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# ABSTRACT HARMONIC ANALYSIS OF GENERALISED FUNCTIONS ON LOCALLY COMPACT SEMIGROUPS WITH APPLICATIONS TO INVARIANT MEANS

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**Abstract**

Let $S$ be a locally compact semigroup and $M(S)$ its measure algebra. It is shown that the dual $M(S)^*$ is isometrically order isomorphic to the space $GL(S)$ of all generalised functions on $S$ first introduced by Šreĭder (1950). Moreover, convolutions of elements in each of the spaces $M(S)^*$ and $GL(S)$ can be defined in such a way that the above isomorphism preserves convolutions. These results on representation of functionals in $M(S)^*$ by generalised functions practically open up a new chapter in abstract harmonic analysis. As an example, some applications to invariant means on locally compact semigroups are given.

## 1. Introduction

Let $S$ be a locally compact semigroup with jointly continuous multiplication and $M(S)$ its measure algebra with convolution as multiplication. In this paper, we show that the dual $M(S)^*$ is isometrically order isomorphic to the space $GL(S)$ of all generalised functions on $S$ introduced by Šreĭder (1950). Moreover, convolutions of elements in each of the spaces $M(S)^*$ and $GL(S)$ by measures in $M(S)$ can be defined in such a way that the isomorphism preserves convolutions (see §2 for definitions and details). As a consequence, we prove that $S$ is left amenable (i.e. $M(S)^*$ has a topological left invariant mean) if and only if $GL(S)^*$ has a topological left invariant mean. Other results in this direction are also obtained.

## 2. Generalised functions

For basic notations and terminologies on integration over locally compact space, we shall follow Hewitt and Ross (1963) unless stated otherwise.

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Let $S$ be a locally compact space (no semigroup structure as yet) and $M(S)$ the Banach space of all bounded regular Borel (signed) measures on $S$ with total variation norm. For each $\mu \in M^+(S) = \{\mu \in M(S): \mu \geqq 0\}$, let $L_\infty(\mu)$ be the Banach space of all bounded Borel measurable (real-valued) functions on $S$ with essential supremum norm $\|f\|_{\mu,\infty} = \inf_{\mu(N)=0} \sup_{x \notin N} |f(x)| = \inf\{\alpha \geqq 0: \{x \in S: |f(x)| > \alpha\}$ is $\mu$-null$\}$ [see Hewitt and Ross (1963), §12.11]. Note that null sets and locally null sets for $\mu$ are the same since measures in $M(S)$ are finite. Hence two functions define the same class in $L_\infty(\mu)$ if they are equal $\mu$-*a.e.* (that is, almost everywhere with respect to $\mu$). If $\mu \in M(S)$ and $f \in L_\infty(|\mu|)$, we write $\|f\|_{\mu,\infty}$ for $\|f\|_{|\mu|,\infty}$ for brevity.

Consider the product linear space $\Pi\{L_\infty(|\mu|): \mu \in M(S)\}$. An element $f = (f_\mu)_{\mu \in M(S)}$ in this product is called a generalised function on $S$ if the following conditions are satisfied:

(a) $\|f\| = \sup_\mu \|f_\mu\|_{\mu,\infty} < \infty$ where the supremum is taken over all $\mu \in M(S)$.

and

(b) If $\mu, \nu \in M(S)$ and $\mu \ll \nu$, then $f_\mu = f_\nu$ $|\mu|$-*a.e.*

Here $\mu \ll \nu$ means $\mu$ is absolutely continuous with respect to $\nu$, that is, $|\mu|$ is absolutely continuous with respect to $|\nu|$ in the sense of Hewitt and Ross (1963, §14.20). Notice that if condition (b) holds for a pair of functions $f_\mu$ and $f_\nu$, then the same holds for any other pair $f'_\mu$, $f'_\nu$ such that $f_\mu$, $f'_\mu$ belong to the same equivalence class in $L_\infty(|\mu|)$ and $f_\nu$, $f'_\nu$ belong to the same equivalence class in $L_\infty(|\nu|)$. This is because $|\nu|(N) = 0$ implies $|\mu|(N) = 0$ for any Borel set $N$. Therefore $f'_\mu$ and $f'_\nu$ determine the same class in $L_\infty(|\mu|)$.

