Glasgow Math. J. **52** (2010) 271–302. © Glasgow Mathematical Journal Trust 2009.
doi:10.1017/S0017089509990346.

ON THE MULTIPLICITY OF SOLUTIONS FOR NON-LINEAR PERIODIC PROBLEMS WITH THE NON-LINEARITY CROSSING SEVERAL EIGENVALUES

SOPHIA TH. KYRITSI

Department of Mathematics, Hellenic Naval Academy, Pireaus 18539, Greece
e-mail: skyrits@math.ntua.gr

and NIKOLAOS S. PAPAGEORGIOU

Department of Mathematics, National Technical University,
Zografou Campus, Athens 15780, Greece
e-mail: npapg@math.ntua.gr

(Received 14 November 2008; accepted 21 September 2009; first published online 25 November 2009)

Abstract. In this paper we consider a non-linear periodic problem driven by the scalar p-Laplacian and with a non-smooth potential. We assume that the multi-valued right-hand-side non-linearity exhibits an asymmetric behaviour at $\pm\infty$ and crosses a finite number of eigenvalues as we move from $-\infty$ to $+\infty$. Using a variational approach based on the non-smooth critical-point theory, we show that the problem has at least two non-trivial solutions, one of which has constant sign. For the semi-linear ($p = 2$), smooth problem, using Morse theory, we show that the problem has at least three non-trivial solutions, again one with constant sign.

2002 *Mathematics Subject Classification.* 34B15, 34C25.

1. Introduction. In this paper, we consider the following non-linear periodic problem driven by the scalar p-Laplacian differential operator and having a non-smooth potential (hemi-variational inequality):

$$\left\{ \begin{array}{l} -\big(|x'(t)|^{p-2}x'(t)\big)' + \beta|x(t)|^{p-2}x(t) \in \partial j\big(t, x(t)\big) \quad \text{a.e. on } T = [0, b], \\ x(0) = x(b),\ \ x'(0) = x'(b),\ \ 1 < p < \infty,\ \ \beta > 0. \end{array} \right\} \tag{1}$$

Here $j : T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function, and for almost all $t \in T$, $x \longrightarrow j(t, x)$ is locally Lipschitz and in general non-smooth. By $\partial j(t, x)$ we denote the *generalised (Clarke) subdifferential* of the locally Lipschitz function $x \longrightarrow j(t, x)$ (see Section 2).

The goal of this paper is to establish a multiplicity result, when the right-hand-side non-linearity (which in our case is multi-valued because of the non-smoothness of the potential $x \longrightarrow j(t, x)$) exhibits an asymmetric behaviour at $\pm\infty$ and crosses a finite number of eigenvalues, as we move from $-\infty$ to $+\infty$ (*crossing or jumping non-linearity*).

Multiplicity results for the scalar periodic p-Laplacian were proved by Aizicovici, Papageorgiou and Staicu [**2**], del Pino, Manásevich and Murúa [**10**], Gasiński and Papageorgiou [**13**], Papageorgiou and Papageorgiou [**22**] and Yang [**25**]. In [**2**], the authors used degree-theoretic methods based on the degree map for certain multi-valued perturbations of $(S)_+$-operators with the non-linearity able to cross only the zero (principal) eigenvalue. A degree-theoretic approach has also been used in [**10**], based on the Leray–Schauder degree map, which is coupled with the use of time maps. In this case the potential is smooth (i.e. $j(t,\cdot) \in C^1(\mathbb{R})$), and the hypotheses on the non-linearity $f(t,x) = \partial j(t,x)$ require partial interaction with the Fučik spectrum of the scalar p-Laplacian. A smooth potential is also assumed in [**22**], and the approach there is variational, based on the second deformation theorem. In the work of Yang [**25**], the right-hand-side non-linearity depends also on x', and so the problem is non-variational. For this reason, the author assumed the existence of an ordered pair of upper and lower solutions, and his method of proof used degree theory, based on Mawhin's coincidence degree (see Mawhin [**19**]). Finally in [**13**] the potential was non-smooth, and the authors used a variational approach based on a non-smooth version of the local linking theorem, owing to Kandilakis, Kourogenis and Papageorgiou [**16**].

Our approach here is variational based on the non-smooth critical-point theory (see Gasiński and Papageorgiou [**14**]), together with the spectrum of a weighted periodic eigenvalue problem for the scalar p-Laplacian, as developed recently by Zhang [**26**]. Although we are dealing with a jumping non-linearity, we are not using the Fučik spectrum of the scalar p-Laplacian. This is in contrast with the work of del Pino, Manásevich and Murúa [**10**]. In Section 5, we consider the semi-linear (i.e. $p = 2$), smooth problem using Morse theory.

2. Mathematical background. As we mentioned in the 'Introduction', our approach is variational based on the non-smooth critical-point theory, which uses the subdifferential theory of locally Lipschitz functions. For easy reference, first we recall some basic definitions and facts from this theory. Details can be found in Clarke [**8**].

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given a locally Lipschitz function $\varphi : X \longrightarrow \mathbb{R}$, the generalised directional derivative $\varphi^0(x;h)$ of φ at $x \in X$, in the direction $h \in X$, is defined by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda},$$

whereas the generalised subdifferential of φ at $x \in X$ is defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \le \varphi^0(x;h) \text{ for all } h \in X\}.$$

If $\varphi : X \longrightarrow \mathbb{R}$ is continuous convex, then φ is locally Lipschitz, and the generalised subdifferential coincides with the subdifferential in the sense of convex analysis, defined by

$$\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \le \varphi(x+h) - \varphi(x) \text{ for all } h \in X\}.$$

Also, if $\varphi \in C^1(X)$, then φ is locally Lipschitz and $\partial\varphi(x) = \{\varphi'(x)\}$ for all $x \in X$.

We say that $x \in X$ is a *a* critical point of φ if $0 \in \partial\varphi(x)$. It is easy to see that a local extremum of φ (i.e. a local minimum or a local maximum) is a critical point.

Using the subdifferential theory for locally Lipscitz functions, we can extend the critical-point theory to non-smooth locally Lipschitz functions. This started with the

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

work of Chang [**5**]. A detailed exposition of the non-smooth critical-point theory can be found in the book by Gasiński and Papageorgiou [**14**].

We say that a locally Lipschitz function $\varphi : X \longrightarrow \mathbb{R}$ satisfies the *Palais–Smale condition at level* $c \in \mathbb{R}$ (the PS_c-condition for short) if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\varphi(x_n) \longrightarrow c \text{ and } m(x_n) = \inf\big[\|x^*\| : x^* \in \partial\varphi(x_n)\big] \longrightarrow 0 \text{ as } n \to \infty$$

has a strongly convergent subsequence. We say that φ satisfies the Palais–Smale condition (PS-condition for short) if it satisfies the PS_c-condition for every $c \in \mathbb{R}$.

Using this notion, we can state a non-smooth version of the well-known 'mountain-pass theorem'.

THEOREM 2.1. **If** *X is a Banach space, $\varphi : X \longrightarrow \mathbb{R}$ is a locally Lipschitz function, $x_0, x_1 \in X$, $r > 0$ satisfy*

$$\max\big\{\varphi(x_0), \varphi(x_1)\big\} < \inf\big[\varphi(x) : \|x - x_0\| = r\big] = \eta, \ \ \|x_1 - x_0\| > r,$$

$\Gamma = \big\{\gamma \in C([0, 1], X) : \gamma(0) = x_0, \gamma(1) = x_1\big\}$, $c = \inf\limits_{\gamma\in\Gamma} \max\limits_{t\in[0,1]} \varphi\big(\gamma(t)\big)$ *and φ satisfies the PS_c–condition,* **then** *$c \geq \eta$ and c is a critical value of φ.*

Given a locally Lipschitz function $\varphi : X \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$, we introduce the following sets:

$\dot{\varphi}^c = \{x \in X : \varphi(x) < c\}$ (the strict sub-level set of φ at $c \in \mathbb{R}$),
$K = \{x \in X : 0 \in \partial\varphi(x)\}$ (the critical set of φ) and
$K_c = \{x \in K : \varphi(x) = c\}$ (the critical set of φ at the level $c \in \mathbb{R}$).

The next theorem is a non-smooth version of the so-called second deformation theorem (see Chang [**6**], p. 23, and Gasiński and Papageorgiou [**15**], p. 628), and it is due to Corvellec [**9**]. In fact, the result of Corvellec is formulated in the more general context of metric spaces and continuous functions, using the so-called weak slope. However, for our purposes, the following particular version of the result suffices.

THEOREM 2.2. **If** *X is a Banach space, $\varphi : X \longrightarrow \mathbb{R}$ is a locally Lipschitz function, $-\infty < a < b \leq +\infty$, φ has no critical points in $\varphi^{-1}(a, b)$, it satisfies the PS_c-condition for every $c \in [a, b)$ and K_a is finite and consists of only local minima of φ,* **then** *there exists a continuous deformation $h : [0, 1] \times \dot{\varphi}^b \longrightarrow \dot{\varphi}^b$ such that*

(a) $h(t, \cdot)\big|_{K_a} = \mathrm{id}\big|_{K_a}$ *for all* $t \in [0, 1]$*;*
(b) $h(1, \dot{\varphi}^b) \subseteq \dot{\varphi}^a \cup K_a$*;*
(c) $\varphi\big(h(t, x)\big) \leq \varphi(x)$ *for all* $t \in [0, 1]$ *and all* $x \in \dot{\varphi}^b$.

REMARK 2.3. In particular, the set $\dot{\varphi}^a \cup K_a$ is a weak deformation retract of $\dot{\varphi}^b$.

In the analysis of problem (1), we shall use the spectrum of a certain weighted periodic eigenvalue problem for the scalar p-Laplacian. First, let us recall the standard spectrum of the negative scalar p-Laplacian with periodic boundary conditions (see Drabek and Manásevich [**12**], Manásevich and Mawhin [**18**] and Gasiński and Papageorgiou [**15**]).

So, we consider the following non-linear eigenvalue problem:

$$\left\{ \begin{array}{l} -\left(|x'(t)|^{p-2}x'(t)\right)' = \lambda |x(t)|^{p-2}x(t) \quad \text{a.e. on } T = [0, b], \\ x(0) = x(b), \ x'(0) = x'(b), \ 1 < p < \infty. \end{array} \right\} \tag{2}$$

A number $\lambda \in \mathbb{R}$, for which problem (2) has a non-trivial solution $x \in C^1(T)$, is said to be eigenvalue, and x is a corresponding eigenfunction. Clearly, a necessary condition for $\lambda \in \mathbb{R}$ to be an eigenvalue is that $\lambda \geq 0$. In fact, $\lambda = 0$ is the smallest eigenvalue, with the corresponding eigenspace $\mathbb{R}$ (the space of constant functions). Moreover, if $u \in C^1(T)$ is an eigenfunction corresponding to an eigenvalue $\lambda > 0$, then u necessarily changes sign (i.e. u is nodal). Also, in this case $u(t) \neq 0$ a.e. on T, and in fact u has finitely many zeros.

Let $\pi_p = \frac{2\pi(p-1)^{\frac{1}{p}}}{p \sin \frac{\pi}{p}}$. Note that if $p = 2$, then $\pi_2 = \pi$. The sequence

$$\left\{ \mu_{2n} = \left(\frac{2n\pi_p}{b} \right)^p \right\}_{n \geq 0},$$

is the set of all eigenvalues for problem (2). If $p = 2$, then we recover the classical spectrum of the negative scalar Laplacian, with periodic boundary conditions, namely

$$\left\{ \mu_{2n} = \left(\frac{2n\pi}{b} \right)^2 \right\}_{n \geq 0}.$$

Another spectrum that we shall use in this paper is the one studied recently by Zhang [**26**]. It concerns the following weighted eigenvalue problem:

$$\left\{ \begin{array}{l} -\left(|x'(t)|^{p-2}x'(t)\right)' = (\lambda + h(t))|x(t)|^{p-2}x(t) \quad \text{a.e. on } T, \\ x(0) = x(b), \ x'(0) = x'(b), \ 1 < p < \infty, \ h \in L^1(T). \end{array} \right\} \tag{3}$$

As before, $\lambda \in \mathbb{R}$ is an eigenvalue if (3) admits a non-trivial solution. Zhang [**26**] showed that (3) has a double sequence of eigenvalues $\left\{\underline{\lambda}_{2n}(h)\right\}_{n\geq 1}$ and $\left\{\overline{\lambda}_{2n}(h)\right\}_{n\geq 0}$, which satisfy

$$\begin{array}{l} -\infty < \overline{\lambda}_0(h) < \underline{\lambda}_2(h) \leq \overline{\lambda}_2(h) \ldots < \underline{\lambda}_{2n}(h) \leq \overline{\lambda}_{2n}(h) < \ldots \text{ and} \\ \underline{\lambda}_{2n}(h) \longrightarrow +\infty \text{ as } n \to \infty. \end{array}$$

If $p = 2$ (linear eigenvalue problem), then the above two sequences are all the eigenvalues of (3) (see Magnus and Winkler [**17**]). If $p \neq 2$ (non-linear eigenvalue problem), then we do not know if this is the case. Nevertheless, we can have the following useful property of this spectrum (see Aizicovici, Papageorgiou and Staicu [**1**]).

PROPOSITION 2.4. **If** $h_1, h_2 \in L^\infty(Z)_+$ *satisfy*

$$\begin{array}{l} \mu_{2n} \leq h_1(t) \leq h_2(t) \leq \mu_{2n+2} \quad \textit{a.e. on } T, \textit{ for some integer } n \geq 0, \textit{ and} \\ h_1 \neq \mu_{2n}, \ h_2 \neq \mu_{2n+2}, \end{array}$$

then *all the eigenvalues of* (3) *are non-zero and do not have zero as a limit point.*

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

In Section 5, for the semi-linear (i.e. $p = 2$), smooth (i.e. $j(t, \cdot) \in C^2(\mathbb{R})$) problem, using Morse theory, we produce additional non-trivial solutions. So, let us briefly recall some basic definitions and facts from Morse theory, which will be used in Section 5.

