum_{j=1}^{n} q_j \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)\right)$.

(b) $\sum_{j=1}^{n} q_j s_j \geqslant 1 - \sum_{i=1}^{m} c_i$.

(c) $\sum_{j=1}^{n} q_j s_j < 1 - \sum_{i=1}^{m} c_i$, *and*

$$e\sum_{j=1}^{n} q_j s_j \geqslant 1 - \sum_{i=1}^{m} c_i \exp\left(r_i \sum_{j=1}^{n} q_j \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} q_j s_j\right)\right).$$

(d) *There exists a nonnegative integer N satisfying*

$$e\sum_{j=1}^{n} q_j s_j \geqslant 1 - \sum_{i=1}^{m} c_i \sum_{k=0}^{N} \left(\left(r_i \sum_{j=1}^{n} q_j\right) \Big/ \left(1 - \sum_{i=1}^{m} c_i\right)\right)^k \Big/ (k!).$$

(e) $\sum_{j=1}^{n} q_j s_j < 1 - \sum_{i=1}^{m} c_i$, *and there exists a nonnegative integer N satisfying*

$$e\sum_{j=1}^{n} q_j s_j \geqslant 1 - \sum_{i=1}^{m} c_i \sum_{k=0}^{N} \left(\left(r_i \sum_{j=1}^{n} q_j\right) \Big/ \left(1 - \sum_{i=1}^{m} c_i - \sum_{j=1}^{n} q_j s_j\right)\right)^k \Big/ (k!).$$

Then all solutions of (1.1) are oscillatory.

Proof: We shall show that the existence of a nonoscillatory solution of (1.1) leads to a contradiction. Suppose y is a nonoscillatory solution of (1.1); we suppose that $y(t) > 0$ for all $t \geqslant T$ for some $T > 0$ (If $y(t) < 0$ eventually the procedure is similar.) It follows from lemma 2.6 that $\lim_{t\to\infty} y(t) = 0$ and we have from (1.1) that

$$\begin{aligned} y(t) &= \sum_{i=1}^{m} c_i y(t - r_i) + \sum_{j=1}^{n} \int_t^{\infty} p_j(s) y(s - s_j)\, ds;\ t \geqslant t_0 \\ &\geqslant \sum_{i=1}^{m} c_i y(t - r_i) + \sum_{j=1}^{n} q_j \int_t^{\infty} y(s - s_j)\, ds;\ t \geqslant t_0 \end{aligned} \tag{3.1}$$

where $t_0 = \max\{T + r_m, T + s_n\}$. It is not difficult to see from (3.1) that

$$y(t) \geqslant c_1 y(t - r_1),\ y(t) = y(t_0 + nr_1 + \tau) \geqslant \alpha \exp(-\mu t) \tag{3.2}$$

for all $t \geqslant t_0$ where n is a nonnegative integer and

$$0 \leqslant \tau < r_1;\ \mu = -(\ln(c_1))/r_1;\ \alpha = \exp(\mu t_0) \min_{t_0 \leqslant t \leqslant t_0 + r_1} \{y(t)\}. \tag{3.3}$$

Define the sequence $\{y_k(t)\}$ by

$$y_0(t) \equiv y(t)$$

$$(3.4)\qquad y_{k+1}(t) \equiv \sum_{i=1}^{m} c_i y_k(t - r_i) + \sum_{j=1}^{n} q_j \int_t^\infty y_k(s - s_j)\,ds;\ t \geqslant t_0.$$

It follows from (3.1) and (3.4) that

$$y_1(t) - y_0(t) \leqslant 0,\ y_2(t) - y_1(t) \leqslant 0,\ \cdots,\ y_{k+1}(t) - y_k(t) \leqslant 0;\ t \geqslant t_0,$$

which implies that

$$(3.5)\qquad y_{k+1}(t) \leqslant y_k(t) \leqslant \cdots \leqslant y_1(t) \leqslant y_0(t),\ t \geqslant t_0.$$

Furthermore we have from (3.4) and (3.2) that

$$y_0(t) \geqslant \alpha \exp(-\mu t),\ t \geqslant t_0,$$

and also one can derive using (3.3) that

$$y_1(t) \geqslant \alpha \exp(-\mu t),\ y_{k+1}(t) \geqslant \alpha \exp(-\mu t),\ k = 1, 2, \cdots,\ t \geqslant t_0.$$

