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Deformation theory of rank one maximal Cohen–Macaulay modules on hypersurface singularities and the Scandinavian complex

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Dedicated to Professor Tor Gulliksen

Abstract

We show that the deformation functor of a maximal Cohen–Macaulay module $M = \operatorname{coker}(\phi)$ over the hypersurface singularity $\det(\phi)$ is given by deformations of the presenting matrix which keep the determinant constant. A simplified expression for an edge map in the canonical five-term exact sequence to a change of rings spectral sequence is obtained, including the tangent and obstruction spaces (H^1 and H^2). We relate the edge map to the *Scandinavian complex* $\mathcal{S}$ of ϕ which yields relations between the homology of $\mathcal{S}$ and H^i for $i = 1, 2$. This gives (infinitesimal) rigidity and non-rigidity results and a dimension estimate for the formally (mini-)versal formal hull H of the deformation functor.

1. Introduction

The *local moduli problem* in algebraic geometry is finding the local rings of the moduli space. The idea is that the algebraic geometric object, e.g. an A-module M, corresponding to a closed point contains all information about the infinitesimal neighbourhoods of any sort of moduli space it may occur in. No *a priori* knowledge about these spaces is necessary. To make this claim precise, one introduces the *deformation functor*. Fix a field k and suppose A is a k-algebra and let Art_k be the category of local Artinian k-algebras R with residue field k such that the composition $k \to R \to k$ is the identity and morphisms are maps of local k-algebras. Then the deformation functor of M is

$$\operatorname{Def}_M : \mathsf{Art}_k \longrightarrow \mathsf{Sets}$$

where $\operatorname{Def}_M(R)$ is the set of equivalence classes of *liftings* (or deformations) of M to R. A lifting of M to R is an $A \otimes_k R$-module M_R, which is flat as R-module, and an $A \otimes_k R$-linear map $\pi : M_R \to M$ with $\pi \otimes_R k : M_R \otimes_R k \xrightarrow{\simeq} M$. Two liftings are equivalent if they are isomorphic above M. Maps are induced by tensorization. More generally, let $F : \mathsf{Art}_k \longrightarrow \mathsf{Sets}$ be a covariant functor with $F(k)$ a one element set. Schlessinger [Sch68] formulated a sufficient and necessary set of criteria for the existence of a complete local ring H called a (pro-representable) *hull*, and a formal versal family $\{M_n\}_{n=1}^{\infty}$, which is a projective system with $M_n \in F(H/\mathfrak{m}_H^{n+1})$, such that the induced map

$$\operatorname{Hom}_{k-\text{alg.}/k}(H, -) \longrightarrow F$$

is an isomorphism or weaker, is a *tangential isomorphism* and is *formally smooth*, a strong inductive surjectivity condition. Most deformation functors satisfy these latter conditions, except possibly

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for finite dimensionality of the Zariski tangent space of F. However, Schlessinger did not give any effective way of constructing the hull. The only known general tool to find H from M is the existence and computability of a natural *obstruction class*.

Definition 1. A *small lifting situation* is a surjective map $\pi : R \to S$ in Art_k where $\ker(\pi)$ is contained in the socle of R, i.e. $\mathfrak{m}_R \cdot \ker(\pi) = 0$, and a lifting M_S of M to S.

The obstruction class is then an element $\mathrm{o} = \mathrm{o}(\pi, M_S) \in H^2 \otimes \ker(\pi)$ where H^2 is the second cohomology group of the object M. If $F = \mathrm{Def}_M$ then $\mathrm{o} = \mathrm{o}_A(\pi, M_S)$ and $H^2 = \mathrm{Ext}^2_A(M, M)$. The obstruction class is natural with respect to morphisms of the lifting situation. There exists a lifting of M_S to R (or a prolongation of the deformation M_S to the 'thicker' Artinian neighbourhood $\operatorname{Spec} R$) if and only if this obstruction class is zero. The obstruction class has been constructed for many deformation functors, e.g. [Ill71, Ill72, Lau79]; for axiomatic approaches, see [Art74, FM98, Ile01].

The starting point of the construction of the hull H, in the case of Def_M, is the 'universal extension' M_1 of M

$$M_1 : 0 \longrightarrow M \otimes_k \mathrm{Ext}^1_A(M, M)^* \longrightarrow M_1 \xrightarrow{\pi_1} M \longrightarrow 0$$

where the extension is given by the image of the identity map under the canonical isomorphism:

$$\mathrm{id} \in \mathrm{End}_k(\mathrm{Ext}^1_A(M, M)) \cong \mathrm{Ext}^1_A(M, M \otimes_k \mathrm{Ext}^1_A(M, M)^*) \ni M_1.$$

If $H_1 = k[\mathrm{Ext}^1_A(M, M)^*] = k \oplus \mathrm{Ext}^1_A(M, M)^*$, we naturally get the equivalence class $[M_1] \in \mathrm{Def}_M(H_1)$. The construction of H then proceeds through successive 'prolongations' of M_1 to thicker Artinian k-algebras through small lifting situations, calculating the obstruction at each step. If this is done correctly, we obtain power series in T^1 of minimal degree greater than or equal to two, one (possibly '0') for each cotangential 'generator' in T^2, where T^i is the completion of the free k-algebra which has $\mathrm{Ext}^i_A(M, M)$ as the Zariski tangent space for $i = 1, 2$. This defines an *obstruction map* $\mathrm{o}^A : T^2 \to T^1$ such that $H = T^1 \otimes_{T^2} k$, see [Ile01, Lau79, Lau86]. An estimate for the Krull dimension of H follows:

$$\dim_k \mathrm{Ext}^1_A(M, M) \geqslant \dim_{\mathrm{Krull}}(H) \geqslant \dim_k \mathrm{Ext}^1_A(M, M) - \dim_k \mathrm{Ext}^2_A(M, M).$$

Here the first inequality is an equality if and only if H is smooth and the second inequality is an equality if and only if the obstruction power series defines a regular sequence and all the second cohomology is hit by obstructions. (See also [Kaw95].)

In practice, it is difficult to calculate H in this way; in fact, very few classes of examples of deformation functors have been given for which anything beyond the general Krull dimension estimate is known. The degree of success in calculation will depend on how explicitly and simply one can represent the cohomology that is involved. In the present paper, we consider a class of modules where the cohomology has a particularly nice and explicit representation. Let us call an affine scheme $X = \mathrm{Spec}(B)$ over a field k a *hypersurface singularity* if $B \cong A/(f)$, where A is a regular local Noetherian k-algebra and f is non-zero and contained in the maximal ideal of A. The main result, Theorem 2, improves the above Krull dimension estimate for maximal Cohen–Macaulay (MCM) modules of rank one on hypersurface singularities in two ways. In the upper bound $\mathrm{Ext}^1_B(M, M)$ is isomorphic to $H_1(\mathcal{S})$, the first homology of the Scandinavian complex $\mathcal{S} = \mathcal{S}(\phi)$, where ϕ is a presenting matrix for M over A with $\det \phi = f$ (all MCMs of rank one on X are given in this way). The Scandinavian complex gives an A-free resolution of the quotient $A/I_{g-1}(\phi)$, where $I_{g-1}(\phi)$ is the ideal of sub-maximal minors of the $g \times g$-matrix ϕ, if $\operatorname{grade} I_{g-1}(\phi) = 4$, the maximal possible value. Hence Def^B_M, the deformation functor of M as a B-module, is non-trivial only if $\operatorname{grade} I_{g-1}(\phi) \leqslant 3$. For the lower bound, we show that there is a natural map $\mathrm{Ext}^2_B(M, M) \to H_2(\mathcal{S})$ which takes the obstruction class to zero, hence the obstruction class resides in the kernel which is determined to be $A/I_{g-1}(\phi)$. Since $\operatorname{grade} I_{g-1}(\phi) = 2$ implies $H_2(\mathcal{S}) \neq 0$, we obtain a strictly better lower estimate for the Krull dimension of the hull than the general inequality. The $\operatorname{rk} M = g-1$ case is covered as well, and by the Knörrer functor the results extend to MCMs of other ranks. Our results fit into a general change of rings framework which we now briefly describe.