Let $GL(S)$ denote the linear subspace of all generalised functions on $S$. It is straightforward to show that $GL(S)$ is a Banach space with norm $\|f\| = \sup_\mu \|f_\mu\|_{\mu,\infty}$. Moreover, because of condition (b), the same norm is also given by $\|f\| = \sup_{\|\mu\| \leqq 1} \|f_\mu\|_{\mu,\infty}$. Since if $\mu \in M(S)$, $\mu \neq 0$, then $\nu = \mu/\|\mu\| \ll \mu$, $\mu \ll \nu$ and $\nu$ has norm 1.

We introduce an order in $GL(S)$ by saying that a generalised function $f$ is non-negative ($f \geqq 0$) if for each $\mu \in M(S)$, $f_\mu \geqq 0$ in $L_\infty(|\mu|)$ (That is $f_\mu \geqq 0$ $|\mu|$-*a.e.*). The generalised function $f$ such that $f_\mu = 1$ for each $\mu \in M(S)$ is again denoted by 1, as is the functional $F \in M(S)^*$ such that $F(\mu) = \int 1 d\mu = \mu(S)$, $\mu \in M(S)$.

The next theorem is due to Šreĭder (1950) who first proved it for locally compact abelian groups (with countable basis). The general case is proved in exactly the same way with an elegant use of the Radon–Nikodym Theorem. We include the proof here for completeness.

THEOREM 2.1. (Šreĭder, 1950). *For each bounded linear functional $F \in M(S)^*$, there is a unique generalised function $f \in GL(S)$ such that*

$$F(\mu) = \int f_\mu d\mu \text{ for any } \mu \in M(S).$$

*Moreover* $\|F\| = \|f\|$.

PROOF. For each $\mu \in M(S)$, $F$ induces a bounded linear functional $F_\mu$ on $\{\nu \in M(S): \nu \ll |\mu|\} = L_1(|\mu|)$ by Radon–Nikodym Theorem. Hence there is a function $f_\mu \in L_\infty(|\mu|) = L_1(|\mu|)^*$ such that $F_\mu(\nu) = F(\nu) = \int f_\mu d\nu$ for any $\nu \in L_1(|\mu|)$. In particular $F(\mu) = \int f_\mu d\mu$. We claim that $f = (f_\mu)_{\mu \in M(S)}$ is a generalised function. Let $\mu, \nu \in M(S)$ and $\mu \ll \nu$. For any $\sigma \in L_1(|\mu|)$, we have $\sigma \ll \mu$ and $\sigma \ll \nu$.
Hence

$$\int f_\mu d\sigma = F_\mu(\sigma) = F(\sigma) = F_\nu(\sigma) = \int f_\nu d\sigma.$$

Therefore $f_\mu = f_\nu$ $|\mu|$-*a.e.*

Also, for any $\mu \in M(S)$, $\|f_\mu\|_{\mu,\infty} = \|F_\mu\| = \sup\{|F_\mu(\nu)|: \nu \ll \mu, \|\nu\| \leqq 1\} \leqq \sup\{|F(\nu)|: \|\nu\| \leqq 1\} \leqq \|F\|$. Thus $f \in GL(S)$ and $\|f\| \leqq \|F\|$. On the other hand $\|F\| = \sup_{\|\mu\| \leq 1} |F(\mu)| = \sup_{\|\mu\| \leq 1} |\int f_\mu d\mu| \leqq \sup_{\|\mu\| \leq 1} \|f_\mu\|_{\mu,\infty} \cdot \|\mu\| \leqq \|f\|$. Consequently, $\|F\| = \|f\|$.

