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ON WEAKLY s-PERMUTABLY EMBEDDED SUBGROUPS OF FINITE GROUPS II

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Abstract

A subgroup H is called weakly s-permutably embedded in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$. In this note, we study the influence of the weakly s-permutably embedded property of subgroups on the structure of G, and obtain the following theorem. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups, and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are s-permutably embedded in G. Also, when $p = 2$ and $|D| = 2$, we suppose that each cyclic subgroup of P of order four is weakly s-permutably embedded in G. Then $G \in \mathcal{F}$.

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1. Introduction

All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [H]. Throughout the paper, G denotes a finite group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$ and G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ *a formation* provided that (i) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation (see [H, p. 713, Satz 8.6]).

A subgroup H of G is said to be *s-permutable* (or *s-quasinormal, π-quasinormal)* [K] in G if H permutes with every Sylow subgroup of G; H is said to be *c-normal* [W] in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$,

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where H_G is the normal core of H in G. More recently, Skiba in [S] introduced the following concept, which covers both s-permutability and c-normality.

Definition 1.1. Let H be a subgroup of G. Then H is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H.

In [LWQ1], the following definition is given. It covers both weakly s-permutability and the s-permutably embedding property [BP] of subgroups.

Definition 1.2. Let H be a subgroup of G. We say that H is weakly s-permutably embedded in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$.

In [S], Skiba proved the following result.

Theorem 1.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every noncyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) are weakly s-permutable in G. Then $G \in \mathcal{F}$.*

Along this line, in [LWQ1], the authors proved the following result by using s-permutably embedded subgroups. This generalised Theorem 1.3 in some sense.

Theorem 1.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. For every noncyclic Sylow subgroup P of $F^*(E)$, suppose that P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are weakly s-permutably embedded in G. When $p = 2$ and $|P : D| > 2$, in addition, suppose that H is weakly s-permutably embedded in G if there exists $D_1 \trianglelefteq H \leq P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order four. Then $G \in \mathcal{F}$.*

On the other hand, in [LWQ2] the authors improved Theorem 1.3 as follows.

Theorem 1.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every noncyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are weakly s-permutable in G; in addition, we suppose that all cyclic subgroups of P of order four are weakly s-permutable in G if P is a nonabelian 2-group and $|D| = 2$ in G. Then $G \in \mathcal{F}$.*

For the sake of convenience of statement, we introduce the following notation.

Let P be a p-subgroup of G. We say P satisfies $(\triangle_1)$ $((\triangle_2)$, respectively) in G if:

$(\triangle_1)$ P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are weakly s-permutably embedded in G; when $p = 2$ and $|D| = 2$, in addition, suppose that each cyclic subgroup of P of order four is weakly s-permutably embedded in G;

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($\triangle_2$) P has a subgroup D such that $1 < |D| < |P|$ and each subgroup H of P with order $|H| = |D|$ is s-permutable in G; when $p = 2$ and $|D| = 2$, in addition, suppose that each cyclic subgroup of P of order four is s-permutable in G.

In this paper, the main purpose is to prove the following result which improves Theorems 1.3, 1.4 and 1.5.

Theorem 1.6. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. If every noncyclic Sylow subgroup of $F^*(E)$ satisfies $\triangle_1$ in G, then $G \in \mathcal{F}$.*

Remark 1.7. This work is a continuation of [LWQ1, LWQ2]. We investigate the influence of the weakly s-permutably embedded property of subgroups on the structure of a finite group.

2. Preliminaries

Lemma 2.1 [K].