Let B be a k-algebra quotient of A and $J = \ker(A \to B)$ and assume M is a B-module as an A-module, i.e. that $J \subseteq \operatorname{Ann}_A(M)$. In [Ile01], the author constructed a new obstruction class $\mathrm{o}_J(\pi, M_S)$ for a lifting situation where $\mathrm{o}_A(\pi, M_S) = 0$. We have

$$\mathrm{o}_B(\pi, M_S) = 0 \Longleftrightarrow \mathrm{o}_A(\pi, M_S) = 0 = \mathrm{o}_J(\pi, M_S).$$

It turns out that with these two classes we can construct *two* obstruction maps which define the hull of Def_M^B. The J-obstruction class resides in the cokernel of a natural map

$$\partial_J : \mathrm{Ext}_A^1(M,M) \longrightarrow \mathrm{Hom}_A(J, \mathrm{End}_A(M)) \tag{1}$$

where ∂_J is induced by the pullback along any homotopy killing the action of J on an A-free resolution of M, hence $\mathrm{o}_J(\pi, M_S) \in \operatorname{coker}(\partial_J) \otimes_k \ker(\pi)$. Moreover, the tangent space of Def_M^B is $\ker \partial_J$, see [Ile01, ch. 1]. Then ∂_J is also the second non-trivial map in the five-term exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}_B^1(M,N) \longrightarrow \mathrm{Ext}_A^1(M,N) \longrightarrow \mathrm{Hom}_A(J, \mathrm{Hom}_A(M,N)) \\ \longrightarrow \mathrm{Ext}_B^2(M,N) \longrightarrow \mathrm{Ext}_A^2(M,N) \end{aligned}$$

of a change of rings spectral sequence

$$E_2^{pq} = \mathrm{Ext}_B^p(M, \mathrm{Ext}_A^q(B,N)) \Rightarrow \mathrm{Ext}_A^*(M,N),$$

see [Ile01, ch. 4] and Lemma 1. Let T_A^2, T_J^2 and T^1 be the completion of free k-algebras with Zariski tangent spaces consisting of the image of the natural map $\mathrm{Ext}_B^2(M,M) \to \mathrm{Ext}_A^2(M,M)$, $\operatorname{coker} \partial_J$ and $\ker \partial_J$, respectively. In the case where these cohomology groups are not finite but of countable dimension as k-vector spaces, one has to introduce a suitable topology in order to allow proper dualization, and a compatible topology in which the T's are complete; see [Lau79, Ile01]. We have the following.

Theorem 1 [Ile01]. Def_M^B *is a functor with two obstructions in*

$$\operatorname{im}(\mathrm{Ext}_B^2(M,M) \longrightarrow \mathrm{Ext}_A^2(M,M)) \quad \textit{and} \quad \operatorname{coker} \partial_J$$

and with tangent space $\ker \partial_J$, *such that if these spaces have countable k-dimension, there are (continuous) obstruction maps*

$$\mathrm{o}^A : T_A^2 \longrightarrow T^1 \quad \textit{and} \quad \mathrm{o}^J : T_J^2 \longrightarrow T^1$$

for the obstructions o_A *and* o_J. *In particular, the hull is given as*

$$H \cong (T^1 \otimes_{T_A^2} k) \otimes_{T_J^2} k.$$

This theorem implies that one can find the hull for Def_M^B by calculating the obstructions as cup and generalized Massey products entirely within an A-free complex; see [Ile01, Theorem 3.1].

For explicit non-trivial calculations of obstructions (given by cup products) for the Hilbert functor of space curves, see [Wal92, Flo93]. Siqveland gave the local equations for the compactified Jacobian of the $\mathbf{E}_6$ curve singularity and found the degeneracy diagram of the rank one torsion free modules in [Siq01a] by calculating the obstruction maps; the Massey product algorithms are given in [Siq01b]. Similar ideas have recently been used by Borge [Bor02] to define a new class of p-groups for which the modular isomorphism problem can be solved.

2. The Scandinavian complex of a matrix factorization

MCM modules over hypersurface singularities are given by matrix factorizations (ϕ,ψ) of the hypersurface f and there is a natural complex $\mathcal{S}$ associated to (ϕ,ψ). If ψ is the adjoint of ϕ, the MCM $B=A/(f)$-module $M=\operatorname{coker}(\phi)$ has rank one. These are the main ingredients in Theorem 2.

Definition 2 [Eis80]. A *matrix factorization* of an element f in a ring A is a pair (ϕ,ψ) of A-linear maps of free modules $\phi: F\to G$, $\psi: G\to F$, with $\phi\psi=f\cdot \mathrm{id}_G$ and $\psi\phi=f\cdot\mathrm{id}_F$.

Let $B=A/(f)$. Then $M=\operatorname{coker}\phi$ is a B-module as an A-module since f annihilates M. If $(f)/(f^2)$ is free as a B-module, then the following 2-periodic complex of free B-modules

$$\overline{G}\xleftarrow{\overline{\phi}}\overline{F}\xleftarrow{\overline{\psi}}\overline{G}\xleftarrow{\overline{\phi}}\overline{F}\xleftarrow{\overline{\psi}}\dots \tag{2}$$

is a free resolution of M where $\overline{G}\xleftarrow{\overline{\phi}}\overline{F}\dots=[G\xleftarrow{\phi}F\dots]\otimes_A B$, [Eis80, Proposition 5.1]. Note that any A-linear map $\alpha: M\to M$ defines a map $L_2:=\overline{G}\to L_0:=\overline{G}$, which gives a cocycle in the complex computing Ext and, therefore, defines an element $\overline{\alpha}\in\operatorname{Ext}^2_B(M,M)$. If A is Noetherian and G is of finite rank, then $\operatorname{rk}F=\operatorname{rk}G$ [Eis80, Proposition 5.3]. We assume the rank(s) of G and F to be finite. Define a *regular* element in A to be $f\in A$ with $\ker(f\cdot)=0$ and $\operatorname{coker}(f\cdot)\neq 0$, where $f\cdot$ is the multiplication with f-map on A.