Thus we have from (3.5) that

$$\alpha \exp(-\mu t) \leqslant y_{k+1}(t) \leqslant y_k(t) \leqslant \cdots \leqslant y_1(t) \leqslant y_0,\ t \geqslant t_0.$$

By Lebesgue's convergence theorem the pointwise limit of $\{y_k(t)\}$ exists and hence

$$\alpha \exp(-\mu t) \leqslant y^*(t) \equiv \sum_{i=1}^{m} c_i y^*(t - r_i) + \sum_{j=1}^{n} q_j \int_t^\infty y^*(s - s_j)\,ds;\ t \geqslant t_0,$$

where

$$y^*(t) = \lim_{k\to\infty} y_k(t),\ t \geqslant t_0.$$

$y^*(t)$ is a nonoscillatory solution of the NDDE

$$(3.6)\qquad \frac{\mathrm{d}}{\mathrm{dt}}\left(x(t) - \sum_{i=1}^{m} c_i x(t - r_i)\right) + \sum_{j=1}^{n} q_j x(t - s_j) = 0,\ t \geqslant t_0,$$

where the coefficients satisfy

$$(3.7)\qquad c_i > 0, i = 1, 2, \cdots, m,\ \sum_{i=1}^{m} c_i < 1 \quad \text{and} \quad 0 < r_1 \leqslant r_2 \leqslant \cdots \leqslant r_m;$$

$$(3.8)\qquad q_j \geqslant 0, j = 1, 2, \cdots, n,\ \sum_{j=1}^{n} q_j > 0 \quad \text{and} \quad 0 < s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n.$$

(A) By Theorem 2.2, (3.6) cannot have a nonoscillatory solution when (a) holds. This contradiction proves all solutions of (1.1) are oscillatory.

(B) By Theorem 2.3 (i), (3.6) cannot have a nonoscillatory solution when (b) holds. This contradiction proves all solutions of (1.1) are oscillatory.

(C) By Theorem 2.3 (ii), (3.6) cannot have a nonoscillatory solution when (c) holds. This contradiction proves all solutions of (1.1) are oscillatory.

(D) By Theorem 2.4, (3.6) cannot have a nonoscillatory solution when (d) holds. This contradiction proves all solutions of (1.1) are oscillatory.

(E) By Theorem 2.5, (3.6) cannot have a nonoscillatory solution when (e) holds. This contradiction proves all solutions of (1.1) are oscillatory.

The proof is complete. ∎

THEOREM 3.2. *Assume that one of the following five conditions holds.*

(a) $e\sum_{j=1}^{n} q_j s_j \geqslant 1 - \sum_{i=1}^{m} u_i \exp\left(r_i \sum_{j=1}^{n} q_j \Big/ \left(1 - \sum_{i=1}^{m} u_i\right)\right)$.

(b) $\sum_{j=1}^{n} q_j s_j \geqslant 1 - \sum_{i=1}^{m} u_i$.

(c) $\sum_{j=1}^{n} q_j s_j < 1 - \sum_{i=1}^{m} u_i$, *and*

$$e\sum_{j=1}^{n} q_j s_j \geqslant 1 - \sum_{i=1}^{m} u_i \exp\left(r_i \sum_{j=1}^{n} q_j \Big/ \left(1 - \sum_{i=1}^{m} u_i - \sum_{j=1}^{n} q_j s_j\right)\right).$$

(d) *There exists a nonnegative integer N satisfying*

$$e\sum_{j=1}^{n} q_j s_j \geqslant 1 - \sum_{i=1}^{m} u_i \sum_{k=0}^{N} \left(\left(r_i \sum_{j=1}^{n} q_j\right) \Big/ \left(1 - \sum_{i=1}^{m} u_i\right)\right)^k \Big/ (k!).$$

(e) $\sum_{j=1}^{n} q_j s_j < 1 - \sum_{i=1}^{m} u_i$, *and there exists a nonnegative integer N satisfying*

$$e\sum_{j=1}^{n} q_j s_j \geqslant 1 - \sum_{i=1}^{m} u_i \sum_{k=0}^{N} \left(\left(r_i \sum_{j=1}^{n} q_j\right) \Big/ \left(1 - \sum_{i=1}^{m} u_i - \sum_{j=1}^{n} q_j s_j\right)\right)^k \Big/ (k!).$$

Then all solutions of (1.2) are oscillatory.