So, let H be a Hilbert space and $\varphi \in C^1(H)$ a function which also satisfies the PS-condition. Let $x_0 \in H$ be an isolated critical point of φ and $c_0 = \varphi(x_0)$. The *critical groups* (over $\mathbb{Z}$) of φ at x_0 are defined by

$$C_k(\varphi, x_0) = H_k\big(\varphi^{c_0} \cap U, \varphi^{c_0} \cap U \setminus \{x_0\}\big) \quad \text{for all } \ k \geq 0,$$

where U is a neighbourhood of x_0; H_k is the $\mathrm{k}^{\underline{\mathrm{th}}}$-singular relative homology group, with integer coefficients; and for every $\eta \in \mathbb{R}$, $\varphi^\eta = \{x \in X : \varphi(x) \leq \eta\}$ (the sub-level set of φ at $\eta \in \mathbb{R}$). From the excision property of singular homology theory, we see that this definition is independent of the neighbourhood U of x_0. Suppose that $-\infty < \inf \varphi(K)$, and let $c < \inf \varphi(K)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(H, \varphi^c) \quad \text{for all } \ k \geq 0$$

(see Bartsch and Li [**4**]). From the deformation lemma, which is valid because we have assumed that φ satisfies the PS-condition, we infer that this definition is independent of c. If the set K is finite, then the *Morse-type numbers* of φ are defined by

$$M_k = \sum_{x \in K} \operatorname{rank} C_k(\varphi, x)$$

and the *Betti-type numbers* of φ are defined by

$$\beta_k = \operatorname{rank} C_k(\varphi, \infty) \quad \text{for all } \ k \geq 0.$$

From the Morse theory (see Bartsch and Li [**4**], Chang [**6**] and Mawhin and Willem [**21**]), we have

$$\sum_{k=0}^{m} (-1)^{m-k} M_k \geq \sum_{k=0}^{m} (-1)^{m-k} \beta_k \text{ and} \tag{4}$$

$$\sum_{k \geq 0} (-1)^k M_k = \sum_{k \geq 0} (-1)^k \beta_k. \tag{5}$$

From (4), we infer that $\beta_k \leq M_k$ for all $k \geq 0$. Therefore, if $\beta_k \neq 0$ for some $k \geq 0$, then φ must have a critical point $x \in H$, and the critical group $C_k(\varphi, x)$ is non-trivial. Equality (5) is known as the *Poincaré–Hopf formula*. If $K = \{x_0\}$, then $C_k(\varphi, \infty) = C_k(\varphi, x_0)$ for all $k \geq 0$.

Finally, if $A : X \longrightarrow X^*$ is a map, we say that A is *of type* $(S)_+$ when for every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X and

$$\limsup_{n \to \infty} \langle A(x_n), x_n - x \rangle \leq 0,$$

one has $x_n \longrightarrow x$ in X.

In the sequel, we use the notation $r^\pm = \max\{\pm r, 0\}$ for all $r \in \mathbb{R}$.

3. Auxiliary results. In this section we prove some general auxiliary results, which will be used in the sequel. Some of them are of independent interest and can be useful in the study of the scalar periodic p-Laplacian, in different settings.

In this paper, we shall use the following two spaces:

$$W^{1,p}_{\text{per}}(0,b) = \{x \in W^{1,p}(0,b) : \ x(0) = x(b)\} \text{ and}$$
$$\widehat{C}^1(T) = \{x \in C^1(T) : \ x(0) = x(b)\} = C^1(T) \cap W^{1,p}_{\text{per}}(0,b).$$

Both are ordered Banach spaces with positive cones given by

$$W_+ = \left\{x \in W^{1,p}_{\text{per}}(0,b) : \ x(t) \geq 0 \quad \text{for all} \quad t \in T\right\} \text{ and}$$
$$\widehat{C}_+ = \left\{x \in \widehat{C}^1(T) : \ x(t) \geq 0 \quad \text{for all} \quad t \in T\right\}.$$

We know that $\text{int}\widehat{C}_+ \neq \emptyset$ and in fact

$$\text{int}\widehat{C}_+ = \{x \in \widehat{C}_+ : x(t) > 0 \quad \text{for all} \quad t \in T\}.$$

LEMMA 3.1. **If** $\vartheta \in L^\infty(T)_+$, $\vartheta(t) \leq \beta$ *a.e. on* T *and* $\vartheta \neq \beta$, **then** *there exists* $\xi_0 > 0$ *such that*

$$\psi(x) = \|x'\|_p^p + \beta\|x\|_p^p - \int_0^b \vartheta(t)|x(t)|^p dt \geq \xi_0\|x\|_p^p \quad \textit{for all} \quad x \in W^{1,p}_{per}(0,b).$$

Proof. Clearly $\psi \geq 0$. We argue by contradiction. So, suppose that the lemma is not true. Since the functional ψ is p-homogeneous, we can find a sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ such that $\|x_n\| = 1$ and $\psi(x_n) \downarrow 0$. By passing to a suitable subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } x_n \longrightarrow x \text{ in } C(T) \text{ as } n \to \infty.$$

We have

$$\|x'\|_p^p \leq \liminf_{n\to\infty} \|x_n'\|_p^p \text{ and}$$
$$\|x_n\|_p^p \longrightarrow \|x\|_p^p, \int_0^b \vartheta|x_n|^p \mathrm{d}t \longrightarrow \int_0^b \vartheta|x|^p \mathrm{d}t \text{ as } n \to \infty.$$

Therefore, in the limit as $n \to \infty$, we obtain

$$\|x'\|_p^p \leq \int_0^b \big(\vartheta(t) - \beta\big)|x(t)|^p \mathrm{d}t \leq 0$$
$$\Rightarrow x \equiv c \in \mathbb{R}. \tag{6}$$

If $c = 0$, then $\|x_n'\|_p \longrightarrow 0$, and so $x_n \longrightarrow 0$ in $W^{1,p}_{\text{per}}(0,b)$, a contradiction to the fact that $\|x_n\| = 1$ for all $n \geq 1$.

Hence $c \neq 0$. Then from (6), we have

$$0 \leq \|x'\|_p^p \leq |c|^p \int_0^b \big(\vartheta(t) - \beta\big)\mathrm{d}t < 0,$$

a contradiction. This proves the lemma. □

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

We consider the non-linear map $A\colon W^{1,p}_{\text{per}}(0,b) \longrightarrow W^{1,p}_{\text{per}}(0,b)^*$ corresponding to the periodic scalar p-Laplacian and defined by

$$\langle A(x), y\rangle = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)\mathrm{d}t \quad \text{for all } \ x, y \in W^{1,p}_{\text{per}}(0,b). \tag{7}$$

LEMMA 3.2. *The map $A : W^{1,p}_{per}(0,b) \longrightarrow W^{1,p}_{per}(0,b)^*$ defined by* (7) *is bounded, continuous and of type* $(S)_+$.

Proof. Clearly A is bounded continuous. Let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ such that $x_n \xrightarrow{w} x$ and

$$\limsup_{n\to\infty}\,\langle A(x_n), x_n - x\rangle \leq 0. \tag{8}$$

Evidently A is monotone (in fact strictly monotone and strongly monotone if $p \geq 2$), and so it is maximal monotone. But a maximal monotone operator is generalised pseudo-monotone (see Gasiński and Papageorgiou [**15**], p. 330). So, from (8) it follows that

$$\|x_n'\|_p^p = \langle A(x_n), x_n\rangle \longrightarrow \langle A(x), x\rangle = \|x'\|_p^p.$$

Since $x_n' \xrightarrow{w} x'$ in $L^p(T)$ and the space $L^p(T)$ is uniformly convex, from the Kadec–Klee property, we have $x_n' \longrightarrow x'$ in $L^p(T)$; hence $x_n \longrightarrow x$ in $W^{1,p}_{\text{per}}(0,b)$. This proves that A is an $(S)_+$-map. □

The next auxiliary result permits us to relate C^1 and Sobolev local minimisers for a large class of locally Lipschitz functionals.

So, we consider a potential function $j_0(t,x)$, which satisfies the following hypotheses:

<u>**$\mathbf{H_0}$:**</u> $j_0\colon T\times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that
  (i) for all $x \in \mathbb{R}$, $t \longrightarrow j_0(t,x)$ is measurable;
  (ii) for almost all $t \in T$, $x \longrightarrow j_0(t,x)$ is locally Lipschitz;
  (iii) for every $r > 0$, there exists $a_r \in L^1(T)_+$ such that

$$|u| \leq a_r(t),$$

for a.a. $t \in T$, all $|x| \leq r$ and all $u \in \partial j_0(t,x)$.

We consider the functional $\varphi_0 : W^{1,p}_{\text{per}}(0,b) \longrightarrow \mathbb{R}$ defined by

$$\varphi_0(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j_0\big(t, x(t)\big)\mathrm{d}t.$$

It is easy to see that φ_0 is Lipschitz continuous on bounded sets; hence it is locally Lipschitz (see Clarke [**8**], p. 83).

PROPOSITION 3.3. **If** *Hypotheses* $\mathbf{H}_0$ *hold and x_0 is a local $\widehat{C}^1(T)$-minimiser of φ_0, i.e. there exists $r_0 > 0$ such that*

$$\varphi_0(x_0) \leq \varphi(x_0 + y) \quad \textit{for all } \ y \in \widehat{C}^1(T),\ \|y\|_{\widehat{C}^1(T)} \leq r_0,$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

then $x_0 \in \widehat{C}^1(T)$ *and it is also a local* $W^{1,p}_{per}(0,b)$*-minimiser of* φ_0, *i.e. there exists* $r_1 > 0$ *such that*

$$\varphi_0(x_0) \le \varphi(x_0 + y) \quad \textit{for all} \;\; y \in W^{1,p}_{per}(0,b),\ \|y\|_{W^{1,p}_{per}(0,b)} \le r_1.$$

Proof. Let $h \in \widehat{C}^1(T)$. If $\lambda > 0$ is small, then by hypothesis

$$\begin{aligned} &0 \le \varphi_0(x_0 + \lambda h) - \varphi_0(x_0),\\ \Rightarrow\ &0 \le \varphi_0'(x_0; h). \end{aligned} \tag{9}$$

Since $\widehat{C}^1(T)$ is dense in $W^{1,p}_{\rm per}(0,b)$ and $\varphi_0'(x_0;\cdot)$ is continuous, we infer that

$$\begin{aligned} &0 \le \varphi_0'(x_0; h) \quad \text{for all} \;\; h \in W^{1,p}_{\rm per}(0,b),\\ &\Rightarrow 0 \in \partial\varphi_0(x_0),\\ &\Rightarrow A(x_0) = u_0, \end{aligned} \tag{10}$$

where $u_0 \in L^1(T)$, $u_0(t) \in \partial j_0\big(t, x_0(t)\big)$ a.e. on T (see Clarke [**8**], p. 83). From the representation theorem for the elements of $W^{-1,p'}(0,b) = W^{1,p}_0(0,b)^*$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ (see for example Gasiński and Papageorgiou [**15**], p. 212), we have

$$\big(|x_0'|^{p-2}x_0'\big)' \in W^{-1,p'}(0,b).$$

In what follows, by $\langle\cdot,\cdot\rangle_0$ we denote the duality brackets for the pair $\big(W^{-1,p'}(0,b), W^{1,p}_0(0,b)\big)$. From the definition of distributional derivative, for every $v \in C^1_c(0,b)$, we have

$$\begin{aligned} &-\Big\langle\big(|x_0'|^{p-2}x_0'\big)', v\Big\rangle_0 = \int_0^b |x_0'|^{p-2}x_0'v'\,\mathrm{d}t = \langle A(x_0), v\rangle\,,\\ \Rightarrow\ &-\Big\langle\big(|x_0'|^{p-2}x_0'\big)', v\Big\rangle_0 = \int_0^b u_0 v\,\mathrm{d}t \quad \text{for all} \;\; v \in C^1_c(0,b) \quad (\text{see (10)}). \end{aligned} \tag{11}$$

Since $C^1_c(0,b)$ is dense in $W^{1,p}_0(0,b)$, we infer that (11) is valid for all $v \in W^{1,p}_0(0,b)$ and so

$$-\big(|x_0'(t)|^{p-2}x_0'(t)\big)' = u_0(t) \quad \text{a.e. on} \;\; T,\ x_0(0) = x_0(b). \tag{12}$$

From (12) we see that

$$|x_0'|^{p-2}x_0' \in W^{1,1}(0,b) \subseteq C(T).$$

Recall that the map $\gamma_p : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\gamma_p(r) = \begin{cases} 0 & \text{if} \;\; r = 0,\\ |r|^{p-2}r & \text{if} \;\; r \ne 0, \end{cases}$$

is a homeomorphism. Note that $\gamma_p^{-1}\big(|x_0'(t)|^{p-2}x_0'(t)\big) = x_0'(t)$ for all $t \in T$. Therefore $x_0' \in C(T)$, and so it follows that $x_0 \in \widehat{C}^1(T)$.

Now, suppose that x_0 is not a local $W^{1,p}_{\rm per}(0,b)$-minimiser of φ_0. Note that φ_0 is sequentially weakly lower semi-continuous, and for every $r>0$ let $\overline{B}_r(x_0)=\{x\in W^{1,p}_{\rm per}(0,b):\|x-x_0\|\le r\}$. Then, by the Weierstrass theorem, φ_0 attains its infimum on $\overline{B}_r(x_0)$. Therefore, for every $n\ge 1$, we can find $v_n\in W^{1,p}_{\rm per}(0,b)$ such that

$$\|x_0-v_n\|\le\frac{1}{n}\quad\text{and}\quad\varphi_0(v_n)=\min\Big[\varphi_0(x):x\in\overline{B}_{\frac{1}{n}}(x_0)\Big]<\varphi_0(x_0).\tag{13}$$

Let $\vartheta:W^{1,p}_{\rm per}(0,b)\longrightarrow\mathbb{R}_+$ be the norm (constraint) functional defined by

$$\vartheta(v)=\frac{1}{p}\|v\|^p=\frac{1}{p}\big(\|v\|_p^p+\|v'\|_p^p\big)\quad\text{for all }\ v\in W^{1,p}_{\rm per}(0,b).$$

By virtue of the Lagrange multiplier rule of Clarke [7], for every $n\ge 1$, we can find $\lambda_n<0$ such that

$$\lambda_n\vartheta'(v_n-x_0)\in\partial\varphi_0(v_n)\quad\text{for all }\ n\ge 1.\tag{14}$$

Let $K_p:L^p(T)\longrightarrow L^{p'}(T)$ be the bounded continuous map defined by $K_p(x)(\cdot)=|x(\cdot)|^{p-2}x(\cdot)$ for all $x\in L^p(T)$. Owing to the compact embedding of $W^{1,p}_{\rm per}(0,b)$ into $L^p(T)$, we see that $K_p\big|_{W^{1,p}_{\rm per}(0,b)}$ is compact. Note that $\vartheta'(x)=A(x)+K_p(x)$ for all $x\in W^{1,p}_{\rm per}(0,b)$. From (14), we have

$$A(v_n)-u_n-\lambda_nA(v_n-x_0)-\lambda_nK_p(v_n-x_0)=0\tag{15}$$

with $u_n\in L^1(T)$, $u_n(t)\in\partial j_0\big(t,v_n(t)\big)$ a.a. on T. We consider two different cases:

Case I: The sequence of Lagrange multipliers $\{\lambda_n\}_{n\ge1}$ is bounded.