Example 1. Let M be a finitely generated MCM module over a local Noetherian ring B. If there is a regular local ring A and a regular element $f\in A$ with $A/(f)\cong B$, then there is a matrix factorization (ϕ,ψ) of f with $\operatorname{coker}\phi\cong M$ and (2) is a B-free resolution of M. Let $\phi: F\to G$ be a finite A-free presentation of M, we may assume that ϕ is injective by the Auslander–Buchsbaum theorem. Multiplication with f on the presentation, considering the presentation as a complex, is homotopic to zero since f annihilates M. Hence there is a $\psi: G\to F$ with $\phi\psi=f\cdot\mathrm{id}_G$. However, then (ϕ,ψ) is a matrix factorization of f by

$$\phi\psi\phi=f\cdot\mathrm{id}_G\,\phi=\phi\, f\cdot\mathrm{id}_F\Longrightarrow\psi\phi=f\cdot\mathrm{id}_F$$

since ϕ is injective.

On the other hand, if (ϕ,ψ) is a matrix factorization with $\operatorname{rk}G<\infty$ of the regular element f in the local Cohen–Macaulay ring A, then $M=\operatorname{coker}\phi$ is a MCM module over $A/(f)$. Since f is regular, both ϕ and ψ are injective, hence $\operatorname{pd}_A(M)=1=\operatorname{depth}A-\operatorname{depth}M$ and $\operatorname{depth}M=\operatorname{depth}A/(f)=\dim A/(f)$.

There is a functorial complex to ϕ due to Gulliksen and Negård, the 'Scandinavian complex' $\mathcal{S}(\phi)$, which approximates an A-free resolution of $A/I_{g-1}(\phi)$; see [GN72]. We define a complex for a matrix factorization in general.

Definition 3. Let $G\underset{\psi}{\overset{\phi}{\rightleftarrows}}F$ be a matrix factorization of $f\in A$ where A is a ring. Then there is a complex $\mathcal{S}(\phi,\psi)$ of free A-modules

$$A\xleftarrow{d_1}\operatorname{Hom}(F,G)\xleftarrow{d_2}E\xleftarrow{d_3}\operatorname{Hom}(G,F)\xleftarrow{d_4}A,$$

which is functorial in A. E is the middle cohomology of the split monad

$$A\xleftarrow{j}\operatorname{End}(G)\oplus\operatorname{End}(F)\xleftarrow{i}A$$

where $i(a)=(aI,aI)$, $j(\rho_0,\rho_1)=\operatorname{tr}(\rho_0)-\operatorname{tr}(\rho_1)$. The differentials are: $d_1(\xi)=\operatorname{tr}(\xi\psi)$; d_2 and d_3 are induced by the differentials in the Yoneda complex,

$$0\longleftarrow\operatorname{Hom}(F,G)\xleftarrow{d_2}\operatorname{End}(G)\oplus\operatorname{End}(F)\xleftarrow{d_3}\operatorname{Hom}(G,F)\longleftarrow 0,$$

i.e. $d_2(\rho_0,\rho_1)=\phi\rho_1-\rho_0\phi$ and $d_3(\tau)=(\phi\tau,\tau\phi)$; and $d_4(r)=r\psi$.

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Remark 1. In the proof of Theorem 2, we only use the complex $\mathcal{S}(\phi, \psi)$ in the case $\psi = \phi^{\mathrm{a}}$, the adjoint of ϕ, and then it is the same as the complex $\mathcal{S}(\phi)$ of [GN72], i.e. $\mathcal{S}(\phi) = \mathcal{S}(\phi, \phi^{\mathrm{a}})$. Apart from (10), we give no further statements concerning the generalization $\mathcal{S}(\phi, \psi)$, but our definition highlights the connection with the Yoneda complex which is convenient for our purposes. Gulliksen and Negård proved that $\mathcal{S}(\phi)$ is a free resolution of $A/I_{g-1}(\phi)$ if $\operatorname{grade} I_{g-1}(\phi)$ has the maximal value, which is 4.

Our main result is a sharpened form of Theorem 1 for modules given by certain matrix factorizations. Let $T^i(X)$ be the completion of the free k-algebra with the Zariski tangent space X (of countable dimension) in the proper topology; see [Ile01, ch. 2] and [Lau79, ch. 4]. Let $\operatorname{grade} I = n$ if a maximal A-regular sequence in I has length n.

THEOREM 2. *Suppose that A is a Noetherian k-algebra, $\phi : F \to G$ is a homomorphism of free A-modules of equal rank g with $\det(\phi) = f$ and $f \in A$ is a regular element. Set $M = \operatorname{coker} \phi$ and $B = A/(f)$. Assume $\operatorname{Ann}_A M = (f)$ and $I_{g-1}(\phi) \neq A$. Then the deformation functor Def_M^B has the tangent space $\operatorname{Def}_M^B(k[\varepsilon]) \cong \operatorname{Ext}_B^1(M, M) \cong H_1(\mathcal{S})$ where $\mathcal{S} = \mathcal{S}(\phi)$ is the Scandinavian complex of ϕ. The first of the two obstruction maps o^A is trivial while the other o^J for $J = (f)$ factors through the quotient*

$$\begin{array}{ccc} T^2(\operatorname{Ext}_B^2(M,M)) & & \\ \downarrow & \searrow^{\mathrm{o}^J} & \\ T^2(A/I_{g-1}(\phi)) & \xrightarrow[\mathrm{o}^f]{} & T^1(H_1(\mathcal{S})) = T^1 \end{array}$$

which is induced by a natural inclusion $A/I_{g-1}(\phi) \hookrightarrow \operatorname{Ext}_B^2(M, M)$; hence the hull is given as

$$H \cong T^1 \otimes_{T^2} k.$$

Moreover, the tangent and obstruction spaces of Def_M^B have finite dimension if and only if $\dim_k A/I_{g-1}(\phi) < \infty$ and then

$$\begin{aligned} \dim_k H_1(\mathcal{S}) \geqslant \dim_{\mathrm{Krull}} H &\geqslant \dim_k H_1(\mathcal{S}) - \dim_k A/I_{g-1}(\phi) \\ &= \dim_k \operatorname{Ext}_B^1(M,M) - \dim_k \operatorname{Ext}_B^2(M,M) + \dim_k H_2(\mathcal{S}). \end{aligned} \tag{3}$$

In particular, $H_2(\mathcal{S}) \neq 0$ if $\operatorname{grade} I_{g-1}(\phi) = 2$ and Def_M^B is infinitesimally rigid if $\operatorname{grade} I_{g-1}(\phi) = 4$.

The main steps in the proof of Theorem 2 (which takes up the remainder of this paper) are as follows. The deformation functor Def_M^B is isomorphic to the deformation functor of the matrix factorization $(\phi, \phi^{\mathrm{a}})$ over A. Since $\operatorname{Ext}_A^2(M, M) = 0$, all obstructions are given by the $J = (f)$-obstruction map; this is a special case of Theorem 1. In Proposition 1, we show that Def_M^B, furthermore, is isomorphic to the functor of deformations $\tilde{\phi}$ of ϕ with $\det \tilde{\phi}$ constant (equal to f), and the argument implies that the (f)-obstruction class resides in the image of the composition of natural maps $B \to \operatorname{End}_B(M) \to \operatorname{Ext}_B^2(M, M)$. The technical heart of the proof is a factorization of the edge map $\partial_f : \operatorname{Ext}_A^1(M, M) \to \operatorname{End}_B(M)$ in the change of rings spectral sequence via a trace map, Proposition 2. This gives a factorization of $B \to \operatorname{Ext}_B^2(M, M)$ via $A/I_{g-1}(\phi)$, hence the factorization of o^J in Theorem 2. The trace map factorization also enables the identification of the Zariski tangent space of Def_M^B with the first homology $H_1(\mathcal{S})$ of the Scandinavian complex of ϕ, and implies the existence of a short exact sequence (11) where $H_2(\mathcal{S})$ is the quotient of $\operatorname{Ext}_B^2(M, M)$ by $A/I_{g-1}(\phi)$. Then the rest follows directly from [GN72].