(a) *An s-permutable subgroup of G is subnormal in G.*
(b) *If $H \le K \le G$ and H is s-permutable in G, then H is s-permutable in K.*
(c) *If H is a subnormal Hall subgroup of G, then $H \lhd G$.*
(d) *Let $K \lhd G$. If H is s-permutable in G, then HK/K is s-permutable in G/K.*
(e) *If P is an s-permutable p-subgroup of G for some prime p, then $O^p(G) \le N_G(P)$.*

Lemma 2.2 [BP, Lemma 1]. *Suppose that U is s-permutably embedded in a group G, and that $H \le G$ and $N \trianglelefteq G$.*

(a) *If $U \le H$, then U is s-permutably embedded in H.*
(b) *We have that UN is s-permutably embedded in G and UN/N is s-permutably embedded in G/N.*

Lemma 2.3 [LWW, Lemma 2.3]. *Suppose that H is s-permutable in G, and that P is a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-permutable in G.*

Lemma 2.4 [LWW, Lemma 2.4]. *Suppose that P is a p-subgroup of G contained in $O_p(G)$. If P is s-permutably embedded in G, then P is s-permutable in G.*

Now we give some basic properties of weakly s-permutably embedded subgroups.

Lemma 2.5 [LWQ1, Lemma 2.5]. *Let U be a weakly s-permutably embedded subgroup of G and N a normal subgroup of G.*

(a) *If $U \le H \le G$, then U is weakly s-permutably embedded in H.*
(b) *If $N \le U$, then U/N is weakly s-permutably embedded in G/N.*
(c) *Let π be a set of primes, U a π-subgroup and N a π'-subgroup. Then $(UN)/N$ is weakly s-permutably embedded in G/N.*
(d) *Suppose that U is a p-group for some prime p and U is not s-permutably embedded in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = MU$.*

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(e) *Suppose that U is a p-group contained in* $O_p(G)$ *for some prime p. Then U is weakly s-permutable in G.*

LEMMA 2.6 [LWQ1, Lemma 2.6]. *Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that* $1 < |D| < |N|$ *and every subgroup H of N satisfying* $|H| = |D|$ *is weakly s-permutably embedded in G. Then some maximal subgroup of N is normal in G.*

LEMMA 2.7 [H, III, 5.2 Satz and IV, 5.4 Satz]. *Suppose that p is a prime and G is minimal non-p-nilpotent, that is, G is not a p-nilpotent group but its proper subgroups are all p-nilpotent. Then the following statements hold:*

(a) *G has a normal Sylow p-subgroup P for some prime p and* $G = PQ$*, where Q is a nonnormal cyclic q-subgroup for some prime* $q \neq p$*;*
(b) $P/\Phi(P)$ *is a minimal normal subgroup of* $G/\Phi(P)$*;*
(c) *the exponent of P is p or* 4.

The generalised Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Its definition and important properties can be found in [HB, X 13]. We would like to give the following basic facts which we will use in our proof.

LEMMA 2.8 [HB, X 13]. *Let G be a group and M a subgroup of G.*

(1) *If M is normal in G, then* $F^*(M) \leq F^*(G)$.
(2) *We have* $F^*(G) \neq 1$ *if* $G \neq 1$*; in fact,* $F^*(G)/F(G) = \mathrm{soc}(F(G)C_G(F(G))/F(G))$.
(3) *We have* $F^*(F^*(G)) = F^*(G) \geq F(G)$*; if* $F^*(G)$ *is solvable, then* $F^*(G) = F(G)$.

3. Main results

THEOREM 3.1. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing* $|G|$*. If P satisfies* $\triangle_1$ *in G, then G is p-nilpotent.*

PROOF. Suppose that the theorem is false and G is a counterexample with minimal order. We will derive a contradiction in several steps.

Step 1: $O_{p'}(G) = 1$. If $O_{p'}(G) \neq 1$. Lemma 2.5(c) guarantees that $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. Thus, $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

Step 2: $|D| > p$. Suppose that $|D| = p$. Since G is not p-nilpotent, G has a minimal non-p-nilpotent subgroup G_1. By Lemma 2.7(a), $G_1 = [P_1]Q$, where $P_1 \in \mathrm{Syl}_p(G_1)$ and $Q \in \mathrm{Syl}_q(G_1)$, $p \neq q$. We use the notation $\Phi = \Phi(P_1)$. Let X/Φ be a subgroup of P_1/Φ of order p, $x \in X \setminus \Phi$ and $L = \langle x \rangle$. Then L is of order p or four by Lemma 2.7(c). By the hypothesis, L is weakly s-permutably embedded in G, and thus in G_1 by Lemma 2.5(a). If L is not s-permutably embedded in G_1, then by Lemma 2.5(d), G_1 has a normal subgroup T such that $G_1 = LT$ and $|G_1 : T| = p$. Since G_1 is a minimal non-p-nilpotent group, T is p-nilpotent. Then T_q char $T \lhd G_1$ and $T_q \lhd G_1$. Therefore, G_1 is p-nilpotent, a contradiction. Hence, L is s-permutably embedded in G_1.