Define

$$\gamma_n(t,r)=|r|^{p-2}r-\lambda_n|r-x_0'(t)|^{p-2}\big(r-x_0'(t)\big)-\lambda_n|x_0'(t)|^{p-2}x_0'(t)$$
$$\text{for all }\ (t,r)\in T\times\mathbb{R},\ \text{ all n}\ge 1.$$

The function γ_n exhibits the following properties:

- $\gamma_n\in C(T\times\mathbb{R})$ for all $n\ge 1$;
- for all $t\in T$ and all $n\ge 1$, $r\longrightarrow\gamma_n(t,r)$ is strictly monotone (strongly monotone if $p\ge 2$);
- there exist $c_1,c_2>0$ such that $|\gamma_n(t,r)|\le c_1+c_2|r|^{p-1}$ for all $(t,r)\in T\times\mathbb{R}$;
- for all $t\in T$ and all $n\ge 1$, we have $\gamma_n(t,0)=0$;
- there exist $c_3,c_4>0$ such that $\gamma_n(t,r)r\ge c_3|r|^p-c_4$ for all $(t,r)\in T\times\mathbb{R}$ and all $n\ge 1$.

Note that $\gamma_n\big(\cdot,v_n'(\cdot)\big)\in L^{p'}(T)$ and

$$\langle A(v_n)-\lambda_nA(v_n-x_0)-\lambda_nA(x_0),y\rangle=\int_0^b\gamma_n\big(t,v_n'(t)\big)y'(t)\mathrm{d}t$$
$$\text{for all }\ y\in W^{1,p}_{\rm per}(0,b).\tag{16}$$

From (10), (15) and (16), as before, we obtain

$$-\Big(\gamma_n\big(t,v_n'(t)\big)\Big)'=u_n(t)+\lambda_n|(v_n-x_0)(t)|^{p-2}(v_n-x_0)(t)+\lambda_nu_0(t)\quad\text{a.e. on }\ T.\tag{17}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

Recall that $v_n \in \overline{B}_{\frac{1}{n}}(x_0)$. Hence, by virtue of Hypothesis $\mathbf{H_0}$(iii) and (17), we have that

$$\begin{aligned} &\left\{\left(\gamma_n\left(\cdot, v_n'(\cdot)\right)\right)'\right\}_{n\geq 1} \subseteq L^{p'}(T) \ \text{ is bounded,}\\ \Rightarrow\ &\left\{\left(\gamma_n\left(\cdot, v_n'(\cdot)\right)\right)'\right\}_{n\geq 1} \subseteq W^{1,p'}(0,b) \ \text{ is bounded,}\\ \Rightarrow\ &\left\{\left(\gamma_n\left(\cdot, v_n'(\cdot)\right)\right)'\right\}_{n\geq 1} \subseteq C(T) \ \text{ is relatively compact} \end{aligned}$$

(we know that $W^{1,p'}(0,b)$ is embedded compactly in $C(T)$).

Fix $n \geq 1$ and $t \in T$. Then because of the strict monotonicity of the function $r \longrightarrow \gamma_n(t,r)$, the inverse map $\gamma_{n,t}^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ is well defined and single valued, and we have

$$\gamma_{n,t}^{-1}(y) = v \ \text{ if and only if } \gamma_n(t,v) = y \quad \text{for all } \ v, y \in \mathbb{R}.$$

Clearly $\gamma_{n,t}^{-1}$ is continuous (see the properties of γ_n stated above). We can define $\sigma_n : C(T) \longrightarrow C(T)$ by

$$\sigma_n(y)(t) = \gamma_{n,t}^{-1}\left(y(t)\right) \quad \text{for all } \ y \in C(T) \text{ and all } t \in T.$$

We have

$$\gamma_n\left(t, \sigma_n(y)(t)\right) = y(t) \quad \text{for all } \ n \geq 1, \ \text{all } t \in T \text{ and all } y \in C(T). \tag{18}$$

Claim: If $\{y_n\}_{n\geq 1} \subseteq C(T)$ is relatively compact, then $\{\sigma_n(y_n)\}_{n\geq 1} \subseteq C(T)$ is relatively compact too.

To this end, from the properties of γ_n stated earlier and (18), we have

$$\begin{aligned} &c_3|\sigma_n(y_n)(t)|^p - c_2\\ \leq\ &\gamma_n\left(t, \sigma_n(y_n)(t)\right)\sigma_n(y_n)(t)\\ \leq\ &|y_n(t)|\,|\sigma_n(y_n)(t)|,\\ \Rightarrow\ &c_3\|\sigma_n(y_n)\|_\infty^p \leq c_2 + c_4\|\sigma_n(y_n)\|_\infty \ \text{ for some } \ c_4 > 0, \ \text{all } n \geq 1,\\ \Rightarrow\ &\{\sigma_n(y_n)\}_{n\geq 1} \subseteq C(T) \ \text{ is bounded } \quad (\text{since } \ p > 1). \end{aligned}$$

Next, let $s, t \in T$. We have

$$\begin{aligned} &\Big(\gamma_n\left(t, \sigma_n(y_n)(t)\right) - \gamma_n\left(s, \sigma_n(y_n)(s)\right)\Big)\left(\sigma_n(y_n)(t) - \sigma_n(y_n)(s)\right)\\ &\quad = \Big(\gamma_n\left(t, \sigma_n(y_n)(t)\right) - \gamma_n\left(t, \sigma_n(y_n)(s)\right)\Big)\left(\sigma_n(y_n) - \sigma_n(y_n)(s)\right)(t)\\ &\qquad + \Big(\gamma_n\left(t, \sigma_n(y_n)(s)\right) - \gamma_n\left(s, \sigma_n(y_n)(s)\right)\Big)\left(\sigma_n(y_n)(t) - \sigma_n(y_n)(s)\right). \end{aligned} \tag{19}$$

From the definition of the function $\gamma_n(t,r)$ and since we assume that $\{\lambda_n\}_{n\geq 1} \subseteq \mathbb{R}_-$ is bounded, given $\varepsilon > 0$, we can find $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$|t-s| < \delta_1 \Rightarrow \left|\gamma_n\left(t, \sigma_n(y_n)(s)\right) - \gamma_n\left(s, \sigma_n(y_n)(s)\right)\right| < \frac{\varepsilon}{2} \quad \text{for all } \ n \geq 1.$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

Hence

$$\Big(\gamma_n\big(t, \sigma_n(y_n)(s)\big) - \gamma_n\big(s, \sigma_n(y_n)(s)\big)\Big)\big(\sigma_n(y_n)(t) - \sigma_n(y_n)(s)\big) \\ \geq -\frac{\varepsilon}{2}\,|\sigma_n(y_n)(t) - \sigma_n(y_n)(s)|. \tag{20}$$

If $p \geq 2$, then

$$\Big(\gamma_n\big(t, \sigma_n(y_n)(t)\big) - \gamma_n\big(t, \sigma_n(y_n)(s)\big)\Big)\big(\sigma_n(y_n)(t) - \sigma_n(y_n)(s)\big) \\ \geq c_5\,|\sigma_n(y_n)(t) - \sigma_n(y_n)(s)|^p \ \text{ for some } c_5 > 0, \text{ all } n \geq 1. \tag{21}$$

If $1 < p < 2$, then

$$\Big(\gamma_n\big(t, \sigma_n(y_n)(t)\big) - \gamma_n\big(t, \sigma_n(y_n)(s)\big)\Big)\big(\sigma_n(y_n)(t) - \sigma_n(y_n)(s)\big) \\ \geq \frac{p-1}{2}\,\big(|\sigma_n(y_n)(t)| + |\sigma_n(y_n)(s)|\big)^{p-2}\,|\sigma_n(y_n)(t) - \sigma_n(y_n)(s)|^2 \\ \geq c_6\,|\sigma_n(y_n)(t) - \sigma_n(y_n)(s)|^2 \ \text{ for some } c_6 > 0, \text{ all } n \geq 1 \tag{22}$$

(since $\{\sigma_n(y_n)\}_{n\geq 1} \subseteq C(T)$ is bounded). In the derivation of (21) and (22), we have used some basic inequalities in $\mathbb{R}^k$, $k \geq 1$ (see Gasiński and Papageorgiou [**15**], p. 740).

Therefore, if $p \geq 2$, then using (20) and (22) in (19), we obtain

$$|y_n(t) - y_n(s)| + \frac{\varepsilon}{2} \geq c_6\,|\sigma_n(y_n)(t) - \sigma_n(y_n)(s)|^{p-1} \quad \text{(see also (18))}. \tag{23}$$

If $1 < p < 2$, then using (20) and (22) in (19), we obtain

$$|y_n(t) - y_n(s)| + \frac{\varepsilon}{2} \geq c_6\,|\sigma_n(y_n)(t) - \sigma_n(y_n)(s)| \quad \text{(see also (18))}. \tag{24}$$

Since, by hypothesis, $\{y_n\}_{n\geq 1} \subseteq C(T)$ is relatively compact, it is equicontinuous. So, we can find $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_1)$ such that

$$|t - s| < \delta_2 \Rightarrow |y_n(t) - y_n(s)| < \frac{\varepsilon}{2} \quad \text{for all } \ n \geq 1. \tag{25}$$

Using (25) in (23) and (24), we infer that

$$\{\sigma_n(y_n)\}_{n\geq 1} \subseteq C(T) \ \text{ is equicontinuous.}$$

Therefore, by the Arzela–Ascoli theorem, we conclude that $\{\sigma_n(y_n)\}_{n\geq 1} \subseteq C(T)$ is relatively compact.

We set $y_n(\cdot) = \gamma_n\big(\cdot, v_n'(\cdot)\big)$ for all $n \geq 1$. Then $\sigma_n(y_n) = v_n'$, and we know that $\{y_n\}_{n\geq 1} \subseteq C(T)$ is relatively compact. So, the Claim given above implies that $\{v_n'\}_{n\geq 1} \subseteq C(T)$ is relatively compact. Also, $\{v_n\}_{n\geq 1}$ being bounded in $W^{1,p}_{\mathrm{per}}(0, b)$, it is relatively compact in $C(T)$. Hence, we conclude that $\{v_n\}_{n\geq 1} \subseteq C^1(T)$ is relatively compact. Since, from (13), we have $v_n \longrightarrow x_0$ in $W^{1,p}_{\mathrm{per}}(0, b)$, it follows that $v_n \longrightarrow x_0$ in $C^1(T)$. Since, by hypothesis x_0 is a local $\widehat{C}^1(T)$-minimiser of φ_0, we can find $n_0 \geq 1$ such that

$$\varphi_0(x_0) \leq \varphi_0(v_n) \quad \text{for all } \ n \geq n_0,$$

which contradicts (13). This proves the proposition when $\{\lambda_n\}_{n\geq 1}$ is bounded.

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

Case II: The sequence $\{\lambda_n\}_{n\geq 1} \subseteq \mathbb{R}_-$ of Lagrange multipliers is unbounded.

We set $z_n = v_n - x_0 \in W^{1,p}_{\text{per}}(0, b)$, $n \geq 1$. From (15), we have

$$\frac{1}{|\lambda_n|} A(z_n + x_0) + A(z_n) = \frac{1}{|\lambda_n|} u_n - K_p(z_n) \quad \text{for all } \ n \geq 1. \tag{26}$$

We define

$$\mu_n(t, r) = \frac{1}{|\lambda_n|} |r + x_0'(t)|^{p-2}\big(r + x_0'(t)\big) + |r|^{p-2}r - \frac{1}{|\lambda_n|} |x_0'(t)|^{p-2}x_0'(t)$$
$$\text{for all } \ (t, r) \in T \times \mathbb{R}, \ \text{ all n} \geq 1.$$

It is easy to see that $\mu_n(t, r)$ has the same properties as $\gamma_n(t, r)$, and from (26), we have

$$-\Big(\mu_n\big(t, v_n'(t)\big)\Big)' = \frac{1}{|\lambda_n|}u_n(t) - |z_n(t)|^{p-2}z_n(t) + \frac{1}{|\lambda_n|}u_0(t) \quad \text{a.e. on } \ T. \tag{27}$$

Reasoning as in Case I, using this time (27), we reach a contradiction. This completes the proof of Proposition 3.3. □

The eigenvalues $\{\mu_{2n}\}_{n\geq 0}$ from Section 2 are all the eigenvalues of the negative periodic scalar p-Laplacian and of course coincide with the eigenvalues provided by the Ljusternik–Schnirelmann theory (see Drabek and Manásevich [**12**], Section 4). The Ljusternik–Schnirelmann theory provides minimax characterisations of the eigenvalues. In the next proposition, for $\mu_2 > 0$, i.e. the second eigenvalue (the first non-zero eigenvalue), we produce an alternative variational (minimax) characterisation, which suits better our purposes.

So, in what follows let

$$\partial B_1^{L^p} = \big\{x \in L^p(T) : \|x\|_p = 1\big\} \text{ and}$$
$$C_1(p) = \Big\{x \in W^{1,p}_{\text{per}}(0, b) \cap \partial B_1^{L^p} : \int_0^b |x|^{p-2}x\,\mathrm{d}t = 0\Big\}.$$

We know that

$$\mu_2 = \inf\big[\|x'\|_p^p : x \in C_1(p)\big] \tag{28}$$

(see Mawhin [**20**] and Gasiński and Papageorgiou [**15**]). Also, let

$$u_0(t) = \frac{1}{b^{\frac{1}{p}}} \quad \text{for all } \ t \in T.$$

This is the L^p-normalised principal eigenfunction for the periodic scalar p-Laplacian.