One obstruction map $\mathrm{o}^x : T^2 \to T^1$, as o^f in Theorem 2 for a functor F with a natural obstruction class o_x in some H^2, should, in addition to being continuous, satisfy the following. Fix a formal versal family $\{M_i\}$, $M_i \in F(H_i)$ where $H = \varprojlim H_i$ is a hull for F. For any lifting situation as in Definition 1 and map $\sigma : H_i \to S$ in Art_k with $\sigma_* M_i = M_S$ (which we know exists by versality), we should have that $\mathrm{o}_x(\pi, M_S) \in H^2 \otimes_k I$ is the adjoint to $(\mathrm{o}^x)^*\theta : H^{2^*} \to I$ in the commutative diagram

$$\begin{array}{ccccccc} & & & & & & I \\ & & & \overset{(\mathrm{o}^x)^*\theta}{\nearrow} & & & \downarrow \\ H^{2^*} & \hookrightarrow & T^2 & \xrightarrow{\mathrm{o}^x} & T^1 & \xrightarrow{\theta} & R \\ & & & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & H_i & \xrightarrow[\sigma]{} & S \end{array} \tag{4}$$

where θ is continuous and lifts σ. In [Ile01, ch. 2] we gave an axiomatic treatment of a functor with n obstructions and proved the existence of n obstruction maps.

Example 2. In the 2×2-case, the Koszul complex $K(x_{ij})$ and $\mathcal{S}(\phi)$ for $\phi = (x_{ij})$ are isomorphic and $I_{g-1}(\phi) = I(\phi) = (x_{ij})$ so we have the following:

- $\operatorname{grade} I(\phi) = 4 \iff K(x_{ij}) \cong \mathcal{S}(\phi)$ are acyclic.
- $\operatorname{grade} I(\phi) = 3 \iff H_1(K) \cong H_1(\mathcal{S}) \neq 0$ and $H_i(K) \cong H_i(\mathcal{S}) = 0$ for $i \geqslant 2$. If $(\underline{x}) = (x_{ij})_{ij \neq i_0 j_0}$ and $\operatorname{grade}(\underline{x}) = 3$, we have $H_1(\mathcal{S}) = \ker(A/(\underline{x}) \xrightarrow{\cdot x_{i_0 j_0}} A/(\underline{x}))$ by the mapping cone sequence of the Koszul complex. In particular, $H_1(\mathcal{S}) = A/I(\phi)$ if $x_{i_0 j_0} \in (\underline{x})$.
- $\operatorname{grade} I(\phi) = 2 \iff H_2(K) \cong H_2(\mathcal{S}) \neq 0$ and $H_i(K) \cong H_i(\mathcal{S}) = 0$ for $i \geqslant 3$. If $(x_{ij}) = (x_1, \dots, x_4)$ such that (x_1, x_2) is a regular sequence then $H_2(\mathcal{S}) \cong ((x_1, x_2) : (x_1, \dots, x_4))/(x_1, x_2)$. In particular if $(x_3, x_4) \subseteq (x_1, x_2)$ then $H_2(\mathcal{S}) = A/I(\phi)$.

3. Deforming matrix factorizations

The deformation functor of the matrix factorization (ϕ, ψ) is isomorphic to Def_M^B. If ψ is the adjoint of ϕ, then Def_M^B is also isomorphic to the functor of deformations $\tilde{\phi}$ of ϕ with $\det \tilde{\phi} = f$. This narrows the obstruction space to the image of B in $\mathrm{Ext}_B^2(M, M)$.

To a matrix factorization (ϕ, ψ) of $f \in A$, where A is a k-algebra, there is a deformation functor $\mathrm{Def}_{(\phi,\psi)}^A$ of equivalence classes of liftings $(\tilde{\phi}, \tilde{\psi})$ of (ϕ, ψ) as matrix factorization, i.e. $\mathrm{Def}_{(\phi,\psi)}^A(R)$ is the set of equivalence classes of commutative diagrams

$$\begin{array}{ccccc} G \otimes_k R & \xleftarrow{\tilde{\phi}} & F \otimes_k R & \xleftarrow{\tilde{\psi}} & G \otimes_k R \\ \downarrow & & \downarrow & & \downarrow \\ G & \xleftarrow{\phi} & F & \xleftarrow{\psi} & G \end{array}$$

such that $\tilde{\phi}\tilde{\psi} = f \cdot \mathrm{id}_{G \otimes R}$. If $B = A/(f)$, there is a map

$$\mathrm{Def}_{(\phi,\psi)}^A \longrightarrow \mathrm{Def}_M^B \tag{5}$$

by taking the cokernel of $\tilde{\phi}$ which gives an R-flat module since ϕ and hence $\tilde{\phi}$ are injective. Moreover, $M_R = \operatorname{coker}(\tilde{\phi})$ is a $B \otimes R$-module as an $A \otimes R$-module since it is annihilated by f which follows from the relation $\tilde{\phi}\tilde{\psi} = f \cdot \mathrm{id}_{G \otimes R}$. The map is a tangential isomorphism and smooth since we can more generally construct the hull of Def_M^B from lifting an A-free resolution (F_*, d_*) of M and a map $\psi : E \otimes_A F_0 \to F_1$ satisfying $d_0 \psi = (f_1, \dots, f_r) \otimes_A F_0$ where $E \cong A^r \twoheadrightarrow J = (f_1, \dots, f_r)$; see [Ile01]. Hence, the map (5) is an isomorphism since it is clear that it is injective.

If f is a regular element, we have that ψ, if it exists, is uniquely determined by ϕ; see [Eis80, Proposition 5.5]. In fact $\det(\phi) \cdot \psi = f \cdot \phi^{\mathrm{a}}$ which uniquely determines ψ since $\det(\phi)$ is regular [*op. cit.*]. Let us therefore define another deformation functor $\mathrm{Def}_{(\phi \mid \det \phi)}^A$ of deformations of ϕ with

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fixed determinant. If f is regular, we have

$$\mathrm{Def}^A_{(\phi|\det\phi)} \hookrightarrow \mathrm{Def}^A_{(\phi,\psi)}. \tag{6}$$

The following result is essential for the proof of Theorem 2.

PROPOSITION 1. *Let (ϕ,ψ) be a matrix factorization of $\det\phi = f \in A$ where A is a k-algebra. Set $M = \mathrm{coker}(\phi)$, $B = A/(f)$ and assume f is regular in A and $\mathrm{Ann}_A M = (f)$. Then the natural maps*

$$\mathrm{Def}^A_{(\phi|\det\phi)} \longrightarrow \mathrm{Def}^A_{(\phi,\psi)} \longrightarrow \mathrm{Def}^B_M$$

are isomorphisms. In particular, there exists a versal lifting $\tilde{\phi} = \varprojlim \phi_n$ of ϕ to the hull $H = \varprojlim H_n$ of Def^B_M such that $\det\tilde{\phi} = f$.