So $X/\Phi = L\Phi/\Phi$ is s-permutably embedded in G_1/Φ by Lemma 2.2(b). Now Lemmas 2.6 and 2.8(b) imply that $|P_1/\Phi| = p$. It follows immediately that P_1 is cyclic. Hence, G_1 is p-nilpotent by [H, IV, Satz 2.8], contrary to the choice of G_1.

Step 3: $|P : D| > p$. This follows by [LWQ1, Theorem 3.1].

Step 4: G has no subgroup of index p. Let M be a subgroup of G of index p. Then $M \lhd G$. By Step 3 together with induction, M is p-nilpotent. It follows that G is p-nilpotent, a contradiction.

Step 5: P satisfies $\triangle_2$ in G. Assume that $H \leq P$ such that $|H| = |D|$ and H is not s-permutably embedded in G. By Lemma 2.5(d), there is a normal subgroup of G such that $|G : M| = p$, contrary to Step 4. Thus, H is s-permutably embedded in G. Let L be an s-permutable subgroup of G such that $H \in \mathrm{Syl}_p(L)$. Since $|L_p| \neq |P|$, $L \neq G$. If $LP = G$, G has a normal subgroup of index p since $|G : L|$ is p-power and $L \lhd\lhd G$, contrary to Step 4. Thus, $LP \neq G$. By induction, LP is p-nilpotent. Then L is p-nilpotent. By Step 1, $L_{p'} \leq O_{p'}(G) = 1$. Then $H = L$ which is s-permutable in G.

Step 6: if P is a nonabelian 2-group, then $|D| > 4$. Suppose that $|D| = 4$. Since P is nonabelian, P has a cyclic subgroup $H := \langle x\rangle$ of order four. By Step 5, $\langle x\rangle$ is s-permutable in G. Thus, $\langle x^2\rangle$ is s-permutable in G. If $\langle x^2\rangle \leq Z(P)$, pick another subgroup $\langle a\rangle$ of P of order two, so $\langle a\rangle \times \langle x^2\rangle$ has order four. Thus, by Step 5, $\langle a\rangle \times \langle x^2\rangle$ is s-permutable in G. Since $\langle x^2\rangle$ is s-permutable in G, then $O^p(G) \leq N_G(\langle x^2\rangle)$. Then $\langle x^2\rangle$ is centralised by $O^p(G)$. By a result of Maschke, $\langle a\rangle \times \langle x^2\rangle$ has a subgroup of order two which is normalised, so centralised, by $O^p(G)$. Then $\langle a\rangle \times \langle x^2\rangle$ is centralised by $O^p(G)$. Thus, $\langle a\rangle$ is centralised in G, and $\langle a\rangle$ is s-permutable in G. Then every minimal subgroup of P is s-permutable in G, contrary to Step 2. If $\langle x^2\rangle \nleq Z(P)$, pick $\langle a\rangle \leq Z(P)$ of order two. Consider the subgroup $\langle a\rangle \times \langle x^2\rangle$; similar analysis with the above will lead to a contradiction. Thus, $|D| > 4$.

Step 7: if $N \leq P$ and N is minimal normal in G, then $|N| \leq |D|$. Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 2.6, N has a maximal subgroup which is normal in G, contrary to the minimality of N.