Proposition 3.4. **If** $\mathcal{S} = W^{1,p}_{per}(0, b) \cap \partial B_1^{L^p}$ *is furnished with the relative* $W^{1,p}_{per}(0, b)$*-norm topology and*

$$\Gamma = \big\{\gamma \in C([-1, 1], \mathcal{S}) : \gamma(-1) = -u_0, \ \gamma(1) = u_0\big\}$$

then $\mu_2 = \inf\limits_{\gamma\in\Gamma} \max\limits_{s\in[-1,1]} \big\| \frac{d}{dt}\gamma(s)\big\|_p^p.$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

Proof. Let $\gamma \in \Gamma$ and set $\eta(s) = \int_0^b |\gamma(s)(t)|^{p-2}\gamma(s)(t)\mathrm{d}t$ for all $s \in [-1, 1]$. Then

$$\eta(-1) < 0 < \eta(1),$$

and so, by the intermediate value theorem, we can find $s_0 \in (0, 1)$ such that

$$\begin{aligned} &\eta(s_0) = 0, \\ \Rightarrow\ &\gamma(s_0) \in C^1(p). \end{aligned}$$

From (28) it follows that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t}\gamma(s_0) \right\|_p^p \geq \mu_2,$$

from which it follows that

$$\inf_{\gamma\in\Gamma} \max_{s\in[-1,1]} \left\| \frac{\mathrm{d}}{\mathrm{d}t}\gamma(s) \right\|_p^p \geq \mu_2. \tag{29}$$

Next, we produce a $\gamma_0 \in \Gamma$ such that

$$\max_{-1\leq s\leq 1} \left\| \frac{\mathrm{d}}{\mathrm{d}t}\gamma_0(s) \right\| = \mu_2. \tag{30}$$

To this end, let $u_1 \in C^1(T) \cap C_1(p)$ be an eigenfunction corresponding to the eigenvalue μ_2, and let $\zeta : \mathbb{R} \longrightarrow \mathcal{S}$ be defined by

$$\zeta(r) = \frac{u_1 + r}{\|u_1 + r\|_p} \quad \text{for all} \ \ r \in \mathbb{R}.$$

We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\zeta(r)(t) &= \frac{u_1'(t)}{\|u_1 + r\|_p}, \\ \Rightarrow \left\| \frac{\mathrm{d}}{\mathrm{d}t}\zeta(r) \right\|_p^p &= \frac{\|u_1'\|_p^p}{\|u_1 + r\|_p^p}. \end{aligned}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}r} \left\| \frac{\mathrm{d}}{\mathrm{d}t}\zeta(r) \right\|_p^p = -p\ \|u_1'\|_p^p\ \|u_1 + r\|_p^{p-1} \frac{\langle \mathcal{F}_p(u_1 + r), 1\rangle_{pp'}}{\|u_1 + r\|_p} \frac{1}{\|u_1 + r\|_p^{2p}},$$

where $\mathcal{F}_p : L^p(T) \longrightarrow L^{p'}(T)$ is the duality map and $\langle\cdot,\cdot\rangle_{pp'}$ denotes the duality brackets for the pair $\big(L^{p'}(T), L^p(T)\big)$. We know that

$$\mathcal{F}_p(v)(\cdot) = \frac{1}{\|v\|_p^{p-2}}\ |v(\cdot)|^{p-2}v(\cdot) \quad \text{for all} \ \ v \in L^p(T)$$

(see, for example, Denkowski, Migórski and Papageorgiou [**11**], p. 43). Therefore

$$\frac{\mathrm{d}}{\mathrm{d}r} \left\| \frac{\mathrm{d}}{\mathrm{d}t}\zeta(r) \right\|_p^p = -p\ \|u_1'\|_p^p \int_0^b |u_1 + r|^{p-2}(u_1 + r)\ \mathrm{d}t\ \frac{1}{\|u_1 + r\|_p^{2p}}. \tag{31}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

From (31), we see that the function $r \longrightarrow \|\frac{d}{dt}\zeta(r)\|_p^p$ has a unique maximum at $r = 0$ (recall $u_1 \in C_1(p)$) and its value is $\|u_1'\|_p^p = \mu_2$. Moreover, for $r \neq 0$, we have

$$\begin{aligned}\zeta(r) &= \frac{u_1}{\|u_1 + r\|_p} + \frac{r}{\|u_1 + r\|_p}\\ &= \frac{u_1}{\|u_1 + r\|_p} + \frac{\operatorname{sgn}(r)}{\left(\int_0^b \left|\frac{u_1}{r} + 1\right|^p dt\right)^{\frac{1}{p}}},\\ &\Rightarrow \zeta(r) \longrightarrow \pm\frac{1}{b^{\frac{1}{p}}} = \pm u_0 \text{ as } |r| \to \infty.\end{aligned}$$

So, if for $s \in (-1, 1)$, we set $\gamma_0(s) = \zeta(\frac{s}{1-s^2})$; then evidently γ_0 is continuous on $(-1, 1)$, and it can be extended continuously to ± 1, by setting $\gamma_0(\pm 1) = \pm u_0$. Moreover, $\gamma_0(0) = \zeta(0) = u_1$ and

$$\max_{-1\leq s\leq 1}\left\|\frac{d}{dt}\gamma_0(s)\right\|_p^p = \mu_2.$$

Hence (30) holds. This combined with (29), concludes the proof of Proposition 3.4. □

4. Multiple solutions. In this section using a variational approach and the auxiliary results of the previous section, we prove a multiplicity theorem for problem (1).

The hypotheses on the non-smooth potential $j(t, x)$ are the following:

H(j): $j: T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that $j(t, 0) = 0$ a.e. on T and

(i) for all $x \in \mathbb{R}$, $t \longrightarrow j(t, x)$ is measurable;

(ii) for almost all $t \in T$, $x \longrightarrow j(t, x)$ is locally Lipschitz;

(iii) for every $r > 0$ there exists $a_r \in L^1(T)_+$ such that

$$|u| \leq a_r(t)$$

for almost all $t \in T$, all $|x| \leq r$ and all $u \in \partial j(t, x)$;

(iv) there exists a function $\vartheta \in L^\infty(T)_+$ such that $\vartheta(t) \leq \beta$ a.e. on T, $\vartheta \neq \beta$ and

$$\limsup_{x\to+\infty}\frac{u}{x^{p-1}} \leq \vartheta(t)$$

uniformly for a.a. $t \in T$ and all $u \in \partial j(t, x)$;

(v) there exist functions $\eta, \widehat{\eta} \in L^\infty(T)_+$ and an integer $n \geq 0$ such that

$$\begin{aligned}&\mu_{2n} + \beta \leq \eta(t) \leq \widehat{\eta}(t) \leq \mu_{2n+2} + \beta \text{ a.e. on } T,\\ &\mu_{2n} + \beta \neq \eta,\ \ \mu_{2n+2} + \beta \neq \widehat{\eta}\end{aligned}$$

and

$$\eta(t) \leq \liminf_{x\to-\infty}\frac{u}{|x|^{p-2}x} \leq \limsup_{x\to-\infty}\frac{u}{|x|^{p-2}x} \leq \widehat{\eta}(t)$$

uniformly for a.a. $t \in T$ and all $u \in \partial j(t, x)$;

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

(vi) there exist $\xi_0 \in L^1(T)_+$ and $\delta > 0$ such that $\beta \le \xi_0(t)$ a.e. on T, $\beta \ne \xi_0$ and

$$\frac{1}{p}\xi_0(t)x^p \le j(t,x) \quad \text{for a.a. } t \in T, \text{ all } 0 \le x \le \delta;$$

(vii) for almost all $t \in T$ and all $x > 0$, we have $u \ge 0$ for all $u \in \partial j(t,x)$.

REMARK 4.1. Hypotheses **H(j)(iv)** and **(v)** dictate an asymmetric behaviour of $x \longrightarrow \partial j(t,x)$ as we approach $-\infty$ and $+\infty$. In fact, as we move from $-\infty$ to $+\infty$, the generalised 'slopes' $\{\frac{u}{|x|^{p-2}x} : u \in \partial j(t,x)\}$ cross a finite number of eigenvalues (*jumping multi-valued non-linearity*).

EXAMPLE 4.2. The following function satisfies Hypotheses **H(j)**. For the sake of simplicity, we drop the t-dependence:

$$j(x) = \begin{cases} \frac{\eta}{p}|x|^p + \ln(1+|x|^r) & \text{if } x < 0, \\ \frac{\eta}{p}|x|^p & \text{if } 0 \le x \le 1, \\ \frac{\vartheta}{p}|x|^p - ce^{-x} + c_0 & \text{if } 1 < x, \end{cases}$$

where $\theta < \beta$, for some integer $n \ge 0$, $\mu_{2n} + \beta < \eta < \mu_{2n+2} + \beta$, $c > 0$ and $c_0 = \frac{\eta - \vartheta}{p} + \frac{c}{e}$. Note that if $c = (\eta - \vartheta)e > 0$, then $j \in C^1(\mathbb{R})$.

Let $\tau_+ : \mathbb{R} \longrightarrow \mathbb{R}$ be the truncation function defined by

$$\tau_+(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \ge 0. \end{cases}$$

We set $j_+(t,x) = j\big(t, \tau_+(x)\big)$ for all $(t,x) \in T \times \mathbb{R}$. Clearly for all $x \in \mathbb{R}$, $t \longrightarrow j_+(t,x)$ is measurable, and for almost all $t \in T$, $x \longrightarrow j_+(t,x)$ is locally Lipschitz. Moreover, from the non-smooth chain rule (see Clarke [**8**], p. 45), we have

$$\partial j_+(t,x) \subseteq \begin{cases} \{0\} & \text{if } x < 0, \\ \{ru : r \in [0,1],\ u \in \partial j(t,0)\} & \text{if } x = 0, \\ \partial j(t,x) & \text{if } 0 < x. \end{cases} \tag{32}$$

We introduce the functionals $\varphi, \varphi_+ : W^{1,p}_{\text{per}}(0,b) \longrightarrow \mathbb{R}$, defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j\big(t, x(t)\big)\mathrm{d}t \text{ and}$$
$$\varphi_+(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j_+\big(t, x(t)\big)\mathrm{d}t \quad \text{for all } x \in W^{1,p}_{\text{per}}(0,b).$$

Both φ and φ_+ are Lipschitz continuous on bounded sets; hence they are locally Lipschitz.

PROPOSITION 4.3. **If** *Hypotheses* **H(j)** *hold,* **then** φ *satisfies the PS-condition.*

Proof. Let $\{x_n\}_{n\ge1} \subseteq W^{1,p}_{\text{per}}(0,b)$ be a sequence such that $\{\varphi(x_n)\}_{n\ge1} \subseteq \mathbb{R}$ is bounded and

$$m(x_n) \longrightarrow +\infty \quad \text{as} \quad n \to \infty. \tag{33}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

Because $\partial\varphi(x_n) \subseteq W^{1,p}_{\mathrm{per}}(0,b)^*$ is w-compact and the norm functional in a Banach space is weakly lower semi-continuous, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi(x_n)$ such that $\|x_n^*\| = m(x_n)$, $n \geq 1$. We know that

$$x_n^* = A(x_n) + \beta K_p(x_n) - u_n,$$

with $u_n \in L^1(T)$, $u_n(t) \in \partial j\big(t, (x_n)\big)$ a.e on T (see Clarke **[8]**, p. 83). From (33), we have

$$|\langle x_n^*, v\rangle| \leq \varepsilon_n\|v\| \quad \text{for all } \; v \in W^{1,p}_{\mathrm{per}}(0,b), \; \text{with } \varepsilon_n \downarrow 0.$$

Let $v = x_n^+ \in W^{1,p}_{\mathrm{per}}(0,b)$. Then

$$\|(x_n^+)'\|_p^p + \beta\|x_n^+\|_p^p - \int_0^b u_n x_n^+ \mathrm{d}t \leq \varepsilon_n\|x_n^+\|. \tag{34}$$

Claim 1: The sequence $\{x_n^+\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is bounded.