Proof. We already have that the last map is an isomorphism and the first map is injective, hence we only have to show the surjectivity of the first map. We proceed by induction on the length of R, the beginning is trivial. Assume $\pi : R \to S$ is surjective in Art_k and $\mathfrak{m}_R \cdot I = 0$ where $I = \ker\pi$. Assume we have a lifting (ϕ_S,ψ_S) of (ϕ,ψ) to S with $\phi_S\psi_S = f\cdot \mathrm{id}_{G\otimes S}$ and $\det\phi_S = \det\phi = f$. Given a further lifting $(\tilde{\phi},\tilde{\psi})$ of (ϕ_S,ψ_S) to R with $\tilde{\phi}\tilde{\psi} = f\cdot\mathrm{id}_{G\otimes R}$. Set $\tilde{M} = \mathrm{coker}\,\tilde{\phi}$. Then $\det\tilde{\phi} = \det\phi_S + u = f + u$ with $u = \sum a_i \otimes u_i \in A \otimes I$. We get $a_i \in \mathrm{Ann}_A M$ since $\det\tilde{\phi} \in \mathrm{Ann}\,\tilde{M}$ and $f \in \mathrm{Ann}\,\tilde{M}$ implies $u \in \mathrm{Ann}\,\tilde{M}$. However, $\mathrm{Ann}_A M = (f)$ by assumption, hence $u = bf$ and we can modify $(\tilde{\phi},\tilde{\psi})$ to $(\tilde{\phi}',\tilde{\psi}')$ where $\tilde{\phi}' = \tilde{\phi} - b\cdot e_{11}\phi$ and $\tilde{\psi}' = \tilde{\psi} + b\cdot\psi e_{11}$ where the endomorphism e_{11} is given by a matrix with 1 in the upper-left corner and 0 elsewhere. Then $(\tilde{\phi}',\tilde{\psi}')$ is a matrix factorization of f equivalent to $(\tilde{\phi},\tilde{\psi})$ with $\det\tilde{\phi}' = \det\tilde{\phi} - b\cdot\mathrm{tr}(e_{11}\phi\phi^{\mathrm{a}}) = f + u - bf = f$ where tr is the trace. □

Proposition 1 states that we only have to lift ϕ and solve the equation $\det\tilde{\phi} = f$ to find the obstructions. If $\det\tilde{\phi} = f$ then $\tilde{\psi} = (\tilde{\phi})^a$, the adjoint of $\tilde{\phi}$.

If $B = A/(f)$ is a domain, i.e. f is prime, the *rank* of $M = \mathrm{coker}\,\phi$ is the dimension of the K-vector space $K \otimes_B M$ where $K = K(B)$ is the field of fractions of B. If (ϕ,ψ) is a matrix factorization of f which is regular and prime, then $\det\phi = xf^r$ with $x \notin (f)$ and $r = \mathrm{rk}_B M$ [Eis80, Proposition 5.6]. Hence, in this case the B-module M of Proposition 1 has rank one. In particular, all MCM modules of rank one on irreducible hypersurface singularities are subsumed by the proposition.

4. An edge map as a trace

The pullback of 1-cocycles by ψ is identified as an edge map in the change of rings spectral sequence and it factors through a trace map. This enables us to establish connections between the cohomology of the module and the homology of the Scandinavian complex.

LEMMA 1. *Let $A \to B$ be a ring homomorphism and N and M be an A- and a B-module, respectively. Then there is a first quadrant cohomological spectral sequence*

$$E_2^{\mathrm{pq}} = \mathrm{Ext}^p_B(M, \mathrm{Ext}^q_A(B,N)) \Rightarrow \mathrm{Ext}^*_A(M,N).$$

In particular, there is a canonical five-term exact sequence which, in the case $B = A/(f)$ and $N = M = \mathrm{coker}\,\phi$ for a matrix factorization (ϕ,ψ) of $f \in A$ where f is regular, reduces to the four-term exact sequence

$$0 \to \mathrm{Ext}^1_B(M,M) \to \mathrm{Ext}^1_A(M,M) \xrightarrow{\partial_f} \mathrm{End}_A(M) \xrightarrow{d_2} \mathrm{Ext}^2_B(M,M) \to 0 \tag{7}$$

where $\partial_f = \psi^*$ is the pullback of cocycles along ψ and d_2 is the differential of the spectral sequence. Moreover, d_2 is the map sending the endomorphism $\alpha \in \mathrm{End}_A(M)$ to $\overline{\alpha}$, defined after (2).

Proof. Let $F = F_. \twoheadrightarrow M$ be a B-projective resolution of M and $N \hookrightarrow I^{\cdot} = I$ an A-injective resolution of N. Then the II-filtration of $\mathrm{Hom}_B(F, \mathrm{Hom}_A(B,I))$ gives ${}^{II}E_1^{pq} = \mathrm{Hom}_A(M, \mathrm{Hom}_A(B, I^q))$ for

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$p = 0$ and zero for $p > 0$, since $\operatorname{Hom}_A(B, I^q)$ is B-injective. We get $\operatorname{Hom}_A(M, \operatorname{Hom}_A(B, I)) \cong \operatorname{Hom}_A(M, I)$ by adjointness so the spectral sequence collapses at stage 2 to a single row with

$$\operatorname{Ext}^q_A(M, N) \cong {}^{II}E_2^{0q} \cong H^q(\operatorname{Hom}_A(F, \operatorname{Hom}_A(B, I))),$$

the total cohomology. The I-filtration gives ${}^{I}E_1^{pq} = \operatorname{Hom}_B(F_p, \operatorname{Ext}^q_A(B, N))$, thus ${}^{I}E_2^{pq} = \operatorname{Ext}^p_B(M, \operatorname{Ext}^q_A(B, N))$. The five-term exact sequence is the standard one $0 \to E_2^{10} \to H^1 \to E_2^{01} \to E_2^{20} \to H^2$ where the last term vanishes in the matrix factorization case. If $B = A/J$, then $E_2^{01} \cong \operatorname{Hom}_A(J, \operatorname{Hom}_A(M, N))$.