PROOF: We shall show that the existence of a nonoscillatory solution of (1.2) leads to a contradiction. Suppose y is a nonoscillatory solution of (1.2). Without loss of generality, we may suppose that $y(t) > 0$ for all $t \geqslant T$ for some $T > 0$. (If $y(t) < 0$

eventually the procedure is similar.) It follows from lemma 2.7 that $\lim_{t\to\infty} y(t) = 0$ and we have from (1.2) that

$$
\begin{aligned}
y(t) &= \sum_{i=1}^{m} c_i(t)y(t-r_i) + \sum_{j=1}^{n} \int_t^{\infty} p_j(s)y(s-s_j)\,ds;\ t \geqslant t_0 \\
&\geqslant \sum_{i=1}^{m} u_i y(t-r_i) + \sum_{j=1}^{n} q_j \int_t^{\infty} y(s-s_j)\,ds;\ t \geqslant t_0
\end{aligned} \tag{3.9}
$$

where $t_0 = \max\{T + r_m, T + s_n\}$. Without loss of generality, we may assume that $u_1 > 0$. It is not difficult to see from (3.9) that

$$
y(t) \geqslant u_1 y(t-r_1),\ y(t) = y(t_0 + nr_1 + \tau) \geqslant \alpha \exp(-\mu t) \tag{3.10}
$$

for all $t \geqslant t_0$ where n is a nonnegative integer and

$$
0 \leqslant \tau < r_1;\ \mu = -(\ln(u_1))/r_1;\ \alpha = \exp(\mu t_0) \min_{t_0 \leqslant t \leqslant t_0 + r_1} \{y(t)\}. \tag{3.11}
$$

Define the sequence $\{y_k(t)\}$ by

$$
\begin{aligned}
&y_0(t) \equiv y(t) \\
&y_{k+1}(t) \equiv \sum_{i=1}^{m} u_i y_k(t-r_i) + \sum_{j=1}^{n} q_j \int_t^{\infty} y_k(s-s_j)\,ds;\ t \geqslant t_0.
\end{aligned} \tag{3.12}
$$

It follows from (3.9) and (3.12) that

$$
y_1(t) - y_0(t) \leqslant 0,\ y_2(t) - y_1(t) \leqslant 0,\ \cdots,\ y_{k+1}(t) - y_k(t) \leqslant 0;\ t \geqslant t_0,
$$

which implies that

$$
y_{k+1}(t) \leqslant y_k(t) \leqslant \cdots \leqslant y_1(t) \leqslant y_0(t),\ t \geqslant t_0. \tag{3.13}
$$

Furthermore we have from (3.12) and (3.10) that

$$
y_0(t) \geqslant \alpha \exp(-\mu t),\ t \geqslant t_0,
$$

and also one can derive using (3.11) that

$$
y_1(t) \geqslant \alpha \exp(-\mu t),\ y_{k+1}(t) \geqslant \alpha \exp(-\mu t),\ k = 1, 2, \cdots,\ t \geqslant t_0.
$$

Thus we have from (3.13) that

$$
\alpha \exp(-\mu t) \leqslant y_{k+1}(t) \leqslant y_k(t) \leqslant \cdots \leqslant y_1(t) \leqslant y_0,\ t \geqslant t_0.
$$

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By Lebesgue's convergence theorem the pointwise limit of $\{y_k(t)\}$ exists and hence

$$\alpha \exp(-\mu t) \leqslant y^*(t) \equiv \sum_{i=1}^{m} u_i y^*(t - r_i) + \sum_{j=1}^{n} q_j \int_t^\infty y^*(s - s_j)\, ds;\ t \geqslant t_0,$$

where

$$y^*(t) = \lim_{k\to\infty} y_k(t),\ t \geqslant t_0.$$

$y^*(t)$ is a nonoscillatory solution of the NDDE

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(x(t) - \sum_{i=1}^{m} u_i x(t - r_i)\right) + \sum_{j=1}^{n} q_j x(t - s_j) = 0,\ t \geqslant t_0, \tag{3.14}$$

where the coefficients satisfy

(3.15)

$$u_i \geqslant 0, i = 1, 2, \cdots, m,\ 0 < \sum_{i=1}^{m} u_i < 1 \quad \text{and} \quad 0 < r_1 \leqslant r_2 \leqslant \cdots \leqslant r_m;$$