Finally, to show uniqueness, let $f, g \in GL(S)$ be such that $F(\mu) = \int f_\mu d\mu = \int g_\mu d\mu$ for any $\mu \in M(S)$. If $\sigma \ll \mu$, then

$$\int f_\mu d\sigma = \int f_\sigma d\sigma = \int g_\sigma d\sigma = \int g_\mu d\sigma$$

which implies that $f_\mu = g_\mu$ in $L_\infty(|\mu|)$.
Hence $f = g$.

As a consequence, we have the following

THEOREM 2.2. *Let $T: GL(S) \to M(S)^*$ be defined by $Tf(\mu) = \int f_\mu d\mu$, $\mu \in M(S)$, $f \in GL(S)$. Then $T$ is an isometric order preserving isomorphism of $GL(S)$ onto $M(S)^*$ such that $T(1) = 1$. Moreover $Tf(\nu) = \int f_\mu d\nu$* if $\nu \ll \mu$.

PROOF. Let $f \in GL(S)$. We first show that $Tf$ is linear. Observe that if $\mu, \nu \in M^+(S)$, then $\mu \ll \mu + \nu$, $\nu \ll \mu + \nu$ and $\mu \ll \alpha\mu$ if $\alpha > 0$. Therefore

$$Tf(\mu + \nu) = \int f_{\mu+\nu} d(\mu + \nu) = \int f_{\mu+\nu} d\mu + \int f_{\mu+\nu} d\nu$$
$$= \int f_\mu d\mu + \int f_\nu d\nu = Tf(\mu) + Tf(\nu)$$

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and

$$Tf(\alpha\mu) = \alpha\int f_{\alpha\mu}d\mu = \alpha\int f_\mu d\mu = \alpha(Tf)(\mu)$$

(which is obvious if $\alpha = 0$). Hence $Tf$ is additive and non-negative homogeneous on $M^+(S)$ and has a unique linear extension to $M(S)$ given by

$$\mu \to Tf(\mu_1) - Tf(\mu_2) = \int f_{\mu_1}d\mu_1 - \int f_{\mu_2}d\mu_2$$

$$= \int f_\mu d\mu_1 - \int f_\mu d\mu_2 = \int f_\mu d\mu = Tf(\mu)$$

where $\mu_1 = (|\mu| + \mu)/2$ and $\mu_2 = (|\mu| - \mu)/2$ (so that $\mu_1 \geqq 0$, $\mu_2 \geqq 0$, $\mu = \mu_1 - \mu_2$, $|\mu| = \mu_1 + \mu_2$ and $\mu_1 \ll \mu$, $\mu_2 \ll \mu$). Thus $Tf$ is linear. It is also bounded. In fact $|Tf(\mu)| \leqq \|f_\mu\|_{\mu,\infty} \cdot \|\mu\|$ for any $\mu \in M(S)$. Hence $\|Tf\| \leqq \sup_{\|\mu\| \leqq 1} \|f_\mu\|_{\mu,\infty} = \|f\|$. Clearly, the map $T$ is bounded linear. Theorem 2.1 shows that $T$ is onto and hence an isometry. Obviously, $T$ preserves order and $T(1) = 1$. This completes the proof.

Let $BM(S)$ be the Banach space of all bounded Borel measurable (real-valued) functions on $S$ with supremum norm. Each $f \in BM(S)$ can be regarded as a generalised function on $S$ if we define $f_\mu = f$ for any $\mu \in M(S)$. Thus $BM(S)$ can be embedded in $GL(S) = M(S)^*$. The restriction of the map $T$ to $BM(S)$ is precisely the same embedding of $BM(S)$ into $M(S)^*$ considered in Wong (1973, §5).

## 3. Convolutions

From now on, $S$ will be a locally compact semigroup with jointly continuous multiplication. For $f \in BM(S)$, $s \in S$, we define as usual the left and right translations $l_s$ and $r_s$ by $l_sf(t) = f(st)$, $r_sf(t) = f(ts)$, $t \in S$. Let $CB(S) \subset BM(S)$ be the space of all bounded continuous functions on $S$. It is known that both $BM(S)$ and $CB(S)$ are translation invariant. Let $LUC(S)$ be the space of left uniformly continuous functions in $CB(S)$. That is $f \in LUC(S)$ if $f \in CB(S)$ and the map $s \to l_sf$ is norm continuous from $S$ into $CB(S)$ with supremum norm. $RUC(S)$ is defined similarly. Again both $LUC(S)$ and $RUC(S)$ are translation invariant.