In the matrix factorization case, let $\xi \in \operatorname{Hom}_A(F, G)$ represent a class $[\xi] \in \operatorname{Ext}^1_A(M, M)$. Then there is a $\rho \in \operatorname{Hom}_A(G, I^0)$ extending $\iota\varepsilon\xi$ where $\varepsilon : G \twoheadrightarrow M$ and $\iota : M \hookrightarrow I^0$ are the augmentation and coaugmentation maps, respectively. There is also a $\tau \in \operatorname{Hom}_A(M, I^1)$ extending $d^0\rho$; clearly $[\tau] = [\xi]$. From $\rho\phi = \iota\varepsilon\xi$ we get $f{\cdot}\rho = \iota\varepsilon\xi\psi$ and the latter represents $\partial_f([\xi])$. If $\overline{\varepsilon} = \varepsilon \otimes_A B$, we are left to show that the connecting $\operatorname{Hom}_B(M, \operatorname{Ext}^1_A(B, M)) \xrightarrow{\simeq} \operatorname{End}_B(M)$ is represented by taking $\tau\overline{\varepsilon}$ to $f{\cdot}\rho$. Applying $\operatorname{Hom}_A(G, \operatorname{Hom}_A(-, I^*))$ to the short exact sequence $0 \to A \xrightarrow{f\cdot} A \to B \to 0$ gives a short exact sequence of complexes; observe that $\operatorname{Hom}_A(G, \operatorname{Hom}_A(B, I^*)) \cong \operatorname{Hom}_A(\overline{G}, I^*)$, hence we obtain the following pointed commutative diagram:

$$\begin{array}{ccccc}
0 \longleftarrow \operatorname{Hom}_A(G, I^1) & \xleftarrow{f\cdot} & \operatorname{Hom}_A(G, I^1) \ni \tau\varepsilon & \longleftarrow & \operatorname{Hom}_A(\overline{G}, I^1) \ni \tau\overline{\varepsilon} \leftarrow 0 \\
\uparrow & & \uparrow & & \uparrow \\
0 \leftarrow \operatorname{Hom}_A(G, I^0) \ni f{\cdot}\rho & \xleftarrow{f\cdot} & \operatorname{Hom}_A(G, I^0) \ni \rho & \longleftarrow & \operatorname{Hom}_A(\overline{G}, I^0) \longleftarrow 0 \\
\uparrow{\scriptstyle \iota_*} & & & & \\
\operatorname{Hom}_A(G, M) \ni \varepsilon\xi\psi & & & &
\end{array}$$

A description of the map d_2 which implies the last part of the lemma is given in much greater generality in [Ile01]. ☐

The following description of the edge map ∂_f is crucial.

Proposition 2. *Let A be a ring and $M = \operatorname{coker}\phi$ an A-module where $\phi : F \to G$ is an A-linear map of free A-modules of equal, finite rank g. If $f = \det\phi$ is A-regular, $B = A/(f)$ and $\operatorname{Ann}_A M = (f)$, then there is an exact sequence mapping to the four-term exact sequence (7):*

$$\begin{array}{ccccccc}
0 \to \operatorname{Ext}^1_B(M, M) & \longrightarrow & \operatorname{Ext}^1_A(M, M) & \xrightarrow{\operatorname{tr}(-\circ\phi^{\mathrm{a}})} & B & \longrightarrow & A/I_{g-1}(\phi) \to 0 \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \cdot\operatorname{id}_M} & & \downarrow \\
0 \to \operatorname{Ext}^1_B(M, M) & \longrightarrow & \operatorname{Ext}^1_A(M, M) & \xrightarrow{\partial_f} & \operatorname{End}_B(M) & \longrightarrow & \operatorname{Ext}^2_B(M, M) \to 0
\end{array}$$

In particular, $\partial_f = (\phi^{\mathrm{a}})^*$ is induced by $\operatorname{tr}(-\circ\phi^{\mathrm{a}})\operatorname{id}_M$ where ϕ^{a} is the adjoint of ϕ and tr is the trace.

Proof. The image of $\operatorname{tr}(-\circ\phi^{\mathrm{a}}) : \operatorname{Hom}_A(F, G) \to A$, $\xi \mapsto \operatorname{tr}(\xi\phi^{\mathrm{a}})$, is $I_1(\phi^{\mathrm{a}}) = I_{g-1}(\phi)$. To get the commutativity of the central square in the diagram, we have to show that $(\phi^{\mathrm{a}})^* = \operatorname{tr}(-\circ\phi^{\mathrm{a}})\operatorname{id}_M$. Let e_{ij} be the $g\times g$-matrix with 1 in ij-position and 0 elsewhere. It is sufficient to find a $g\times g$-matrix ϕ^{ij} with

$$e_{ij}\phi^{\mathrm{a}} + \phi\phi^{ij} = c_{ij}{\cdot}\operatorname{id}_G \tag{8}$$

where c_{ij} is the ij-cofactor; $c_{ij} = (-1)^{i+j}m_{ij}(\phi)$ where $m_{ij}(\phi)$ is the ij-minor of ϕ. We have $\operatorname{tr}(e_{ij}\phi^{\mathrm{a}}) = c_{ij}$ and since any $\xi \in \operatorname{Hom}_A(F, G)$ is an A-linear combination of e_{ij}, we get $\partial_f(\xi) = \operatorname{tr}(\xi\phi^{\mathrm{a}})\operatorname{id}_M$. If $m_{ik}(\phi_{lj})$ is the ik-minor of the matrix ϕ_{lj} obtained from ϕ by deleting

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the lth row and jth column, we define ϕ^{ij} by

$$\phi^{ij}=(-1)^{i+j}\begin{pmatrix} m_{i1}(\phi_{1j}) & -m_{i1}(\phi_{2j}) & \dots & \pm m_{i1}(\phi_{i-1j}) & 0 & \mp m_{i1}(\phi_{i+1j}) & \dots & \pm m_{i1}(\phi_{gj}) \\ -m_{i2}(\phi_{1j}) & m_{i2}(\phi_{2j}) & \dots & \mp m_{i2}(\phi_{i-1j}) & 0 & \pm m_{i2}(\phi_{i+1j}) & \dots & \mp m_{i2}(\phi_{gj}) \\ \vdots & \vdots & & & \vdots & & & \vdots \\ \pm m_{ij-1}(\phi_{1j}) & \mp m_{ij-1}(\phi_{2j}) & \dots & & 0 & & & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \mp m_{ij+1}(\phi_{1j}) & \pm m_{ij+1}(\phi_{2j}) & \dots & & 0 & & & \\ \vdots & \vdots & & & \vdots & & & \vdots \\ \pm m_{ig}(\phi_{1j}) & \mp m_{ig}(\phi_{2j}) & \dots & & 0 & & \dots & \pm m_{ig}(\phi_{gj}) \end{pmatrix}$$

Then the kl-entry of $\phi\phi^{ij}$ is

$$(\phi\phi^{ij})_{kl} = \begin{cases} 0 & k \neq i \neq l \neq k \quad \text{(Laplace relation)} \\ -c_{lj} & k = i \neq l \quad \text{(Laplace expansion of } \det \phi_{lj}) \\ c_{ij} & k = l \neq i \quad (\, m_{it}(\phi_{kj}) = m_{kt}(\phi_{ij})\,) \\ 0 & l = i \end{cases}$$

hence (8) follows and the rest follows from Lemma 1. □

Corollary 1. *With the assumptions as in Proposition 2,*

$$\ker(\partial_f) \cong \operatorname{Ext}^1_B(M,M) \cong H_1(\mathcal{S})$$

where $\partial_f = (\phi^{\mathrm{a}})^*$ is the pullback map in (7) and $\mathcal{S} = \mathcal{S}(\phi)$ is the Scandinavian complex of ϕ.