Step 8: if $N \leq P$ and N is minimal normal in G, then G/N is p-nilpotent. If $|N| < |D|$, G/N satisfies the hypothesis of the theorem by Lemma 2.2. Thus, G/N is p-nilpotent by the minimal choice of G. So we may suppose that $|N| = |D|$ by Step 6. Now we show every cyclic subgroup of P/N of order p or order four is s-permutable in G/N. Let $K \leq P$ and $|K/N| = p$. By Step 2, N is noncyclic, so are all subgroups containing N. Hence, there is a maximal subgroup $L \neq N$ of K such that $K = NL$. Of course, $|N| = |D| = |L|$. Since L is s-permutable in G by Step 5, $K/N = LN/N$ is s-permutable in G/N by Lemma 2.2(b). If $p = 2$, take $X/N \leq P/N$ which is cyclic of order four. Since N is noncyclic, X is noncyclic. This yields that there is a maximal subgroup X_1 such that $X = X_1N$. If X_1 is noncyclic, then X_1 has at least two maximal subgroups X_{11}, X_{12} of order $|D|$ which are s-permutable in G by Step 4. Then $X_1 = X_{11}X_{12}$ is s-permutable in G, and

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so $X/N = X_1N/N$ is s-permutable in G. Now suppose that X_1 is cyclic. Therefore, $|X_1 \cap N| \le 2$ as N is elementary abelian. By the computation of $|X| = |X_1||N|/|X_1 \cap N|$, we have that $|N| = 2$ or 4. Since N is noncyclic, we have $|N| = |D| = 4$, contrary to Step 6.

Step 9: a final contradiction. By Step 5, $O_p(G) \ne 1$. Take a minimal normal subgroup N of G contained in $O_p(G)$. By Step 6, G/N is p-nilpotent. It is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$. Hence, $O_p(G)$ is an elementary abelian p-group. On the other hand, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $O_p(G) \cap M = 1$, $N = O_p(G)$ and $M \cong G/N$ is p-nilpotent. Then G can be written as $G = N(M \cap P)M_{p'}$, where $M_{p'}$ is the normal p-complement of M. First, if $M \cap P = 1$, then $N = P$, a contradiction. Second, suppose that $1 < |M \cap P| \le |D|$. Pick $H \le P$ such that $M \cap P \le H$ and $|H| = |D|$. By Step 4, H is s-permutable in G. It follows that $M \cap P \le H \le O_p(G) = N$, then $M \cap P \le M \cap N = 1$, a contradiction. Finally, suppose that $|M \cap P| \ge |D|$. Pick $H \le M \cap P$ with $|H| = |D|$. By Step 4, H is s-permutable in G. It follows that $H \le O_p(G) = N$, and so $H \le N \cap M = 1$, a final contradiction.

This finishes the proof. □

Corollary 3.2. *Suppose that G is a group. If every noncyclic Sylow subgroup of G satisfies $\triangle_1$ in G. Then G has a Sylow tower of supersolvable type.*

Theorem 3.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups, and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every noncyclic Sylow subgroup of E satisfies $\triangle_1$ in G. Then $G \in \mathcal{F}$.*

Proof. Suppose that P is a Sylow p-subgroup of E for all $p \in \pi(E)$. Since P satisfies $\triangle_1$ in G by hypotheses, P satisfies $\triangle_1$ in E by Lemma 2.5. Applying Corollary 3.2, we have that E has a Sylow tower of supersolvable type. Let q be the maximal prime divisor of $|E|$ and $Q \in \mathrm{Syl}_q(E)$. Then $Q \trianglelefteq G$. Since $(G/Q, E/Q)$ satisfies the hypothesis of the theorem, by induction, $G/Q \in \mathcal{F}$. For any subgroup H of Q with $|H| = |D|$, since $Q \le O_q(G)$, H is weakly s-permutable in G by Lemma 2.5(e). By [LWQ2, Theorem 3.3], we get $G \in \mathcal{F}$. □

Now, we are ready to prove the main theorem of this paper.

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups, and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every noncyclic Sylow subgroup of $F^*(E)$ satisfies $\triangle_1$ in G. Then $G \in \mathcal{F}$.*

Proof. We distinguish two cases.

Case 1: $\mathcal{F} = \mathcal{U}$. Let (G, E) be a counterexample with $|G||E|$ minimal.