Let $f \in BM(S)$ and $\mu \in M(S)$. We define left and right convolutions $l_\mu$ and $r_\mu$ by $l_\mu f = \mu \odot f$ and $r_\mu f = f \odot \mu$ where

$$\mu \odot f(s) = \int f(ts)d\mu(t) = \int r_sf d\mu$$

$$f \odot \mu(s) = \int f(st)d\mu(t) = \int l_sf d\mu.$$

Note that $\int f(ts)d\mu(t)$ may not be defined for every $s \in S$. However, by Fubini's Theorem, for each $\nu \in M(S)$, it is defined everywhere outside some $|\nu|$-null set. Putting it equal to zero where it is not defined, we obtain a bounded Borel measurable function $\mu \odot f$ with $\|\mu \odot f\| \leqq \|\mu\| \cdot \|f\|$. This function depends on the $|\nu|$-null set but it is easy to see that $\mu \odot f$ determines uniquely an equivalence class in $L_\infty(|\nu|)$.

If $f$ belongs to $CB(S)$ or $LUC(S)$ or $RUC(S)$, then $\mu \odot f(s) = \int f(ts)d\mu(t)$ is defined everywhere on $S$ and is a function of the same type (see for example Williamson (1967) and Glicksberg (1961)). Similar remarks hold for $f \odot \mu(s) = \int f(st)d\mu(t)$.

Convolutions of functionals in $M(S)^*$ and measures in $M(S)$ are defined as in Wong (1969). If $F \in M(S)^*$, $\mu \in M(S)$, we define $l_\mu F = \mu \odot F$ and $r_\mu F = F \odot \mu$ by $\mu \odot F(\nu) = F(\mu * \nu)$ and $F \odot \mu(\nu) = F(\nu * \mu)$, $\nu \in M(S)$. Again $\|\mu \odot F\| \leqq \|\mu\| \cdot \|F\|$ and $\|F \odot \mu\| \leqq \|F\| \cdot \|\mu\|$.

To define convolutions of generalised functions and measures, we need the following result also due to Šreĭder (for commutative groups).

LEMMA 3.1. (Šreĭder) *Let $\mu$, $\nu$ and $\sigma$ be measures in $M^+(S)$. If $\mu \ll \nu$, then $\sigma * \mu \ll \sigma * \nu$.*

PROOF. Let $E$ be a Borel set with $\sigma * \nu(E) = 0$. If $\xi_E$ denotes the characteristic function of $E$, then by Hewitt and Ross (1963, Theorem 19.10),

$$\begin{aligned}\sigma * \nu(E) &= \int \xi_E d\sigma * \nu \\ &= \int \sigma(Et^{-1})d\nu(t) = 0.\end{aligned}$$

Hence $\sigma(Et^{-1}) = 0$, $\nu$-*a.e.* But $\mu \ll \nu$. Therefore $\sigma(Et^{-1}) = 0$ $\mu$-*a.e.* and $\sigma * \mu(E) = \int \sigma(Et^{-1})d\mu(t) = 0$. This completes the proof.

REMARKS. Theorem 19.10 as proved by Hewitt and Ross (1963) for locally compact groups is also valid for locally compact semigroups with jointly continuous multiplication. The proof carries over without change. This extension of Theorem 19.10 will be used again very often without mention. Of course, here $Et^{-1}$ is the set of all elements $s$ in $S$ such that $st$ belongs to $E$.