Proof. By (8), for each ξ there is a ϕ^ξ with $\xi\phi^{\mathrm{a}} + \phi\phi^\xi = (\sum r_{ij}c_{ij})\cdot \mathrm{id}_G$ if $\xi = \sum r_{ij}e_{ij}$. However, clearly $\operatorname{tr}(\xi\phi^{\mathrm{a}}) = \sum r_{ij}c_{ij}$ hence every class $[\xi'] \in \operatorname{Ext}^1_B(M,M)$, i.e. with $\partial_f([\xi']) = 0$, may be represented by a cocycle with $\operatorname{tr}(\xi'\phi^{\mathrm{a}}) = xf$ for some $x \in A$. Set $\xi = \xi' - x\cdot e_{11}\phi$, then $\xi \in Z_{\mathrm{tr}} := \ker d_1$ and $[\xi] = [\xi'] \in \operatorname{Ext}^1_B(M,M)$. Let $B_{\mathrm{tr}} = B \cap Z_{\mathrm{tr}}$, where B is the set of Yoneda coboundaries. Since $(\phi\rho_1 - \rho_0\phi)\phi^{\mathrm{a}} = \phi\rho_1\phi^{\mathrm{a}} - f\rho_0$ we get $\operatorname{tr}((\phi\rho_1 - \rho_0\phi)\phi^{\mathrm{a}}) = f\cdot(\operatorname{tr}\rho_1 - \operatorname{tr}\rho_0)$ which equals zero if and only if $\operatorname{tr}\rho_1 = \operatorname{tr}\rho_0$, i.e. $B_{\mathrm{tr}} = \operatorname{im}(d_2)$ and $\operatorname{Ext}^1_B(M,M) \cong H_1(\mathcal{S})$. By Lemma 1, $\operatorname{Ext}^1_B(M,M) \cong \ker(\partial_f)$. □

We say that an A-module M is *infinitesimally rigid* if $\operatorname{Ext}^1_A(M,M) = 0$. Whenever the deformation functor of M is defined, e.g. if A is a k-algebra, every lifting M_R of M to R in Art_k is equivalent to the trivial lifting $M \otimes R$ of M if and only if M is infinitesimally rigid.

Corollary 2. *In addition to the assumptions in Proposition 2, assume that A is Noetherian and $I_{g-1}(\phi) \neq A$. Then M is infinitesimally rigid as a B-module if* $\operatorname{grade} I_{g-1}(\phi) = 4$, *the maximal value. Moreover, M is not infinitesimally rigid as a B-module if* $\operatorname{grade} I_{g-1}(\phi) = 3$.

Proof. By [GN72, Théorèm 1], $c + q = 4$ where $c = \operatorname{grade} I_{g-1}(\phi)$ and q is maximal with $H_q(\mathcal{S}(\phi)) \neq 0$. The result then follows from Corollary 1. □

There is also a relation between the obstruction group and the second homology group of the Scandinavian complex, which we may obtain from the commutative diagram in Proposition 2.

Corollary 3. *With the assumptions as in Proposition 2, there is an exact sequence*

$$0 \to \operatorname{Ext}^1_B(M,M) \to \operatorname{Ext}^1_A(M,M) \xrightarrow{\operatorname{tr}(-\circ\phi^{\mathrm{a}})} B \to \operatorname{Ext}^2_B(M,M) \to H_2(\mathcal{S}) \to 0 \tag{9}$$

where ϕ^{a} is the adjoint to ϕ and $\mathcal{S} = \mathcal{S}(\phi)$ is the Scandinavian complex of ϕ. In the case A is Noetherian, $I_{g-1}(\phi) \neq A$ and $\operatorname{grade} I_{g-1}(\phi) \geqslant 3$ *we have*

$$\operatorname{coker}(\partial_f) \cong \operatorname{Ext}^2_B(M,M) \cong A/I_{g-1}(\phi).$$

Proof. The map $B \to \operatorname{Ext}^2_B(M,M)$ is the composition

$$B \longrightarrow A/I_{g-1}(\phi) \longrightarrow \operatorname{Ext}^2_B(M,M),$$

see Proposition 2. The map $B \to \operatorname{End}_B(M)$ is injective since $\operatorname{Ann}_A M = (f)$, and hence $A/I_{g-1}(\phi) \to \operatorname{Ext}^2_B(M,M)$ is also injective, and thus (9) is exact at B. The map $\operatorname{Ext}^2_B(M,M) \to H_2(\mathcal{S})$ is defined via the natural $d_2 : \operatorname{End}_B(M) \to \operatorname{Ext}^2_B(M,M)$ in Lemma 1 and the second non-trivial map h in the sequence

$$0 \longrightarrow A/\operatorname{Ann}_A M \longrightarrow \operatorname{End}_B(M) \xrightarrow{h} H_2(\mathcal{S}) \longrightarrow 0 \tag{10}$$

which is defined and short exact for any matrix factorization (ϕ,ψ) of a regular $f \in A$ and with $\mathcal{S} = \mathcal{S}(\phi,\psi)$. If $\rho_0 : G \to G$ and $\rho_1 : F \to F$ represent an endomorphism $[\rho]$, i.e. $\rho_0\phi = \phi\rho_1$, we have $\psi\rho_0\phi = \psi\phi\rho_1 = f\cdot\rho_1$ which implies $f\cdot\operatorname{tr}(\rho_1) = f\cdot\operatorname{tr}(\rho_0)$ and hence $\operatorname{tr}(\rho_1) = \operatorname{tr}(\rho_0)$. Let $h([\rho])$ be defined as the class represented by (ρ_0,ρ_1) in $\mathcal{S}_2$, which is clearly independent of representatives by the definition of d_3 as essentially the Yoneda differential. The surjectivity of h is immediate since d_2 is induced by the Yoneda differential. To show exactness in the middle, assume that $h([\rho]) = 0$, then $(\rho_0,\rho_1) = (\phi\tau,\tau\phi) + (a\cdot\operatorname{id}_G, a\cdot\operatorname{id}_F)$ for some $\tau \in \operatorname{Hom}_A(G,F)$ and $a \in A$, but then $[\rho] = a\cdot\operatorname{id}_M$. The exactness at $\operatorname{Ext}^2_B(M,M)$ follows because the cokernels of $B \to \operatorname{End}_B(M)$ and $A/I_{g-1}(\phi) \to \operatorname{Ext}^2_B(M,M)$ are isomorphic to $H_2(\mathcal{S})$. The remaining follows from Proposition 2 and the argument in Corollary 2. □

Corollary 4. *With the assumptions as in Corollary 2, we have the following assertions:*

$$l(A/I_{g-1}(\phi)) < \infty \Longrightarrow l(\operatorname{Ext}^i_B(M,M)) < \infty \quad \text{for } i \geqslant 1$$

and

$$l(\operatorname{Ext}^2_B(M,M)) < \infty \Longrightarrow l(A/I_{g-1}(\phi)) < \infty.$$

In particular, if the deformation functor of a rank one MCM module over an irreducible hypersurface singularity X has finite-dimensional obstruction space then $\dim X \leqslant 3$. If in addition $\dim X = 3$, then the module is infinitesimally rigid.