We argue indirectly. So, suppose that $\{x_n^+\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is unbounded. We may assume that $\|x_n^+\| \longrightarrow 0$. Set $y_n = \frac{x_n^+}{\|x_n^+\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$, and so we may assume that

$$y_n \xrightarrow{w} y \;\text{ in }\; W^{1,p}_{\mathrm{per}}(0,b) \;\text{ and }\; y_n \longrightarrow y \;\text{ in }\; C(T), \; y \geq 0.$$

Let $\widehat{u}_n(t) = u_n(t)\chi_{\{x_n \geq 0\}}(t)$. Clearly $u_n(t)x_n^+(t) = \widehat{u}_n(t)x_n^+(t)$ for a.a. $t \in T$. From Hypotheses **H(j)**(iii), (iv) and (vii), we have

$$\begin{aligned} &0 \leq \widehat{u}_n(t) \leq \widehat{a}(t) + \widehat{c}(t)x_n^+(t)^{p-1} \text{ for a.a. } t \in T, \text{ with } \widehat{a}, \widehat{c} \in L^1(T)_+, \\ &\Rightarrow \left\{h_n = \frac{\widehat{u}_n}{\|x_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^1(T) \text{ is uniformly integrable.} \end{aligned} \tag{35}$$

Hence, by virtue of the Dunford–Pettis theorem, we may say that

$$h_n \xrightarrow{w} h \;\text{ in }\; L^1(T) \;\text{ as }\; n \to \infty. \tag{36}$$

Note that $x_n^+(t) \longrightarrow +\infty$ for all $t \in \{y > 0\}$ as $n \to \infty$. For every $\varepsilon > 0$ and $n \geq 1$, we define

$$D_{\varepsilon,n} = \left\{t \in T : x_n(t) = x_n^+(t) > 0, \; \frac{\widehat{u}_n(t)}{x_n^+(t)^{p-1}} \leq \vartheta(t) + \varepsilon\right\}.$$

Hypothesis **H(j)**(iv) implies that

$$\chi_{D_{\varepsilon,n}}(t) \longrightarrow 1 \quad \text{a.e. on } \; \{y > 0\}.$$

Using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} &\|(1 - \chi_{D_{\varepsilon,n}})h\|_{L^1(\{y>0\})} \longrightarrow 0, \\ &\Rightarrow \; \chi_{D_{\varepsilon,n}} h_n \xrightarrow{w} h \;\text{ in }\; L^1(\{y > 0\}) \quad (\text{see (36)}). \end{aligned} \tag{37}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

From the definition of the set $D_{\varepsilon,n}$, we have

$$\begin{aligned}\chi_{D_{\varepsilon,n}}(t)h_n(t) &= \chi_{D_{\varepsilon,n}}(t)\frac{\widehat{u}_n(t)}{x_n^+(t)^{p-1}}\, y_n(t)^{p-1}\\ &\leq \chi_{D_{\varepsilon,n}}(t)\big(\vartheta(t)+\varepsilon\big)y_n(t)^{p-1} \quad \text{a.e. on } \ T.\end{aligned}$$

Passing to the limit as $n \to \infty$, using (37) and Mazur's lemma and letting $\varepsilon \downarrow 0$, we obtain

$$h(t) \leq \vartheta(t)y(t)^{p-1} \quad \text{a.e. on } \ \{y>0\}. \tag{38}$$

Moreover, from (35) it is clear that

$$h(t) = 0 \quad \text{a.e. on } \ \{y=0\}. \tag{39}$$

Since $y \geq 0$, from (38) and (39), we infer that

$$h(t) \leq \vartheta(t)y(t)^{p-1} \quad \text{a.e. on } \ T. \tag{40}$$

We return to (34) and divide with $\|x_n^+\|^p$. We have

$$\|y_n'\|_p^p + \beta\|y_n\|_p^p - \int_0^b \frac{\widehat{u}_n}{\|x_n^+\|^{p-1}}\, y_n \mathrm{d}t \leq \varepsilon_n' \ \text{ with } \varepsilon_n' \downarrow 0. \tag{41}$$

Passing to the limit as $n \to \infty$ and using (36) and (40), we obtain

$$\|y'\|_p^p + \beta\|y\|_p^p \leq \int_0^b hy \mathrm{d}t \leq \int_0^b \vartheta y^p \mathrm{d}t.$$

Lemma 3.1 implies that $y \equiv 0$. Hence, from (41), it is clear that $y_n \longrightarrow 0$ in $W^{1,p}_{\mathrm{per}}(0,b)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$. This proves Claim 1.

Claim 2: The sequence $\{x_n^-\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is bounded.

Again we proceed by contradiction. So, suppose that $\|x_n^-\| \longrightarrow \infty$. We set $v_n = \frac{x_n^-}{\|x_n^-\|}$, $n \geq 1$. Then $\|v_n\| = 1$ for all $n \geq 1$, and so we may assume that

$$v_n \xrightarrow{w} v \ \text{ in } \ W^{1,p}_{\mathrm{per}}(0,b) \ \text{ and } \ v_n \longrightarrow v \ \text{ in } \ C(T).$$

From (33), we have

$$\left|\left\langle \frac{x_n^*}{\|x_n^-\|^{p-1}}, w\right\rangle\right| \leq \varepsilon_n'\|w\| \quad \text{for all } \ w \in W^{1,p}_{\mathrm{per}}(0,b), \ \text{with } \varepsilon_n' \downarrow 0. \tag{42}$$

Note that

$$A(x_n) = A(x_n^+) - A(x_n^-) \ \text{ and } \ K_p(x_n) = K_p(x_n^+) - K_p(x_n^-). \tag{43}$$

We set

$$\widehat{u}_n = \chi_{\{x_n \geq 0\}}u_n \ \text{ and } \ \overline{u}_n = \chi_{\{x_n<0\}}u_n \quad \text{for all } \ n \geq 1.$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

We have

$$\widehat{u}_n(t) \in \partial j\big(t, x_n^+(t)\big) \quad \text{a.e. on } \{x_n > 0\} \quad \text{(see (32))}, \tag{44}$$

$$\overline{u}_n(t) \in \partial j\big(t, x_n(t)\big) \quad \text{a.e. on } \{x_n < 0\} \text{ and} \tag{45}$$

$$|\widehat{u}_n(t)| \leq \widehat{a}(t) \quad \text{a.e. on } \{x_n = 0\}, \text{ with } \widehat{a} \in L^1(T)_+. \tag{46}$$

From (42) and (43), we have

$$\left| \left\langle \frac{A(x_n^+)}{\|x_n^-\|^{p-1}}, w \right\rangle - \langle A(v_n), w\rangle + \frac{\beta}{\|x_n^-\|^{p-1}} \int_0^b |x_n^+|^{p-2} x_n^+ w \, dt \right.$$
$$\left. -\beta \int_0^b |v_n|^{p-2} v_n w \, dt - \int_0^b \frac{\widehat{u}_n}{\|x_n^-\|^{p-1}} w \, dt - \int_0^b \frac{\overline{u}_n}{\|x_n^-\|^{p-1}} w \, dt \right| \leq \varepsilon_n' \|w\|. \tag{47}$$

Because of the boundedness of $\{x_n^+\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$, we have

$$\left\langle \frac{A(x_n^+)}{\|x_n^-\|^{p-1}}, w \right\rangle \longrightarrow 0, \quad \frac{1}{\|x_n^-\|^{p-1}} \int_0^b |x_n^+|^{p-2} x_n^+ w \, dt \longrightarrow 0,$$
$$\int_0^b \frac{\widehat{u}_n}{\|x_n^-\|^{p-1}} w \, dt \longrightarrow 0 \quad \text{(see (44) and (46))}.$$

Moreover, from (45) and Hypothesis **H(j)**(iii) and (v), we see that

$$\left\{ g_n = \frac{\overline{u}_n}{\|x_n^-\|^{p-1}} \right\}_{n\geq 1} \subseteq L^1(T) \text{ is uniformly integrable.}$$

As before, invoking the Dunford–Pettis theorem, we may assume that

$$g_n \xrightarrow{w} g \quad \text{in} \quad L^1(T). \tag{48}$$

Given $\varepsilon > 0$ and $n \geq 1$, we introduce the set

$$E_{\varepsilon,n} = \left\{ t \in T : x_n(t) < 0, \ \eta(t) - \varepsilon \leq \frac{\overline{u}_n(t)}{|-x_n^-(t)|^{p-2}\big(-x_n^-(t)\big)} \leq \widehat{\eta}(t) + \varepsilon \right\}.$$

Note that $x_n(t) \longrightarrow -\infty$ for all $t \in \{v > 0\}$ as $n \to \infty$. So, Hypothesis **H(j)**(v) and (45) imply

$$\chi_{E_{\varepsilon,n}}(t) \longrightarrow 1 \quad \text{a.e. on } \{v > 0\}.$$

Arguing as in the proof of Claim 1, we show that

$$-\widehat{\eta}(t)v(t)^{p-1} \leq g(t) \leq -\eta(t)v(t)^{p-1} \quad \text{a.e. on } T. \tag{49}$$

From (49), it follows that

$$g(t) = -\xi(t)v(t)^{p-1} \quad \text{a.e. on } T, \tag{50}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

where $\xi \in L^\infty(T)_+$, $\eta(t) \le \xi(t) \le \widehat{\eta}(t)$ a.e. on T. Let $\xi_1(t) = \xi(t) - \beta$. In (47), we let $w = v_n - v \in W^{1,p}_{\text{per}}(0,b)$. Then

$$\begin{aligned}
&\left| \left\langle \frac{A(x_n^+)}{\|x_n^-\|^{p-1}}, v_n - v \right\rangle - \langle A(v_n), v_n - v\rangle + \frac{\beta}{\|x_n^-\|^{p-1}} \int_0^b |x_n^+|^{p-2} x_n^+ (v_n - v)\mathrm{d}t \right.\\
&\quad \left. -\beta \int_0^b v_n^{p-1}(v_n - v)\mathrm{d}t - \int_0^b \frac{\widehat{u}_n}{\|x_n^-\|^{p-1}}(v_n - v)\mathrm{d}t - \int_0^b g_n(v_n - v)\mathrm{d}t \right|\\
&\le \varepsilon_n' \|v_n - v\|.
\end{aligned} \tag{51}$$

The boundedness of $\{x_n^+\}_{n\ge1} \subseteq W^{1,p}_{\text{per}}(0,b)$ implies

$$\begin{aligned}
&\left\langle \frac{A(x_n^+)}{\|x_n^-\|^{p-1}}, v_n - v \right\rangle \longrightarrow 0,\ \int_0^b |x_n^+|^{p-2}x_n^+(v_n - v)\mathrm{d}t \longrightarrow 0 \text{ and}\\
&\int_0^b \frac{\widehat{u}_n}{\|x_n^-\|^{p-1}}(v_n - v)\mathrm{d}t \longrightarrow 0 \quad \text{(see (44) and (46))}.
\end{aligned} \tag{52}$$

Also, it is clear that

$$\int_0^b v_n^{p-1}(v_n - v)\mathrm{d}t \longrightarrow 0 \ \text{ and } \ \int_0^b g_n(v_n - v)\mathrm{d}t \longrightarrow 0 \ \text{ as } n \to \infty. \tag{53}$$

So, if we pass to the limit as $n \to \infty$ in (51) and use (52) and (53), we obtain

$$\lim_{n\to\infty} \langle A(v_n), v_n - v\rangle = 0.$$

This, by virtue of Lemma 3.2, implies that

$$v_n \longrightarrow v \ \text{ in } \ W^{1,p}_{\text{per}}(0,b), \ \text{ i.e. } \ \|v\| = 1. \tag{54}$$

We return to (42), and passing to the limit as $n \to \infty$, we obtain

$$\begin{aligned}
&\langle A(v), w\rangle + \beta \int_0^b v^{p-1} w \mathrm{d}t = \int_0^b \xi v^{p-1}\mathrm{d}t \quad \text{for all } \ w \in W^{1,p}_{\text{per}}(0,b)\\
&\Rightarrow \left\{ \begin{array}{ll} -\left(|v'(t)|^{p-2}v'(t)\right)' = \xi_1(t)|v(t)|^{p-2}v(t) & \text{a.e. on } T,\\ v(0) = v(b),\ v'(0) = v'(b). & \end{array} \right\}
\end{aligned} \tag{55}$$

From Hypothesis **H(j)**(v), we have

$$\mu_{2n} \le \eta(t) - \beta \le \xi_1(t) \le \widehat{\eta}(t) - \beta \le \mu_{2n+2},\ \ \mu_{2n} \ne \eta(\cdot) - \beta,\ \ \mu_{2n+2} \ne \widehat{\eta}(\cdot) - \beta.$$

Then, from (55) and Proposition 2.4, it follows that $v = 0$, a contradiction to (54). This proves that $\{x_n^-\}_{n\ge1} \subseteq W^{1,p}_{\text{per}}(0,b)$ is bounded.

Claims 1 and 2 imply that $\{x_n\}_{n\ge1} \subseteq W^{1,p}_{\text{per}}(0,b)$ is bounded, and so, we may assume that

$$x_n \xrightarrow{w} x \ \text{ in } \ W^{1,p}_{\text{per}}(0,b) \ \text{ and } \ x_n \longrightarrow x \ \text{ in } \ C(T).$$

As above, we can have

$$\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0,$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

which via Lemma 3.2 implies $x_n \longrightarrow x$ in $W^{1,p}_{\mathrm{per}}(0,b)$, and so φ satisfies the PS-condition. $\square$

PROPOSITION 4.4. **If** *Hypotheses* **H(j)** *hold,* **then** *φ_+ is coercive on $W^{1,p}_{per}(0,b)$.*

Proof. By virtue of Hypothesis **H(j)**(iv), given $\varepsilon > 0$, we can find $M = M(\varepsilon) > 0$ such that

$$u \le \big(\vartheta(t) + \varepsilon\big)x^{p-1} \quad \text{for a.a. } t \in T, \text{ all } x \ge M \text{ and all } u \in \partial j_+(t,x) \quad (\text{see (32)}).$$

From this, Hypothesis **H(j)**(iii) and (32), it follows that we can find $a_\varepsilon \in L^1(T)_+$ such that

$$u \le \big(\vartheta(t) + \varepsilon\big)|x|^{p-1} + a_\varepsilon(t) \quad \text{for a.a. } t \in T, \text{ all } x \in \mathbb{R} \text{ and all } u \in \partial j_+(t,x). \tag{56}$$

Hypothesis **H(j)**(i) and (ii) and Rademacher's theorem imply that for almost all $t \in T$, $r \longrightarrow j_+(t,r)$ is differentiable almost everywhere on $\mathbb{R}$, and at every point of differentiability $r \in \mathbb{R}$, we have $\frac{\mathrm{d}}{\mathrm{d}r} j_+(t,r) \in \partial j_+(t,r)$ (see Clarke [**8**]). Combining this with (56), we have

$$\frac{\mathrm{d}}{\mathrm{d}r} j_+(t,r) \le \big(\vartheta(t) + \varepsilon\big)|r|^{p-1} + a_\varepsilon(t) \quad \text{for a.a. } t \in T \setminus D,\ |D| = 0 \text{ and}$$
$$\text{for all } \ r \in \mathbb{R} \setminus D(t),\ |D(t)| = 0$$

(by $|\cdot|$ we denote the Lebesgue measure on $\mathbb{R}$). Integrating this inequality and recalling that $j_+(t,x) = 0$ for a.a. $t \in T$ and all $x \le 0$, we obtain

$$j_+(t,x) \le \frac{1}{p}\big(\vartheta(t) + \varepsilon\big)|x|^p + a_\varepsilon(t)|x| \quad \text{for a.a. } t \in T, \text{ all } x \in \mathbb{R}. \tag{57}$$