Now let $f \in GL(S)$ and $\mu \in M^+(S)$. Define $\mu \odot f \in \Pi\{L_\infty(|\nu|): \nu \in M(S)\}$ as follows:

If $\nu \in M^+(S)$, we let $(\mu \odot f)_\nu = \mu \odot f_{\mu * \nu} \in L_\infty(\nu)$. This is independent of the representative $f_{\mu * \nu}$ in $L_\infty(\mu * \nu)$. For if $f_{\mu * \nu} = f'_{\mu * \nu}$ $\mu * \nu$-*a.e.*, then for any $\sigma \in M^+(S)$, $\sigma \ll \nu$, we have, by Lemma 3.1, $\mu * \sigma \ll \mu * \nu$. Hence $f_{\mu * \nu} = f'_{\mu * \nu}$ $\mu * \sigma$-*a.e.* Therefore

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$$\int \mu \odot f'_{\mu*\nu} d\sigma = \int f'_{\mu*\nu} d\mu * \sigma$$

$$= \int f_{\mu*\nu} d\mu * \sigma = \int \mu \odot f_{\mu*\nu} d\sigma$$

for any $\sigma \in M^+(S)$, $\sigma \ll \nu$. This means that $\mu \odot f'_{\mu*\nu} = \mu \odot f_{\mu*\nu}$ $\nu$-*a.e.*

In general, if $\nu \in M(S)$, we define $(\mu \odot f)_\nu = (\mu \odot f)_{|\nu|}$. We claim that $\mu \odot f$ is a generalised function.

Suppose $\sigma, \nu \in M^+(S)$ and $\sigma \ll \nu$. Then $\mu * \sigma \ll \mu * \nu$ by Lemma 3.1. Hence $f_{\mu*\sigma} = f_{\mu*\nu}$ $\mu * \sigma$-*a.e.* Now for each $\tau \in M^+(S)$, $\tau \ll \sigma$, then $\mu * \tau \ll \mu * \sigma$ and so $f_{\mu*\sigma} = f_{\mu*\nu}$ $\mu * \tau$-*a.e.* Consequently

$$\int \mu \odot f_{\mu*\sigma} d\tau = \int f_{\mu*\sigma} d\mu * \tau$$

$$= \int f_{\mu*\nu} d\mu * \tau = \int \mu \odot f_{\mu*\nu} d\tau.$$

This implies that $\mu \odot f_{\mu*\sigma} = \mu \odot f_{\mu*\nu}$ $|\sigma|$-*a.e.* or $(\mu \odot f)_\sigma = (\mu \odot f)_\nu$ $|\sigma|$-*a.e.* The same is true if $\sigma, \nu$ are in $M(S)$.

On other hand, for each $\nu \in M(S)$, we have $\|(\mu \odot f)_\nu\|_{\nu,\infty} \leqq \|\mu\| \cdot \|f_{\mu*\nu}\|_{\mu*\nu,\infty}$. So $\mu \odot f$ is a generalised function and

$$\|\mu \odot f\| \leqq \|\mu\| \cdot \|f\|.$$

By similar arguments, it is easy to show that the map $\mu \to \mu \odot f$ is additive and non-negative homogeneous on $M^+(S)$ into $GL(S)$ and hence has a unique linear extension also denoted by $\mu \odot f = l_\mu f$. Clearly $\mu \odot f$ is bilinear and $\|\mu \odot f\| \leqq \|\mu\| \cdot \|f\|$.

Similarly, we can define $f \odot \mu = r_\mu f$ and obtain similar results.

THEOREM 3.2. *The isomorphism* $T: GL(S) \to M(S)^*$ *commutes with convolutions. More precisely* $T(\mu \odot f) = \mu \odot Tf$ *and* $T(f \odot \mu) = Tf \odot \mu$ *for any* $f \in GL(S)$ *and* $\mu \in M(S)$.

PROOF. If $f \in GL(S)$ and $\mu \in M(S)$, we have

$$T(\mu \odot f)(\nu) = \int (\mu \odot f)_\nu d\nu$$

$$= \int \mu \odot f_{\mu*\nu} d\nu = \int f_{\mu*\nu} d\mu * \nu$$

$$= Tf(\mu * \nu) = (\mu \odot Tf)(\nu)$$

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for any $\nu \in M(S)$. Hence $T(\mu \odot f) = \mu \odot Tf$. Similarly for $T(f \odot \mu) = Tf \odot \mu$.