(3.16)

$$q_j \geqslant 0, j = 1, 2, \cdots, n,\ \sum_{j=1}^{n} q_j > 0 \quad \text{and} \quad 0 < s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n.$$

(A) By Theorem 2.2, (3.14) cannot have a nonoscillatory solution when (a) holds. This contradiction proves all solutions of (1.2) are oscillatory.
(B) By Theorem 2.3 (i), (3.14) cannot have a nonoscillatory solution when (b) holds. This contradiction proves all solutions of (1.2) are oscillatory.
(C) By Theorem 2.3 (ii), (3.14) cannot have a nonoscillatory solution when (c) holds. This contradiction proves all solutions of (1.2) are oscillatory.
(D) By Theorem 2.4, (3.14) cannot have a nonoscillatory solution when (d) holds. This contradiction proves all solutions of (1.2) are oscillatory.
(E) By Theorem 2.5, (3.14) cannot have a nonoscillatory solution when (e) holds. This contradiction proves all solutions of (1.2) are oscillatory.

The proof is complete. ∎

Remark. The sufficient conditions for oscillation of first order NDDEs (1.1) and (1.2) in this section include and are weaker than those of [**5, 6**], so the results of this section develop the results of [**5, 6**]. See Example 3, Example 4 and Example 5 in the Section 4 of this paper.

4. Examples

In this section, we shall apply the results of this paper to some examples; furthermore we shall show from these examples that the sufficient conditions for oscillation of first order NDDEs which we obtained in this paper include and are in many cases weaker than those known and these sufficient conditons can be verified when a NDDE is given.

Example 1. We consider the following NDDE

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(x(t)-\frac{1}{4}x\left(t-\frac{3}{2}\right)\right)+\frac{1}{2}x\left(t-\frac{1}{e}\right)=0, \tag{4.1}$$

where

$$m=n=1,\ c_1=\frac{1}{4},\ r_1=\frac{3}{2},\ p_1=\frac{1}{2},\ s_1=\frac{1}{e}.$$

Note that

$$p_1s_1=\frac{1}{2e}<\frac{3}{4}=1-c_1,\ ep_1s_1=\frac{1}{2}=1-c_1\left(1+\frac{r_1p_1}{1-c_1}\right).$$

Then the condition

$$p_1s_1e>1-c_1 \quad \text{or} \quad p_1s_1e>1-c_1\left(1+\frac{r_1p_1}{1-c_1}\right)$$

of [**2, 3, 4, 5**] does not hold. Hence the results of [**2, 3, 4, 5**] can not be applied to (4.1). But condition (2.7) (or (2.5), or (2.6), or (2.8)) holds when $m=n=1$. It follows from Theorem 2.4 (or Theorem 2.2, or Theorem 2.3, or Theorem 2.5) that all solutions of (4.1) are oscillatory.

Example 2. We consider the following NDDE

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(x(t)-\frac{1}{4}x\left(t-\frac{1}{2}\right)-\frac{1}{8}x(t-1)\right)+\frac{1}{2}x\left(t-\frac{1}{e}\right)+\frac{1}{8}x\left(t-\frac{2}{e}\right)=0, \tag{4.2}$$

where

$$m=n=2,\ c_1=\frac{1}{4},\ c_2=\frac{1}{8};\ r_1=\frac{1}{2},$$
$$r_2=1;\ p_1=\frac{1}{2},\ p_2=\frac{1}{8};\ s_1=\frac{1}{e},\ s_2=\frac{2}{e}.$$

Note that

$$p_1s_1+p_2s_2=\frac{3}{4e}<\frac{5}{8}=1-c_1-c_2;$$
$$e(p_1s_1+p_2s_2)=\frac{3}{4}>\frac{3}{8}=1-c_1\left(1+\frac{r_1(p_1+p_2)}{1-c_1-c_2}\right)-c_2\left(1+\frac{r_2(p_1+p_2)}{1-c_1-c_2}\right).$$

Then condition (2.5) (or (2.6), or (2.7), or (2.8)) holds when $m=n=2$. It follows from Theorem 2.2 (or Theorem 2.3, or Theorem 2.4, or Theorem 2.5) that all solutions of (4.2) are oscillatory.