Then, for $x \in W^{1,p}_{\mathrm{per}}(0,b)$, we have

$$\begin{aligned}
\varphi_+(x) &= \frac{1}{p}\|x'\|_p^p + \frac{\beta}{p}\|x\|_p^p - \int_0^b j_+(t,x)\mathrm{d}t \\
&\ge \frac{1}{p}\|x'\|_p^p + \frac{\beta}{p}\|x\|_p^p - \frac{1}{p}\int_0^b \vartheta|x|^p \mathrm{d}t - \frac{\varepsilon}{p}\|x\|^p - c_\varepsilon\|x\| \quad \text{for some } c_\varepsilon > 0 \\
&\quad (\text{see (57)}) \\
&\ge \frac{\xi_0 - \varepsilon}{p}\|x\|^p - c_\varepsilon\|x\|.
\end{aligned} \tag{58}$$

Choosing $\varepsilon < \xi_0$, from (57) we conclude that φ_+ is coercive. $\square$

PROPOSITION 4.5. **If** *Hypotheses* **H(j)** *hold,* **then** *problem* (1) *has a solution $x_0 \in \operatorname{int}\widehat{C}_+$ which is a local minimiser of φ.*

Proof. Exploiting the compact embedding of $W^{1,p}_{\mathrm{per}}(0,b)$ into $C(T)$, we can easily check that φ_+ is sequentially weakly lower semi-continuous. This fact together with Proposition 4.4 and the Weierstrass theorem implies that we can find $x_0 \in W^{1,p}_{\mathrm{per}}(0,b)$ such that

$$\varphi_+(x_0) = \inf\Big[\varphi_+(x) : x \in W^{1,p}_{\mathrm{per}}(0,b)\Big] = m_+. \tag{59}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

Let $r \in (0, \delta]$, where $\delta > 0$ is as in Hypothesis **H(j)**(vi). Then

$$\begin{aligned} \varphi_+(r) &= \frac{\beta}{p} r^p b - \int_0^b j_+(t, r)\mathrm{d}t \\ &\le \frac{r^p}{p} \int_0^b \big(\beta - \xi_0(t)\big)\mathrm{d}t < 0, \\ &\Rightarrow m_+ < 0 = \varphi_+(0), \\ &\Rightarrow x_0 \neq 0 \quad \text{(see (59))}. \end{aligned}$$

From (59), we have

$$\begin{aligned} &0 \in \partial\varphi_+(x_0), \\ \Rightarrow\ & A(x_0) + \beta K_p(x_0) = u_+, \end{aligned} \tag{60}$$

with $u_+ \in L^1(T)$, $u_+(t) \in \partial j_+\big(t, x_0(t)\big)$ a.a. on T. On (60), we act with the test function $-x_0^- \in W^{1,p}_{\mathrm{per}}(0, b)$. We obtain

$$\begin{aligned} &\|(-x_0^-)'\|_p^p + \beta\| - x_0^-\|_p^p = 0 \quad \text{(see (32))} \\ &\Rightarrow -x_0^- = 0, \text{ i.e. } x_0 \ge 0,\ x_0 \neq 0. \end{aligned}$$

From (60), as in Gasiński and Papageorgiou [**13**] (see also the proof of Proposition 3.3), we obtain

$$\left\{ \begin{array}{l} -\big(|x_0'(t)|^{p-2}x_0'(t)\big)' + \beta|x_0(t)|^{p-2}x_0(t) = u_+(t) \quad \text{a.e. on } T, \\ x_0(0) = x_0(b),\ x_0'(0) = x_0'(b). \end{array} \right\} \tag{61}$$

Since $x_0'(t) = 0$ a.a. on $\{x_0 = 0\}$, from (61) and Hypothesis **H(j)**(vii), we see that

$$\big(|x_0'(t)|^{p-2}x_0'(t)\big)' \le 0 \quad \text{a.e. on } T. \tag{62}$$

Invoking the non-linear strong maximum principle of Vazquez [**24**], from (62) we infer that

$$x_0(t) > 0 \quad \text{for all } t \in (0, b).$$

Moreover, if $x_0(0) = x_0(b) = 0$, then

$$x_0'(b^-) < 0 < x_0'(0^+),$$

a contradiction to (61). This means that

$$x(t) > 0 \quad \text{for all } t \in T, \text{ and hence } x_0 \in \mathrm{int}\widehat{C}_+.$$

Therefore, we can find $r > 0$ small such that

$$\overline{B}_r^{\widehat{C}^1(T)}(x_0) = \big\{x \in \widehat{C}^1(T) : \|x - x_0\|_{\widehat{C}^1(T)} \le r\big\} \subseteq \widehat{C}_+. \tag{63}$$

Note that

$$\varphi\big|_{\widehat{C}_+} = \varphi_+\big|_{\widehat{C}_+}. \tag{64}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

But x_0 is a minimiser of φ_+. So, from (63) and (64), it follows that x_0 is a local $\widehat{C}^1(T)$-minimiser of φ. Invoking Proposition 3.3, we conclude that x_0 is a local minimiser of φ. $\square$

Without any loss of generality, we can assume that x_0 is an isolated critical point (and local minimiser) of φ. Otherwise, we already have a whole sequence of distinct non-trivial positive solutions of (1). Hence, we can find $\varrho > 0$ small such that

$$\varphi(x_0) < \varphi(y) < 0 \quad \text{and} \quad 0 \notin \partial\varphi(y) \quad \text{for all} \quad y \in \overline{B}_\varrho \setminus \{x_0\}. \tag{65}$$

Here $\overline{B}_\varrho(x_0) = \left\{y \in W^{1,p}_{\text{per}}(0,b) : \|y - x_0\| \leq \varrho\right\}$.

PROPOSITION 4.6. **If** *Hypotheses* **H(j)** *hold and* $x_0 \in \text{int}\widehat{C}_+$ *is the solution of problem* (1) *obtained in Proposition 4.5,* **then** $\varphi(x_0) < \inf[\varphi(x) : \|x - x_0\| = \varrho] = c_\varrho$, *with* $\varrho > 0$ *as in (65).*

Proof. We argue by contradiction. So, suppose that Proposition 4.6 is not true. Then, we can find $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ such that

$$\|x_n - x_0\| = r \quad \text{and} \quad \varphi(x_n) \downarrow \varphi(x_0) \quad \text{as } n \to \infty. \tag{66}$$

By passing to a suitable subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in} \quad W^{1,p}_{\text{per}}(0,b) \quad \text{and} \quad x_n \longrightarrow x \quad \text{in} \quad C(T) \quad \text{as } n \to \infty.$$

The sequential weak lower semi-continuity of φ and that of the norm functional imply

$$\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n) = \varphi(x_0), \quad \|x - x_0\| \leq \varrho \quad \text{(see (66))}. \tag{67}$$

From (65) and (67), it follows that

$$x = x_0.$$

From the non-smooth mean-value theorem (see Clarke **[8]**, p. 41), we have

$$\varphi(x_n) - \varphi\left(\frac{1}{2}(x_n + x_0)\right) = \left\langle y_n^*, \frac{1}{2}(x_n + x_0)\right\rangle \tag{68}$$

with $y_n^* \in \partial\varphi\left(s_n x_n + (1 - s_n)\frac{x_n + x_0}{2}\right)$, $s_n \in (0,1)$. Recall that

$$y_n^* = A\left(s_n x_n + (1 - s_n)\frac{x_n + x_0}{2}\right) + \beta K_p\left(s_n x_n + (1 - s_n)\frac{x_n + x_0}{2}\right) - u_n, \tag{69}$$

with $u_n \in L^1(T)$, $u_n(t) \in \partial j(t, s_n x_n(t) + (1 - s_n)\frac{x_n + x_0}{2}(t))$ a.e. on T. We use (69) in (68) and then pass to the limit as $n \to \infty$. We obtain

$$\limsup_{n\to\infty} \left\langle A\left(s_n x_n + (1 - s_n)\frac{x_n + x_0}{2}\right), x_n - x_0\right\rangle \leq 0. \tag{70}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

We have

$$\left\langle A\Big(s_n x_n + (1-s_n)\frac{x_n+x_0}{2}\Big), s_n x_n + (1-s_n)\frac{x_n+x_0}{2} - x_0\right\rangle$$
$$= \frac{1+s_n}{2}\left\langle A\Big(s_n x_n + (1-s_n)\frac{x_n+x_0}{2}\Big), x_n - x_0\right\rangle,$$
$$\Rightarrow \limsup_{n\to\infty}\left\langle A\Big(s_n x_n + (1-s_n)\frac{x_n+x_0}{2}\Big), s_n x_n + (1-s_n)\frac{x_n+x_0}{2} - x_0\right\rangle \le 0$$
$$\text{(see (70)).} \tag{71}$$

Assuming that $s_n \longrightarrow s \in [0,1]$ and since $x_n \xrightarrow{w} x_0$ in $W^{1,p}_{\mathrm{per}}(0,b)$, we infer from (71) and Lemma 3.2 that

$$s_n x_n + (1-s_n)\frac{x_n+x_0}{2} \longrightarrow x_0 \ \text{ in } \ W^{1,p}_{\mathrm{per}}(0,b). \tag{72}$$

But note that

$$\Big\| s_n x_n + (1-s_n)\frac{x_n+x_0}{2} - x_0\Big\| = (1+s_n)\Big\|\frac{x_n-x_0}{2}\Big\| \ge \frac{\varrho}{2} > 0,$$

which contradicts (72). This completes the proof. □

To produce a second non-trivial solution for problem (1), we need to strengthen the hypotheses on $j(t,\cdot)$ near the origin and on the negative half-line. So, the new hypotheses on $j(t,x)$ are the following.

H(j)′: $j\colon T\times\mathbb{R} \longrightarrow \mathbb{R}$ is a function such that $j(t,0)=0$ a.e. on T, $0\in\partial j(t,0)$ a.e. on T, Hypotheses **H(j)′**(i)⟶(v) and (vii) are the same as the corresponding Hypotheses **H(j)**(i)⟶(v) and (vii) and

(vi) there exist $\delta>0$ and $\sigma>0$ such that $\sigma>\mu_2+\beta$ and $\frac{\sigma}{p}|x|^p \le j(t,x)$ for a.a. $t\in T$, all $|x|\le\delta$ and $j(t,x)\ge\frac{\eta(t)}{p}|x|^p$ for a.a. $t\in T$, all $x\le 0$, with $\eta\in L^\infty(T)_+$ as in **H(j)′**(v).

REMARK 4.7. *The potential function $x\longrightarrow j(x)$ produced in the Example 4.2, satisfies also Hypotheses* **H(j)′**.

THEOREM 4.8. **If** *Hypotheses* **H(j)′** *hold,* **then** *problem* (1) *has at least two non-trivial solutions, namely $x_0\in\mathrm{int}\widehat{C}_+$ and $v_0\in C^1(T)$.*

Proof. From Proposition 4.5, we already have one solution $x_0\in\mathrm{int}\widehat{C}_+$.
Also, by virtue of Proposition 4.6, we can find $\varrho>0$ small such that

$$\varphi(x_0) < \inf\Big[\varphi(x) : \|x-x_0\| = \varrho\Big] = c_\varrho. \tag{73}$$

Using Hypotheses **H(j)′**(iii) and (v), we see that given $\varepsilon>0$, we can find $a_\varepsilon\in L^1(T)_+$, such that

$$j(t,x) \ge \frac{1}{p}\Big(\eta(t)+\varepsilon\Big)|x|^p - a_\varepsilon(t)|x| \ \text{ for a.a. } t\in T \text{ and all } x\le 0. \tag{74}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

Let $r<0$. Then

$$\begin{aligned}\varphi(r) &= \frac{\beta}{p}\,|r|^p b - \int_0^b j(t,r)\mathrm{d}t \\ &\leq \frac{\beta}{p}\,|r|^p b - \frac{1}{p}|r|^p \int_0^b \eta(t)\mathrm{d}t + \frac{\varepsilon}{p}|r|^p b + |r|\,\|a_\varepsilon\|_1 \quad \text{(see (74))} \\ &= \frac{|r|^p}{p}\left[\int_0^b \big(\beta-\eta(t)\big)\mathrm{d}t + \varepsilon b\right] + |r|\,\|a_\varepsilon\|_1. \end{aligned} \tag{75}$$

Note that $\gamma = \int_0^b \big(\eta(t)-\beta\big)\mathrm{d}t > 0$. So, if we choose $\varepsilon>0$ small such that $\varepsilon b<\gamma$, then from (75) it follows that

$$\varphi(r) \longrightarrow -\infty \text{ as } r \longrightarrow -\infty.$$

Then, for $r_0<0$ with $|r_0|>0$ large, we can have

$$\|r_0 - x_0\| > \varrho \text{ and } \varphi(r_0) < c_\varrho. \tag{76}$$

Because of (73), (76) and Proposition 4.3, we can apply Theorem 2.1 and obtain $v_0 \in W^{1,p}_{\mathrm{per}}(0,b)$ such that

$$0 \in \partial\varphi(v_0) \text{ and } c_\varrho \leq \varphi(v_0) = \inf_{\gamma_0\in\Gamma_0}\max_{0\leq t\leq 1}\varphi\big(\gamma_0(t)\big), \tag{77}$$

where $\Gamma_0 = \big\{\gamma_0 \in C\big([0,1], W^{1,p}_{\mathrm{per}}(0,b)\big) : \gamma_0(0)=r_0,\ \gamma_0(1)=x_0\big\}$.

From the inclusion $0\in\partial\varphi(v_0)$ in (77), we infer that $v_0 \in C^1(T)$ solves (1). It remains to show that v_0 is non-trivial.