## 4. Applications to invariant means

A linear functional $M$ on $M(S)^*$ is called a mean if $M(F) \geqq 0$ whenever $F \geqq 0$ and $M(1) = 1$. It is called topological left invariant if $M(\mu \odot F) = M(F)$ for any $F \in M(S)^*$ and $\mu \in M_0(S) = \{\mu \in M(S): \mu \geqq 0 \text{ and } \|\mu\| = 1\}$. Topological left invariant means on $CB(S)$ or $LUC(S)$ or $RUC(S)$ can be defined in a similar way.

A linear functional $m$ in $GL(S)^*$ is called a mean if $m(f) \geqq 0$ whenever $f \geqq 0$ (in $GL(S)$) and $m(1) = 1$. It is topological left invariant if $m(\mu \odot f) = m(f)$ for any $f \in GL(S)$ and $\mu \in M_0(S)$. Since $T$ is an isometric order isomorphism of $GL(S)$ onto $M(S)^*$ which commutes with left convolution and $T(1) = 1$, it follows that $M(S)^*$ has a topological left invariant mean ($TLIM$) if and only if $GL(S)$ has one. This gives yet another characterisation of a locally compact left amenable semigroup (i.e. one for which $M(S)^*$ has a $TLIM$, see Wong (1969) for more details). We summarise this discussion in the following:

THEOREM 4.1. *$GL(S)$ has a TLIM if and only if $M(S)^*$ has a TLIM. In this case, the adjoint $T^*$ of $T$ maps the set of all TLIM on $M(S)^*$ onto that of $GL(S)$.*

For each $\mu \in M(S)$, let $L_\infty(|\mu|) = L_1(|\mu|)^*$ be endowed with the weak* topology. The product weak* topology of $\Pi\{L_\infty(|\mu|): \mu \in M(S)\}$ is called the weak* operator topology.

THEOREM 4.2. *The map $T: GL(S) \to M(S)^*$ is a homeomorphism when $GL(S)$ has the weak* operator topology and $M(S)^*$ has the weak* topology.*

PROOF. Suppose $f^\alpha$ is a net in $GL(S)$ such that $f^\alpha \to f$ in weak* operator topology of $GL(S)$. Let $\mu \in M(S)$, then $f^\alpha_\mu \to f_\mu$ weak* in $L_\infty(|\mu|)$. In particular, $\int f^\alpha_\mu \, d\mu \to \int f_\mu \, d\mu$. Hence $Tf^\alpha(\mu) \to Tf(\mu)$ for each $\mu \in M(S)$ or $Tf^\alpha \to Tf$ weak* in $M(S)^*$. Conversely, assume this is true. Let $\mu \in M^+(S)$ and $\nu \in L_1(\mu)$. Then $\nu \ll \mu$ and

$$\int f^\alpha_\mu d\nu = \int f^\alpha_\nu d\nu = Tf^\alpha(\nu) \to Tf(\nu) = \int f_\nu d\nu = \int f_\mu d\nu.$$

That is $f^\alpha_\mu \to f_\mu$ weak* in $L_\infty(\mu)$. For general $\mu \in M(S)$, we have $f^\alpha_\mu = f^\alpha_{|\mu|} \to f_{|\mu|} = f_\mu$ weak* in $L_\infty(|\mu|)$. This completes the proof.

In Wong (1969), a locally compact semigroup $S$ is called topological right stationary if for each $F \in M(S)^*$, there is a net $\mu_\alpha \in M_0(S)$ such that $F \odot \mu_\alpha$

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converges weak* to a constant function in $M(S)^*$. It is shown that $S$ is topological right stationary if and only if $M(S)^*$ has a *TLIM* Wong (1969, Theorem 3.1). Therefore we have

THEOREM 4.3. *The following statements are equivalent*:

(a) *$M(S)^*$ has a TLIM*

(b) *$S$ is topological right stationary*

(c) *$GL(S)$ has a TLIM*

(d) *For each $f \in GL(S)$, there is a net $\mu_\alpha \in M_0(S)$ such that $f \odot \mu_\alpha$ converges to a constant function in weak* operator topology of $GL(S)$.*

PROOF. The equivalence of (a) and (b) follows from Wong (1969, Theorem 3.1) and that of (a) and (c) follows from Theorem 4.1. Also (b) and (d) are equivalent by Theorems 4.2 and 3.2 and the fact that $T(1) = 1$.