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Step 1: every proper normal subgroup N (if it exists) of G containing $F^*(E)$ is supersolvable.

If N is a proper normal subgroup of G containing $F^*(E)$, we have that $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 2.8(c), $F^*(E) = F^*(F^*(E)) \le F^*(E \cap N) \le F^*(E)$, so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup P of $F^*(E \cap N) = F^*(E)$, P satisfies $\triangle_1$ in G by hypotheses. Hence, P satisfies $\triangle_1$ in N by Lemma 2.5. So $(N, N \cap H)$ satisfies the hypotheses of the theorem, the minimal choice of G implies that N is supersolvable.

Step 2: $E = G$. If $E < G$, then $E \in \mathcal{U}$ by Step 1. Hence, $F^*(E) = F(E)$ by Lemma 2.8. It follows that every Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 2.5(e), (G, E) satisfies the hypotheses of Theorem 1.2 of [LWQ2]. By [LWQ2, Theorem 1.2], $G \in \mathcal{U}$, a contradiction.

Step 3: $F^*(G) = F(G) < G$. If $F^*(G) = G$, then $G \in \mathcal{F}$ by Theorem 3.3, contrary to the choice of G. So $F^*(G) < G$. By Step 1, $F^*(G) \in \mathcal{U}$ and $F^*(G) = F(G)$ by Lemma 2.8.

Step 4: the final contradiction. Since $F^*(G) = F(G)$, by Lemma 2.5(e), (G, E) satisfies the hypotheses of Theorem 1.2 of [LWQ2]. By [LWQ2, Theorem 1.2], $G \in \mathcal{U}$, a final contradiction.

Case 2: $\mathcal{F} \neq \mathcal{U}$. By hypotheses, every noncyclic Sylow subgroup of $F^*(E)$ satisfies $\triangle_1$ in G, thus in E by Lemma 2.5. Applying case 1, $E \in \mathcal{U}$. Then $F^*(E) = F(E)$ by Lemma 2.8. It follows that each Sylow subgroup of $F^*(E)$ is normal in G, by Lemma 2.5(e).

Since $F^*(G) = F(G)$, by Lemma 2.5(e), (G, E) satisfies the hypotheses of Theorem 1.2 of [LWQ2]. By [LWQ2, Theorem 1.2], $G \in \mathcal{F}$.

This finishes the proof of the theorem. □

4. Some applications

From the definition of weakly *s*-permutably embedded subgroup, we can see that [S, Corollaries 5.1–5.5.24] are corollaries of our Theorem 3.1. Furthermore, we have the following corollary.

COROLLARY 4.1 [LW, Theorem 1.1]. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(E)$ are s-permutably embedded in G.*

COROLLARY 4.2 [LW, Theorem 1.2]. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of prime order or order four of $F^*(E)$ are s-permutably embedded in G.*

COROLLARY 4.3 [LW, Theorem 3.8]. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E*

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such that $G/E \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(E)$ are either s-permutably embedded or c-normal in G.

Corollary 4.4 [LW, Theorem 4.3]. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of prime order or order four of $F^*(E)$ are either s-permutably embedded or c-normal in G.*

Corollary 4.5 [WW, Theorem 4.1]. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(E)$ are c^*-normal in G.*

Corollary 4.6. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of prime order or order four of $F^*(E)$ are c^*-normal in G.*

A generalisation of Theorem 3.1 is also interesting. In a routine way, we can generalise it as follows.

Theorem 4.7. *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If P satisfies $\triangle_1$ in G, then G is p-nilpotent.*

Corollary 4.8 [WW, Theorem 3.1]. *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is c^*-normal in G, then G is p-nilpotent.*

Corollary 4.9 [LWW, Theorem 3.1]. *Let G be a group and P a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is s-permutably embedded in G, then G is p-nilpotent.*

Corollary 4.10 [AH, Theorem 3.1]. *Let G be a group and P a Sylow p-subgroup of G, where p is the minimal prime divisor of $|G|$. If every maximal subgroup of P is s-permutably embedded in G, then G is p-nilpotent.*

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