According to the minimax expression in (77), in order to show the non-triviality of v_0, it suffices to produce a path $\gamma_0^* \in \Gamma_0$, such that $\varphi\big|_{\gamma_0^*}<0$. To this end, let $\nu>0$ be such that

$$\mu_2 + \beta + \nu < \sigma. \tag{78}$$

By virtue of Proposition 3.4, we can find

$$\gamma_1 \in \Gamma = \big\{\gamma \in C([-1,1],\mathcal{S}) : \gamma(-1) = -u_0,\ \gamma(1)=u_0\big\}$$

such that

$$\max\Big[\|x'\|_p^p : x\in\gamma_1([-1,1])\Big] \leq \mu_2+\nu. \tag{79}$$

Since $W^{1,p}_{\mathrm{per}}(0,b)$ is embedded (compactly) in $C(T)$, we can find $\varepsilon>0$ small such that

$$|\varepsilon x(t)| \leq \delta \quad \text{for all } t\in T \text{ and all } x\in\gamma([-1,1]),$$

with $\delta>0$ as in Hypothesis **H(j)**$'$(vi). Hence

$$\frac{\sigma}{p}\,\varepsilon^p|x(t)|^p \leq j\big(t,\varepsilon x(t)\big) \text{ for a.a. } t\in T, \text{ all } x\in\gamma_1([-1,1]). \tag{80}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

So, if $x \in \gamma_1([-1, 1])$, then

$$\begin{aligned}\varphi(\varepsilon x) &= \frac{\varepsilon^p}{p}\|x'\|_p^p + \frac{\beta\varepsilon^p}{p}\|x\|_p^p - \int_0^b j\big(t, \varepsilon x(t)\big)\mathrm{d}t \\ &\le \frac{\varepsilon^p}{p}\|x'\|_p^p + \frac{\varepsilon^p}{p}\beta\|x\|_p^p - \frac{\varepsilon^p}{p}\sigma\|x\|_p^p \quad \text{(see (80))} \\ &\le \frac{\varepsilon^p}{p}\big[\mu_2 + \nu + \beta - \sigma\big] \quad \text{(see (79) and recall that } \|x\|_p = 1) \\ &< 0 \quad \text{(see (79)).}\end{aligned}$$

Therefore, if $\gamma_1^\varepsilon = \varepsilon\gamma_1$, then

$$\varphi\big|_{\gamma_1^\varepsilon} < 0. \tag{81}$$

Next, let $\gamma_-(t) = tr_0 + (1-t)(-\varepsilon u_0)$, and let $v \in \gamma_-([-1, 1]) \subseteq \mathbb{R}_-$. Then

$$\begin{aligned}\varphi(v) &= \frac{\beta}{p}|v|^p b - \int_0^b j(t, v)\mathrm{d}t \\ &\le \frac{|v|^p}{p}\int_0^b \big(\beta - \eta(t)\big)\mathrm{d}t \quad \text{(see Hypothesis } \mathbf{H(j)}'\text{(vi))} \\ &< 0 \\ &\Rightarrow \varphi\big|_{\gamma_-} < 0.\end{aligned} \tag{82}$$

Finally, we shall produce a continuous path γ_+ which joins εu_0 with x_0 and along which φ is strictly negative. To this end, we proceed as follows.

We may assume that $\{0, x_0\}$ are the only critical points of φ_+. Indeed, if this is not the case, then we can find a third critical point v_0 of φ_+, distinct from 0 and x_0. Arguing as in the proof of Proposition 4.5, we can show that $v_0 \in \operatorname{int}\widehat{C}_+$ and that it solves problem (1). So, we have produced a second non-trivial solution for (1), and we are done.

Let $a = m_+ = \varphi_+(x_0) = \varphi(x_0) = \inf \varphi_+ < 0 = b$. Note that $K_a(\varphi_+) = \{x_0\}$ and x_0 is a minimiser of φ_+. Also, from Proposition 4.4, we know that φ_+ is coercive. From this, it follows easily that φ_+ satisfies the PS-condition. So, we can apply Theorem 2.2 and produce a continuous deformation $h : [0, 1] \times \dot{\varphi}_+^b \longrightarrow \dot{\varphi}_+^b$, such that $h(t, \cdot)\big|_{K_a(\varphi_+)} = \mathrm{id}\big|_{K_a(\varphi_+)}$ and

- $h(1, \dot{\varphi}_+^b) \subseteq \dot{\varphi}_+^a \cup K_a(\varphi_+) = \{x_0\},$ (83)
- $\varphi_+\big(h(t, x)\big) \le \varphi_+(x) \quad \text{for all } t \in [0, 1] \text{ and all } x \in \dot{\varphi}_+^b.$ (84)

We set

$$\gamma_+(t) = h(t, \varepsilon u_0) \quad \text{for all } t \in [0, 1].$$

Evidently, $\gamma_+ \in C\big([0, 1], W^{1,p}_{\mathrm{per}}(0, b)\big)$. Also,

$$\begin{aligned}&\gamma_+(0) = h(0, \varepsilon u_0) = \varepsilon u_0 \quad \text{(because } h \text{ is a deformation),} \\ &\gamma_+(1) = h(1, \varepsilon u_0) = x_0 \quad \text{(see (83)).}\end{aligned}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

Therefore, γ_+ is a continuous path which joins εu_0 with x_0. Moreover,

$$\begin{aligned}\varphi_+\big(\gamma_+(t)\big) &= \varphi_+\big(h(t, \varepsilon u_0)\big)\\ &\leq \varphi_+(\varepsilon u_0) \quad \text{(see (84))}\\ &< 0 \quad \text{(see (81))},\\ &\Rightarrow \varphi_+\big|_{\gamma_+} < 0.\end{aligned} \tag{85}$$

Note that for all $x \in \gamma_+([0, 1])$, we have

$$\begin{aligned}&j\big(t, x(t)\big) = j_+\big(t, x(t)\big) \ \text{ for a.a. } \ t \in \{x \geq 0\} \text{ and}\\ &j\big(t, x(t)\big) \geq j_+\big(t, x(t)\big) \ \text{ for a.a. } \ t \in \{x < 0\} \quad \text{(see Hypothesis } \mathbf{H(j)}'\text{(vi))}.\end{aligned}$$

Therefore

$$\begin{aligned}&\varphi\big|_{\gamma_+} \leq \varphi_+\big|_{\gamma_+},\\ \Rightarrow\ &\varphi\big|_{\gamma_+} < 0 \quad \text{(see (85))}.\end{aligned} \tag{86}$$

We concatenate paths γ_-, γ_1^ε and γ_+ and this way we produce a path $\gamma_0^* \in \Gamma_0$ such that $\varphi\big|_{\gamma_0^*} < 0$ (see (81), (82) and (86)). Then from (77) it follows that $\varphi(x_0) < 0 = \varphi(0)$, and hence $v_0 \neq 0$. □

5. Semi-linear and smooth problems. In this section, we consider the semi-linear (i.e. $p = 2$), smooth (i.e. $j(t, \cdot) \in C^2(\mathbb{R})$) problem. By strengthening the hypotheses near the origin and using Morse theory in our approach, we show the existence of three non-trivial solutions.

Now, the problem under consideration is the following:

$$\left\{\begin{array}{l} -x''(t) + \beta x(t) = f\big(t, x(t)\big) \quad \text{a.e. on } T, \\ x(0) = x(b), \ x'(0) = x'(b). \end{array}\right\} \tag{87}$$

The hypotheses on the $f(t, x)$ are the following:

H(f): $f : T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that $f(t, 0) = 0$ a.e. on T and

(i) for all $x \in \mathbb{R}$, $t \longrightarrow f(t, x)$ is measurable;
(ii) for almost all $t \in T$, $x \longrightarrow f(t, x)$ is C^1;
(iii) for every $r > 0$ there exists $a_r \in L^1(T)_+$ such that

$$|f'_x(t, x)| \leq a_r(t) \ \text{ for almost all } t \in T \text{ and all } |x| \leq r;$$

(iv) there exists a function $\vartheta \in L^\infty(T)_+$ such that $\vartheta(t) \leq \beta$ a.e. on T, $\vartheta \neq \beta$ and

$$\limsup_{x \to +\infty} \frac{f(t, x)}{x} \leq \vartheta(t) \ \text{ uniformly for a.a. } t \in T;$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

(v) there exist functions $\eta, \widehat{\eta} \in L^\infty(T)_+$ and an integer $n \geq 0$ such that

$$\mu_{2n} + \beta \leq \eta(t) \leq \widehat{\eta}(t) \leq \mu_{2n+2} + \beta, \ \eta \neq \mu_{2n} + \beta \text{ and } \widehat{\eta} \neq \mu_{2n+2} + \beta$$

and

$$\eta(t) \leq \liminf_{x \to -\infty} \frac{f(t,x)}{x} \leq \limsup_{x \to -\infty} \frac{f(t,x)}{x} \leq \widehat{\eta}(t)$$
uniformly for a.a. $t \in T$;

(vi) there exists an integer $k \geq 0$

$$\mu_{2k} + \beta \leq f'_x(t,0) = \lim_{x \to 0} \frac{f(t,x)}{x} \leq \mu_{2k+2} + \beta$$
uniformly for a.a. $t \in T$,
$$\mu_{2k} + \beta \neq f'_x(\cdot,0), \ \mu_{2k+2} + \beta \neq f'_x(\cdot,0),$$
while if $k = 0$, then $\mu_2 + \beta < \operatorname{ess\,inf} f'_x(\cdot,0)$,

and $\frac{1}{2}\eta(t)x^2 \leq F(t,x)$

for a.a. $t \in T$, all $x \leq 0$, with $\eta \in L^\infty(T)_+$, as in **H(f)**(v);

(vii) for almost all $t \in T$ and all $x > 0$, we have $f(t,x) \geq 0$.

EXAMPLE 5.1. The following function satisfies Hypotheses **H(f)**. For the sake of simplicity, we drop the t-dependence:

$$f(x) = \begin{cases} \eta x & \text{if } x < 0, \\ \eta \ln(1+x) & \text{if } x \geq 0, \end{cases} \quad \text{with } \mu_{2n} + \beta < \eta < \mu_{2n+2} + \beta.$$

We introduce the functional $\varphi : W^{1,2}_{\text{per}}(0,b) \longrightarrow \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{2}\|x'\|_2^2 + \frac{\beta}{2}\|x\|_2^2 - \int_0^b F\big(t, x(t)\big)\mathrm{d}t \quad \text{for all } x \in W^{1,2}_{\text{per}}(0,b).$$

Note that in this case $\varphi \in C^2\big(W^{1,2}_{\text{per}}(0,b)\big)$.

PROPOSITION 5.2. **If** *Hypotheses* **H(f)** *hold,* **then** $C_m(\varphi, \infty) = 0$ *for all* $m \geq 0$.

Proof. Let $h \in L^\infty(T)_+$ and $h \neq 0$ and consider the one-parameter family of C^1 functionals $\varphi_t : W^{1,2}_{\text{per}}(0,b) \longrightarrow \mathbb{R}$, $0 \leq t \leq 1$, defined by

$$\varphi_t(x) = \frac{1}{2}\|x'\|_2^2 + \frac{\beta}{2}\|x\|_2^2 - t\int_0^b F\big(t, x(t)\big)\mathrm{d}t - \frac{1-t}{2}\int_0^b \eta(x^-)^2 \mathrm{d}t$$
$$+ (1-t)\int_0^b hx\mathrm{d}t, \quad \text{for all } x \in W^{1,2}_{\text{per}}(0,b).$$

Claim: There exists $R > 0$ such that $\inf\big[\|\varphi'_t(x)\| : t \in [0,1], \ \|x\| > R\big] > 0$.

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

We proceed by contradiction. So, suppose that the Claim is not true. Then we can find sequences $\{t_n\}_{n\geq 1} \subseteq [0, 1]$ and $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(0, b)$ such that

$$t_n \longrightarrow t \in [0, 1], \ \ \|x_n\| \longrightarrow +\infty \ \text{ and } \ \varphi'_{t_n}(x_n) \longrightarrow 0 \ \text{ in } \ W^{1,2}_{\text{per}}(0, b)^*.$$

So, we have

$$\begin{aligned}
&\left|\langle\varphi'_{t_n}(x_n), v\rangle\right| \leq \varepsilon_n\|v\| \quad \text{for all } \ v \in W^{1,2}_{\text{per}}(0, b) \ \text{ with } \ \varepsilon_n \downarrow 0,\\
&\quad\Rightarrow \Big| \langle A(x_n), v\rangle + \beta\int_0^b |x_n|^{p-2}x_n v\mathrm{d}t - t_n\int_0^b f(t, x_n)v\mathrm{d}t + (1-t_n)\int_0^b \eta(x_n^-)^2\mathrm{d}t\\
&\qquad + (1-t_n)\int_0^b hv\mathrm{d}t\Big| \leq \varepsilon_n\|v\|. \qquad (88)
\end{aligned}$$

In (88) we use as a test function $v = x_n^+ \in W^{1,2}_{\text{per}}(0, b)$. We obtain

$$\Big|\|(x_n^+)'\|_2^2 + \beta\|x_n^+\|_2^2 - t_n\int_0^b f(t, x_n)x_n^+\mathrm{d}t + (1-t_n)\int_0^b hx_n^+\mathrm{d}t\Big| \leq \varepsilon_n\|x_n^+\|. \qquad (89)$$

Hypotheses **H(f)**(iii), (iv) and (vii) imply that given $\varepsilon > 0$, we can find $\xi_\varepsilon \in L^1(T)_+$ such that

$$\begin{aligned}
&0 \leq f(t, x) \leq \big(\vartheta(t) + \varepsilon\big)x + \xi_\varepsilon(t) \ \text{ for a.a. } t \in T \text{ and all } x \geq 0,\\
&\quad\Rightarrow 0 \leq f(t, x)x \leq \big(\vartheta(t) + \varepsilon\big)x^2 + \xi_\varepsilon(t)x \ \text{ for a.a. } t \in T \text{ and all } x \geq 0. \qquad (90)
\end{aligned}$$

Using (90) in (89), we obtain

$$\begin{aligned}
&\|(x_n^+)'\|_2^2 + \beta\|x_n^+\|_2^2 - \int_0^b \vartheta(x_n^+)^2\mathrm{d}t - \varepsilon\|x_n^+\|_2^2 \leq \gamma\ \|x_n^+\| \ \text{ for some } \gamma > 0,\\
&\quad \text{all } n \geq 1 \ \ (\text{since } h \geq 0 \ \text{ and } \ 0 \leq t_n \leq 1),\\
&\quad \Rightarrow (\xi_0 - \varepsilon)\|x_n^+\|^2 \leq \gamma\ \|x_n^+\| \quad (\text{see Lemma 3.1}). \qquad (91)
\end{aligned}$$

Choosing $\varepsilon \in (0, \xi_0)$, from (91), we see that

$$\{x_n^+\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(0, b) \ \text{ is bounded.}$$

Since $\|x_n\| \longrightarrow \infty$, we must have $\|x_n^-\| \longrightarrow \infty$. We set $y_n = \frac{x_n^-}{\|x_n^-\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$, and so by passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \ \text{ in } \ W^{1,2}_{\text{per}}(0, b) \ \text{ and } \ y_n \longrightarrow y \ \text{ in } \ C(T).$$

In (88) we use the test function $v = y_n - y \in W^{1,2}_{\text{per}}(0, b)$ and divide with $\|x_n\|$. Exploiting the boundedness of $\{x_n^+\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(0, b)$ and the boudedness of $\{\frac{f(\cdot, -x_n^-(\cdot))}{\|x_n^-\|}\} \subseteq L^2(T)$ (see Hypotheses **H(f)**(iii) and (v)), we obtain

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0. \qquad (92)$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

Invoking Lemma 3.2, from (92) we infer that

$$y_n \longrightarrow y \ \text{ in } \ W^{1,2}_{\mathrm{per}}(0,b), \ \text{ i.e. } \ \|y\| = 1. \tag{93}$$

Moreover, arguing as in the proof of Proposition 4.3 (Claim 2), we can show that

$$\left\{\frac{f\big(\cdot, -x_n^-(\cdot)\big)}{\|x_n^-\|}\right\} \longrightarrow g \ \text{ in } \ L^1(T) \tag{94}$$

with $g(t) = -\xi(t)y(t)$, $\xi \in L^\infty(T)_+$, $\eta(t) \le \xi(t) \le \widehat{\eta}(t)$ a.e. on T.