Next, we want to generalise a well-known result for locally compact groups which states that if $G$ is a locally compact group, then $L_\infty(G)$ has a topological left invariant mean if $LUC(G)$ has a topological left invariant mean (see Greenleaf (1969) where $LUC(G)$ is denoted by $UCB_r(G)$ and functions in $UCB_r(G)$ are called right uniformly continuous).

For locally compact semigroups of course, we consider the space $M(S)^*$ instead of $L_\infty(S)$ since the latter is not available in the absence of a Haar measure. (However for the group case, existence of *TLIM* on $L_\infty(G)$ or $M(G)^*$ are equivalent, see Wong (1969, Theorem 3.1). Also our result is valid for only a special class of locally compact semigroups which admit absolutely continuous probability measures.

A measure $\mu \in M(S)$ is called left absolutely continuous if the map $s \to \varepsilon_s * \mu$ of $S$ into $M(S)$ is norm continuous, where $\varepsilon_s$ is the Dirac measure at $s$. Let $M_a^l(S)$ denote the space of all left absolutely continuous measures in $M(S)$. In case $G$ is a locally compact group, $M_a^l(G) = M_a(G) = L_1(G)$. [See Hewitt and Ross (1963, §19.27) and Wong (1975).]

First we establish the following, a special case of which can be found in Hart (1970). The proof of the general case is the same.

LEMMA 4.4. *Let $\mu \in M_a^l(S) \cap M_0(S)$ and $\nu \in M^+(S)$. If $x \in \operatorname{supp} \nu$ (support of $\nu$), then $\varepsilon_x * \mu \ll \nu * \mu$.*

PROOF. Again we include the proof for completeness. Let $E$ be any Borel set, $x \in \operatorname{supp} \nu$ and assume $\varepsilon_x * \mu(E) > 0$. For any $s \in S$, $\varepsilon_s * \mu(E) = \int \xi_E(st) d\mu(t)$ and

$$\nu * \mu(E) = \int \int \xi_E(st) d\nu(s) d\mu(t) = \int \varepsilon_s * \mu(E) d\nu(s).$$

Now the function $s \to \varepsilon_s * \mu(E)$ is also continuous. Therefore there is some compact neighbourhood $K$ of $s$ such that $\varepsilon_s * \mu(E) \geqq \delta > 0$ for any $s \in K$. Hence

$$\nu * \mu(E) \geqq \int_K \varepsilon_s * \mu(E) d\nu(s) \geqq \delta \nu(K) > 0,$$

since $K$ contains an open set which intersects supp $\nu$.

LEMMA 4.5. *If* $F \in M(S)^*$, $\mu \in M_a^l(S) \cap M_0(S)$, *then* $F(\nu * \mu) = \int F(\varepsilon_s * \mu) d\nu(s)$ *for any* $\nu \in M(S)$.

PROOF. First observe that the function $s \to F(\varepsilon_s * \mu)$ is in $CB(S)$ and the integral is therefore finite. Let $f \in GL(S)$ be such that $F(\sigma) = \int f_\sigma d\sigma$, $\sigma \in M(S)$. Then for $\nu \in M^+(s)$,

$$\begin{aligned} F(\nu * \mu) &= \int f_{\nu*\mu}\, d\nu * \mu = \int \int f_{\nu*\mu}(st) d\mu(t) d\nu(s) \\ &= \int_{\text{supp } \nu} \int f_{\nu*\mu}(t) d(\varepsilon_s * \mu)(t) d\nu(s) \\ &= \int_{\text{supp } \nu} \int f_{\varepsilon_s*\mu}\, d\varepsilon_s * \mu d\nu(s) = \int F(\varepsilon_s * \mu) d\nu(s) \end{aligned}$$

where we have used the preceding Lemma in the fourth equality. It follows that the same is true for all $\nu \in M(S)$.