EXAMPLE 3. We consider the following NDDE

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(x(t)-\frac{1}{4}x\left(t-\frac{3}{2}\right)\right)+\left(\frac{1}{2}+t\right)x\left(t-\frac{1}{e}\right)=0,\ t\geqslant 0, \tag{4.3}$$

where

$$m=n=1,\ c_1=\frac{1}{4},\ r_1=\frac{3}{2},\ p_1(t)=\frac{1}{2}+t\geqslant q_1=\frac{1}{2},\ s_1=\frac{1}{e}.$$

Note that

$$eq_1s_1=\frac{1}{2}=1-c_1\left(1+\frac{r_1p_1}{1-c_1}\right).$$

Then condition (2.29) of [**5**] and $p_1(t)<w$ where w is a positive constant do not hold, and hence [**5**, Theorem 2.3] and [**6**, Theorem 2.7] can not be applied to (4.3). But condition (d) of Theorem 3.1 holds when $m=n=1$, so it follows from Theorem 3.1 that all solutions of (4.3) are oscillatory.

EXAMPLE 4. We consider the following NDDE

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(x(t)-\left(\frac{1}{4}+\frac{1}{t}\right)x\left(t-\frac{3}{2}\right)\right)+\left(\frac{1}{2}+\frac{1}{t}\right)x\left(t-\frac{1}{e}\right)=0,\ t\geqslant 1, \tag{4.4}$$

where

$$m=n=1,\ c_1(t)=\frac{1}{4}+\frac{1}{t}\geqslant u_1=\frac{1}{4},\ r_1=\frac{3}{2},\ p_1(t)=\frac{1}{2}+\frac{1}{t}\geqslant q_1=\frac{1}{2},\ s_1=\frac{1}{e}.$$

Note that

$$eq_1s_1=\frac{1}{2}=1-u_1\left(1+\frac{r_1q_1}{1-u_1}\right).$$

Then condition (d) of Theorem 3.2 holds when $m=n=1$, so it follows from Theorem 3.2 that all solutions of (4.4) are oscillatory.

EXAMPLE 5. We consider the following NDDE

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\left(x(t)-\left(\frac{1}{4}+\frac{2}{t}\right)x\left(t-\frac{1}{2}\right)-\left(\frac{1}{8}+\frac{1}{t}\right)x(t-1)\right)\\ +\frac{1}{2}x\left(t-\frac{1}{e}\right)+\left(\frac{1}{8}+2t\right)x\left(t-\frac{2}{e}\right)=0,\ t>0\end{aligned} \tag{4.5}$$

where

$$m=n=2,\ c_1(t)=\frac{1}{4}+\frac{2}{t}\geqslant u_1=\frac{1}{4},\ c_2(t)=\frac{1}{8}+\frac{1}{t}\geqslant u_2=\frac{1}{8};\ r_1=\frac{1}{2},$$
$$r_2=1;\ p_1(t)=q_1=\frac{1}{2},\ p_2(t)=\frac{1}{8}+2t\geqslant q_2=\frac{1}{8};\ s_1=\frac{1}{e},\ s_2=\frac{2}{e}.$$

Note that

$$e(q_1s_1+q_2s_2)=\frac{3}{4}>\frac{3}{8}=1-u_1\left(1+\frac{r_1(q_1+q_2)}{1-u_1-u_2}\right)-u_2\left(1+\frac{r_2(q_1+q_2)}{1-u_1-u_2}\right).$$

Then condition (d) (or (a), or (c), or (e)) of Theorem 3.2 holds when $m=n=2$, so it follows from Theorem 3.2 that all solutions of (4.5) are oscillatory.

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