Hence, if we divide (88) with $\|x_n^-\|$, pass to the limit as $n \to \infty$ and use (93) and (94), we obtain

$$\begin{aligned} &\langle A(y), v\rangle + \beta\int_0^b yv\mathrm{d}t = \int_0^b \big(t\xi + (1-t)\eta\big)yv\mathrm{d}t \quad \text{for all } \ v \in W^{1,2}_{\mathrm{per}}(0,b),\\ &\Rightarrow A(y) + \beta y = \widehat{\xi}y \ \text{ with } \ \widehat{\xi} = t\xi + (1-t)\eta \in L^\infty(T)_+. \end{aligned} \tag{95}$$

From (95), it follows that

$$\left\{\begin{array}{l} -y''(t) + \beta y(t) = \widehat{\xi}(t)y(t) \quad \text{a.e. on } T,\\ y(0) = y(b), \ \ y'(0) = y'(b). \end{array}\right\} \tag{96}$$

Invoking Proposition 2.4, from (96) we infer that $y = 0$, a contradiction to (93). So, the Claim is true.

Clearly, we also have

$$\inf\Big[\varphi_t(y) : t \in [0,1], \ \|y\| \le R\Big] > -\infty.$$

Finally note that $x \longrightarrow \partial_t\varphi_t(x)$ and $x \longrightarrow \varphi_t'(x)$ are both locally Lipschitz maps. Therefore, we can apply Lemma 2.4 of Perera and Schechter [**23**] and can obtain

$$C_m(\varphi_0, \infty) = C_m(\varphi_1, \infty) \quad \text{for all } \ m \ge 0. \tag{97}$$

We have

$$\begin{aligned} &\varphi_0(x) = \frac{1}{2}\|x'\|_2^2 + \frac{\beta}{2}\|x\|_2^2 - \frac{1}{2}\int_0^b \eta(x^-)^2\mathrm{d}t + \int_0^b hx\mathrm{d}t\\ &\text{and } \ \varphi_1(x) = \varphi(x) \quad \text{for all } \ x \in W^{1,2}_{\mathrm{per}}(0,b). \end{aligned}$$

Let $x \in W^{1,2}_{\mathrm{per}}(0,b)$, be a critical point of φ_0. Then

$$\begin{aligned} &\varphi_0'(x) = 0,\\ &\Rightarrow A(x) + \beta x = -\eta(x^-) - h. \end{aligned} \tag{98}$$

On (98), we act with the test function $x^+ \in W^{1,2}_{\mathrm{per}}(0,b)$. Then

$$\begin{aligned} &\|(x^+)'\|_2^2 + \beta\|x^+\|_2^2 \le 0 \quad (\text{since } \ h \ge 0)\\ &\Rightarrow x^+ = 0, \ \text{ i.e. } \ x \le 0, \ \ x \ne 0 \quad (\text{see (98) and recall } \ h \ne 0). \end{aligned}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

Let $u = -x \geq 0$. Then from (98), we have

$$A(u) = (\eta - \beta)u + h,$$

$$\Rightarrow \left\{ \begin{array}{l} -u''(t) = \big(\eta(t) - \beta\big)u(t) + h(t) \quad \text{a.e. on } T, \\ u(0) = u(b),\ u'(0) = u'(b). \end{array} \right\} \tag{99}$$

As in the proof of Proposition 4.5, using the non-linear strong maximum principle of Vazquez [**24**], since $u \neq 0$, we obtain $u \in \operatorname{int}\widehat{C}_+$. Let $v \in C^1(T)$, $v(0) = v(b)$, $v'(0) = v'(b)$ and $v \geq 0$ and set

$$R(u, v)(t) = v'(t) - u'(t)\Big(\frac{v^2}{u}\Big)'(t) \quad \text{for all } \ t \in [0, 1].$$

Then, from Picone's identity (see Allegretto and Huang [**3**] and Gasiński and Papageorgiou [**15**], p. 785), we have

$$\begin{aligned} 0 &\leq \int_0^b R(u, v)(t)\mathrm{d}t \\ &= \|v'\|_2^2 - \int_0^b u'(t)\Big(\frac{v^2}{u}\Big)'(t)\mathrm{d}t \\ &= \|v'\|_2^2 - \int_0^b -u''(t)\Big(\frac{v^2}{u}\Big)(t)\mathrm{d}t \quad \text{(by integration by parts)} \\ &= \|v'\|_2^2 - \int_0^b \big(\eta(t) - \beta\big)u(t)\Big(\frac{v^2}{u}\Big)(t)\mathrm{d}t - \int_0^b h(t)\Big(\frac{v^2}{u}\Big)(t)\mathrm{d}t \quad \text{(see (99))} \\ &= \|v'\|_2^2 - \int_0^b \big(\eta(t) - \beta\big)v(t)^2\mathrm{d}t - \int_0^b h(t)\Big(\frac{v^2}{u}\Big)(t)\mathrm{d}t. \end{aligned} \tag{100}$$

Let $v \equiv 1$. Then from (100), we have

$$0 \leq -\int_0^b \big(\eta(t) - \beta\big)\mathrm{d}t - \int_0^b h(t)\mathrm{d}t < 0,$$

a contradiction. Hence φ_0 has no critical points, and so

$$\begin{aligned} &C_m(\varphi_0, \infty) = 0 \quad \text{for all } \ m \geq 0, \\ \Rightarrow\ &C_m(\varphi, \infty) = 0 \quad \text{for all } \ m \geq 0 \quad \text{(see (97) and recall } \ \varphi_1 = \varphi). \end{aligned}$$

$\square$

PROPOSITION 5.3. **If** *Hypotheses* **H(f)** *hold,* **then** $C_m(\varphi, 0) = \delta_{m,k}\mathbb{Z}$ *for all* $m \geq 0$.

Proof. We know that $\varphi \in C^2\big(W^{1,2}_{\text{per}}(0, b)\big)$, and for all $u, v \in W^{1,2}_{\text{per}}(0, b)$, we have

$$\big\langle \varphi''(0)u, v\big\rangle = \int_0^b u'(t)v'(t)\mathrm{d}t + \beta\int_0^b u(t)v(t)\mathrm{d}t - \int_0^b f'_x(t, 0)u(t)v(t)\mathrm{d}t.$$

Hence, if $u \in \ker\varphi''(0)$, then

$$\left\{ \begin{array}{l} -u''(t) = \big(f'_x(t, 0) - \beta\big)u(t) \quad \text{a.e. on } T, \\ u(0) = u(b),\ u'(0) = u'(b). \end{array} \right\} \tag{101}$$

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

By virtue of Hypothesis **H(f)**(v) and Proposition 2.4, we have $u \equiv 0$. Therefore, 0 is a non-degenerate critical point, with Morse index k. Hence

$$C_m(\varphi, 0) = \delta_{m,k}\mathbb{Z} \quad \text{for all} \quad m \geq 0$$

(see Chang [**6**], p. 33, and Mawhin and Willem [**21**], p. 188). □

Now we are ready for multiplicity result concerning problem (87).

THEOREM 5.4. **If** *Hypotheses* **H(f)** *hold,* **then** *problem (87) has at least three non-trivial solutions*

$$x_0 \in \operatorname{int}\widehat{C}_+, \;\; v_0, \;\; y_0 \in C^1(T).$$

Proof. From Theorem 4.8, we already have two non-trivial solutions $x_0 \in \operatorname{int}\widehat{C}_+$ and $v_0 \in C^1(T)$. From Proposition 4.5, we know that $x_0 \in \operatorname{int}\widehat{C}_+$ is a local minimiser of φ. Hence

$$C_m(\varphi, x_0) = \delta_{m,0}\mathbb{Z} \quad \text{for all} \quad m \geq 0 \tag{102}$$

(see Chang [**6**], p. 33, and Mawhin and Willem [**21**], p. 175). Moreover, from the proof of Theorem 4.8, we have that v_0 is a critical point of φ of the mountain-pass type. Hence

$$C_m(\varphi, v_0) = \delta_{m,1}\mathbb{Z} \quad \text{for all} \quad m \geq 0 \tag{103}$$

(see Chang [**6**], p. 9, and Mawhin and Willem [**21**], p. 195).

Suppose that $\{0, x_0, v_0\}$ are the only critical points of φ. Then from Propositions 5.2 and 5.3, from (102) and (103) and from the Poincaré–Hopf formula (see (5)), we have

$$\begin{aligned} &(-1)^0 + (-1)^1 + (-1)^k = 0, \\ &\Rightarrow (-1)^k = 0, \quad \text{a contradiction.} \end{aligned}$$

Hence, φ has one more critical point $y_0 \in W^{1,2}_{\text{per}}(0, b)$ distinct from $\{0, x_0, v_0\}$. Then $y_0 \in C^1(T)$ and solves (87). □

REFERENCES

1. S. Aizicovici, N. S. Papageorgiou and V. Staicu, Periodic solutions for second order differential inclusions with the p-Laplacian, *J. Math. Anal. Appl.* **322** (2006), 913–929.

2. S. Aizicovici, N. S. Papageorgiou and V. Staicu, Multiple nontrival solutions for nonlinear periodic problems with the p-Laplacian, *J. Diff. Eq.* **243** (2007), 504–535.

3. W. Allegretto and Y. Huang, A Picone's identity for the p-Laplacian and applications, *Nonlin. Anal.* **32** (1998), 819–830.

4. T. Bartsch and S. Li, Critical point theory for asymptotically quadratic functionals and applications to problems with resonance, *Nonlin. Anal.* **28** (1997), 419–441.

5. K. C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* **80** (1981), 102–129.

6. K. C. Chang, *Infinite dimensional Morse theory and multiple solution problems* (Birkhäuser, Boston, 1993).

7. F. Clarke, A new approach to Lagrange multipliers, *Math. Oper. Res.* **1** (1976), 165–174.

8. F. H. Clarke, *Optimization and nonsmooth analysis* (Wiley, New York, 1983).

https://doi.org/10.1017/S0017089509990346 Published online by Cambridge University Press

9. J. N. Corvellec, On the second deformation lemma, *Topol. Meth. Nonlin. Anal.* **17** (2001), 55–66.

10. M. A. del Pino, R. Manásevich and A. Murúa, Existence and multiplicity of solutions with prescribed period for a second order quasilinear ODE, *Nonlin. Anal.* **18** (1992), 79–92.

11. Z. Denkowski, S. Migórski and N. S. Papageorgiou, *An introduction to nonlinear analysis: applications* (Kluwer Academic, Boston, 2003).

12. P. Drabek and R. Manásevich, On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian, *Diff. Integral Eq.* **12** (1999), 773–788.

13. L. Gasiński and N. S. Papageorgiou, On the existence of multiple periodic solutions for equations driven by the p-Laplacian and a nonsmooth potential, *Proc. Edinburgh Math. Soc.* **46** (2003), 229–249.

14. L. Gasiński and N. S. Papageorgiou, *Nonsmooth critical point Ttheory and nonlinear boundary value problems* (Chapman & Hall/CRC Press, Boca Raton, FL, 2005).

15. L. Gasiński and N. S. Papageorgiou, *Nonlinear analysis* (Chapman & Hall/CRC Press, Boca Raton, Fl, 2006).

16. D. Kandilakis, N. Kourogenis and N. S. Papageorgiou, Two nontrivial critical points for nonsmooth functionals via local linking and applications, *J. Global Optim.* **34** (2006), 219–244.

17. W. Magnus and S. Winkler, *Hill's equation* (Dover, New York, 1979).

18. P. Manásevich and J. Mawhin, Boundary value problems for nonlinear perturbations of vector p-Lapalcian-like operators, *J. Kor. Math. Soc.* **37** (2000), 665–685.

19. J. Mawhin, *Topological degree methods in nonlinear boundary value problems*, Regional Conference Series in Mathematics, Vol. 40 (AMS, Providence, RI, 1979).

20. J. Mawhin, Periodic solutions of systems with p-Lapalcian-like operators. In *Nonlinear Analysis and Its Applications to Differential Equations (Lisbon 1998)* (Ramos, M. and Sanchez, L., Editors) (Birkhäuser Verlag, Boston, MA, 2001), 37–63.

21. J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, vol. 74, (Springer, New York, 1989).

22. E. H. Papageorgiou and N. S. Papageorgiou, Two nontrivial solutions for quasilinear periodic problems, *Proc. AMS* **132** (2004), 429–434.

23. K. Perera and M. Schechter, Solution of nonlinear equations having asymptotic limits at zero and infinity, *Calc. Var. Partial Diff. Eq.* **12** (2001), 359–369.

24. J. Vazquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12** (1984), 191–202.

25. X. Yang, Multiple periodic solutions of a class of p-Laplacian, *J. Math. Anal. Appl.* **314** (2006), 17–29.

26. M. Zhang, The rotation number approach to eigenvalues of the one-dimensional p-Laplacian with periodic potentials, *J. Lond. Math. Soc.* **64** (2001), 125–143.