Proof. If A is Noetherian and the rank of G and F is finite, the homology modules $H_*(\mathcal{S})$ are finitely generated. By Lemme 2 and Lemme 4 in [GN72] $I_{g-1}(\phi)^i\cdot H_i(\mathcal{S}) = 0$ for $i = 1,2$, the result then follows from Corollary 1 and from the short exact sequence

$$0 \longrightarrow A/I_{g-1}(\phi) \longrightarrow \operatorname{Ext}^2_B(M,M) \longrightarrow H_2(\mathcal{S}) \longrightarrow 0 \tag{11}$$

derived from the exact sequence (9) in Corollary 3. For the last part see Corollary 2 and remarks after Proposition 1. □

Remark 2. If we are mainly interested in modules with non-trivial deformation functors where the tangent and obstruction spaces are finite dimensional, Corollary 4 states that there are not too many places to look for such MCM-modules of rank one on hypersurface singularities. However, this is certainly not the case for higher rank MCMs. To a matrix factorization (ϕ,ψ) of f, Knörrer defined a matrix factorization

$$\left(\begin{pmatrix} v\cdot\operatorname{id} & \phi \\ \psi & -u\cdot\operatorname{id}\end{pmatrix}, \begin{pmatrix} u\cdot\operatorname{id} & \phi \\ \psi & -v\cdot\operatorname{id}\end{pmatrix}\right)$$

of $f + uv$ which induces a functor of MCMs modulo stable equivalence; see [Kno87]. In [Ile90] we proved that this functor gives a natural transformation of the corresponding deformation functors inducing isomorphisms of the tangent and obstruction spaces (see also [Ile01, Theorem 7.4.18]). Hence, starting with a rank one MCM in the 'interesting' range, applying the Knörrer functor one or more times produces MCMs in higher dimensions with the same deformation theory. However, these new MCMs naturally cannot be of rank one. In fact $\operatorname{rk} K(M) = \operatorname{rk} G$, where K is the Knörrer functor.

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The deformation theory of rank one MCM modules also subsumes the deformation theory of their syzygies which have rank $g-1$ by the following lemma.

LEMMA 2. *If M is a MCM $B = A/(f)$-module, A is a local, regular k-algebra and f is regular and prime such that the minimal presentation $\psi : G \to F$ of M has* $\operatorname{rk} F = g$ *while* $\operatorname{rk}_B M = g-1$, *then there is a* $\operatorname{rk} 1$ *MCM B-module M' such that*

$$\mathrm{Def}_M^B \cong \mathrm{Def}_{M'}^B.$$

Proof. To ψ there is a $\phi : F \to G$ such that (ψ, ϕ) is a matrix factorization of f. Since $\det\psi \cdot \det\phi = f^g$ and $\operatorname{rk} M = g-1$, we may assume $\det\phi = f$. Hence $M' := \operatorname{coker}\phi$ is an MCM B-module of $\operatorname{rk} 1$. Now the following isomorphisms, which hold for any matrix factorization of a regular f, concludes the argument:

$$\mathrm{Def}_M^B \cong \mathrm{Def}_{(\psi,\phi)}^A \cong \mathrm{Def}_{(\phi,\psi)}^A \cong \mathrm{Def}_{M'}^B. \tag{12}$$

□

Proof of Theorem 2. By Proposition 1, $\mathrm{Def}_M^B \cong \mathrm{Def}_{(\phi|\det\phi)}^A$ and $\mathrm{Def}_{(\phi|\det\phi)}^A$ has, by the proof of Proposition 1, the obstruction induced by $\det\tilde{\phi} - f$, where $\tilde{\phi}$ is a lifting of ϕ_S (with $\det\phi_S = f$) to R in a small lifting situation (Definition 1). This implies that the obstruction is always in the image of B in (9), hence maps to zero in $H_2(\mathcal{S})$ and is therefore naturally found in A/I_{g-1} by (11). The tangent space is given by Corollary 1, which gives the upper estimate of the dimension of the hull since the obstruction power series has minimal degree greater than or equal to two by construction. At 'worst' they give a regular sequence with a maximal number of elements; this gives the lower estimate. The construction of the obstruction map is done as in [Ile01, Lau86]. The rest follows from Corollaries 2 and 4 and [GN72, Théorèm 1]. □

Example 3. If $\phi = (x_{ij})$ is the generic $g \times g$-matrix, then $M = \operatorname{coker}\phi$ is a rigid $\operatorname{rk} 1$ MCM $B = k[x_{ij}]/(\det\phi)$-module for all $g \geqslant 1$. By (12), $M' = \operatorname{coker}(\phi^{\mathrm{a}})$ is also a rigid MCM module, but of B-rank $g-1$. Let $g = 2$. Then the equation $\operatorname{tr}((a_{ij})\phi^{\mathrm{a}}) = a_{11}x_{22} - a_{12}x_{21} - a_{21}x_{12} + a_{22}x_{11} = 0$ has solutions generated by Koszul relations; they are all coboundaries. However, if we set $x_{12} = x_{21}$, then $a_{12} + a_{21} = 0$ is a non-trivial solution and we obtain $\operatorname{Ext}_B^1(M, M) \cong k$, generated by

$$\xi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We try to lift the universal extension $M_1 \in \mathrm{Def}_M^B(k[u]/(u^2))$ given by $\phi + \xi u$ to $k[u]/(u^3)$ and calculate the obstruction given by $\det(\phi + \xi u) = \det\phi + \det(\xi)u^2 = f + u^2$ where $f = \det\phi$. The cup product $\xi \cup \xi = d_2(\mathrm{id}_M)$ is non-zero, hence the hull is $H = k[u]/(u^2)$ as there can be no further liftings.

Example 4. Let $\phi = (x_{ij})$ be the 3×3 generic matrix, with the restriction that $x_{ii} = 0$ for $i = 1, 2, 3$. Then $\det\phi = x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32}$ and $\operatorname{tr}((a_{ij})\phi^{\mathrm{a}}) = -a_{11}x_{23}x_{32} + a_{12}x_{23}x_{31} + a_{13}x_{21}x_{32} + a_{21}x_{23}x_{32} - a_{22}x_{13}x_{31} + a_{23}x_{12}x_{31} + a_{31}x_{12}x_{23} + a_{32}x_{13}x_{21} - a_{33}x_{12}x_{21}$, but we do not get finite-dimensional tangent and obstruction spaces. Set $x_{13} = x_{21} = x_{32} = y$. Then $f = \det\phi = x_{12}x_{23}x_{31} + y^3$ and we get a two-dimensional tangent space for the *graded* deformation functor given by the relation $a_{13} + a_{21} + a_{32} = 0$. Deform ϕ to

$$\tilde{\phi} = \begin{pmatrix} 0 & x_{12} & y+u+v \\ y-u & 0 & x_{23} \\ x_{31} & y-v & 0 \end{pmatrix}.$$

Then $\det\tilde{\phi} = f - y(u^2 - uv + v^2) + (u+v)uv$ which gives the second-order obstruction polynomial $g = u^2 - uv + v^2$ 'carried' by the class of $-y$ in the obstruction space $A/\mathfrak{m}^2$, hence

$H_2 = k[u,v]/((g) + (u\,,v)^3)$. The obstruction to lift M_2 along $\pi : k[u,v]/((g)(u,v) + (u,v)^4) \twoheadrightarrow H_2$ is $-y{\cdot}g + 1{\cdot}h$ where $h = (u+v)uv$. The cohomology classes $-y$ and 1 are independent over k and there are no further obstructions. Hence $H \cong k[[u,v]]/(u^2 - uv + v^2, (u+v)uv)$.

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