REMARKS. **1.** Lemma 4.5 is also proved in Baker and Baker (1972, Lemma 2.2) under slightly different assumption on $|\mu|$, namely, the continuity of the maps $s \to \varepsilon_s * |\mu|(K)$ and $s \to |\mu| * \varepsilon_s(K)$ for each compact set $K$ (continuity of $s \to |\mu| * \varepsilon_s(K)$ is really not needed). Whereas here we require the continuity of the map $s \to \varepsilon_s * \mu$ in the norm topology of $M(S)$ (note the presence of $\mu$ instead of $|\mu|$). From the proof of Lemma 4.4 (on which Lemma 4.5 depends), it is easy to see that all we need here is the continuity of the map $s \to \varepsilon_s * \mu(K)$ for each compact set $K$ since absolute continuity of measures can be defined in terms of compact sets [see Hewitt and Ross (1963, Theorem 14.19 and Definition 14.20)]. Of course, the function $F(\varepsilon_s * \mu)$ of Lemma 4.5 is no longer continuous, but is bounded measurable $|\nu|$-*a.e.* Also, the continuity of the map $s \to \varepsilon_s * |\mu|(K)$ implies that of $s \to \varepsilon_s * \mu(K)$ since $\mu \ll |\mu|$ [see Baker and Baker (1970, Theorem 3.2)]. However, the converse is not known [cf. Hart (1970, Lemma 3.5 and Theorem 3.8)].

**2.** For the special case of a locally compact abelian group, Saka (1974, Lemma 3) obtains the same result under yet another different but more general assumption on $|\mu|$. Namely, continuity of the map $s \to \varepsilon_s * |\mu|(C)$ for each $C$ in a pre-Raikov system for $M(S)$ [see Saka (1974) and Šreĭder (1950) for definition] which in particular includes the system of compact sets. Both Baker and Baker (1972) and Saka (1974) make use of this result to study properties of certain subalgebras of $M(S)$ analogous to those of group algebras (e.g. approximate identities, semi-characters and semi-simplicity).

We now present yet another application of Lemma 4.5:

THEOREM 4.6. *Let $S$ be a locally compact semigroup with jointly continuous multiplication such that $M_a^l(s) \cap M_0(S) \neq \emptyset$. Then $M(S)^*$ has a topological left invariant mean if and only if $RUC(S)$ has a topological left invariant mean.*

PROOF. Assume that $RUC(S)$ has a *TLIM*. Let $\mu \in M_a^l(S) \cap M_0(S)$ be fixed. For $F \in M(S)^*$, define $f(s) = F(\varepsilon_s * \mu)$. Then $f \in RUC(S)$ since $\| r_s f - r_t f \| \leqq \| F \| \cdot \| \varepsilon_s * \mu - \varepsilon_t * \mu \|$.

Let $m$ be a *TLIM* on $RUC(S)$. Define $M(F) = m(f)$. Clearly $M$ is a mean on $M(S)^*$. Now for each $\nu \in M_0(S)$, we have

$$\begin{aligned} \nu \odot f(s) &= \int f(ts) d\nu(t) = \int F(\varepsilon_{ts} * \mu) d\nu(t) \\ &= \int F(\varepsilon_t * \mu) d(\nu * \varepsilon_s)(t) = F(\nu * \varepsilon_s * \mu) \\ &= \nu \odot F(\varepsilon_s * \mu) \end{aligned}$$

by the preceding Lemma. Therefore $M(\nu \odot F) = m(\nu \odot f) = m(f) = M(F)$ for any $\nu \in M_0(S)$ and $F \in M(S)^*$ on $M$ is a *TLIM* on $M(S)^*$.

The converse is obvious (by restriction) and is true even if the assumption that $M_a^l(S) \cap M_0(S) \neq \emptyset$ is dropped. [See Wong (1975) for examples of locally compact semigroups which admit absolutely continuous probability measures].

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