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# Determining Fitting ideals of minus class groups via the equivariant Tamagawa number conjecture

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Abstract

We assume the validity of the equivariant Tamagawa number conjecture for a certain motive attached to an abelian extension $K/k$ of number fields, and we calculate the Fitting ideal of the dual of $cl_K^-$ as a Galois module, under mild extra hypotheses on $K/k$. This builds on concepts and results of Tate, Burns, Ritter and Weiss. If $k$ is the field of rational numbers, our results are unconditional.

## Introduction

Let $G$ be a finite abelian group, $K/k$ a $G$-Galois extension of number fields where $k$ is totally real and $K$ a CM field, and write $c$ for the unique complex conjugation in $G$. For every $\mathbb{Z}G$-module $M$ we let $M^- = R \otimes_{\mathbb{Z}G} M$, where $R$ is defined as $\mathbb{Z}[\frac{1}{2}][G]/(1+c)$. (This notation, which includes inversion of 2, is non-standard but practical.) If for example $M$ is a class group, then $M^-$ is just the non-2-part of the minus class group.

Our original goal was to calculate the initial Fitting ideal $\mathrm{Fitt}_R(cl_K^-)$ in terms of an appropriate generalisation of Stickelberger's ideal. (Higher Fitting ideals will never occur in this paper.) We start from the assumption that the equivariant Tamagawa number conjecture (ETNC) is true for the motive $h^0(K)$ with coefficients in $\mathbb{Z}G$, and we use techniques of Burns, Ritter and Weiss in order to obtain information on $cl_K^-$. This means that our results are conditional unless $k = \mathbb{Q}$ (cf. Theorem 1.1). As a main technical tool we use metrised complexes and their refined Euler characteristics, which lie in a relative $K$-group. Our calculations lead to an explicit ideal called $\mathrm{SKu}(K/k)$, which in the absolutely abelian case is related to Sinnott's version of Stickelberger's ideal, and which was first introduced by Kurihara in a slightly more special setting. We expected to end up with this type of ideal, but on the other side of the equation an unexpected twist occurs. Our final results (Theorems 8.5 and 8.8) say that $\mathrm{SKu}(K/k)^-$ is the Fitting ideal of the *Pontryagin dual* $cl_K^{\vee -}$ of $cl_K^-$. For cyclic $G$ this implies that $cl_K^-$ has the same Fitting ideal. Joint work with Kurihara (in preparation) actually shows that, in general, $\mathrm{Fitt}_R(cl_K^-)$ does differ from $\mathrm{Fitt}(cl_K^{\vee -})$, and that $\mathrm{Fitt}_R(cl_K^-)$ is in general not equal to $\mathrm{SKu}(K/k)^-$. On the other hand, it is remarkable that previous results of Kurihara in the cyclotomic setting do yield $\mathrm{Fitt}_R(cl_K^-)$ (no dual), and the ideal obtained is the same as in the present paper. We do not have a convincing explanation for this phenomenon yet.

We now give the precise statement of the main result (Theorem 8.8):

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*If (under the blanket assumptions on $K/k$ explained above) ETNC holds for the motive $h^0(K)$ with coefficients in $\mathbb{Z}[G]$ (more briefly, ETNC($h^0(K), \mathbb{Z}[G]$) holds), and if the $R$-module $\mu_{K,\mathrm{odd}}$ of odd-order roots of unity in $K$ is of projective dimension at most 1, then*

$$\mathrm{Fitt}_R(cl_K^{\vee -}) = \mathrm{SKu}(K/k)^-.$$

(Compare with the remark after Theorem 8.5.)

The ideal $\mathrm{SKu}(K/k)$, which will be called the Sinnott–Kurihara ideal, is defined in § 2, where we also explain its relation to other constructions; § 1 recalls Tate sequences and equivariant Tamagawa numbers; § 3 offers an outline describing the main steps of the central argument, and also explains the link between refined Euler characteristics and Fitting ideals in a special case. The details of our constructions are carried out in §§ 4–7. An important ingredient is a kind of multiplicativity property of the refined Euler characteristic (see § 7) proved by Burns. In § 8 we finally complete the transition from ETNC to Fitting ideals and prove our main result. The paper closes with an application towards Brumer's conjecture (Corollary 8.11) and a comparison with previous results of Kurihara.

Partial results on Fitting ideals of class groups, which do not involve the Pontryagin dual, were proved previously by the author [Gre00] (assuming cohomological triviality of $cl_K^-$) and Kurihara [Kur03a] (assuming $k = \mathbb{Q}$ and more conditions). The approach in [Gre00] was somewhat related to the present one, but certainly more naive; Kurihara's techniques were quite different. We should also mention a more recent result of Kurihara [Kur03b] which gives the Fitting ideal of the projective limit $\lim(cl_{K_n}\{p\}^{\vee -})$ attached to the cyclotomic tower $K_\infty/k$, under some hypotheses on $K$ and the prime $p$. This again indicates that duals of minus class groups are the 'better' objects to be studied.

We close this introduction by discussing some issues of notation. By $S$ we always denote a finite $G$-stable set of places of $K$, which contains $S_\infty$, the set of infinite places. The set of $k$-places below places in $S$ will consistently be written $S_k$. The symbol $S'$ will always denote a finite $G$-stable set of places of $K$ which contains $S$, and which is 'larger' in the sense of [RW96] (see also [Wei96]), which means: $S'$ contains all ramified primes; the $S'$-class group of $K$ is trivial; and $G$ is the union of the decomposition groups $G_{\mathfrak{P}}$ attached to $\mathfrak{P}$ running over $S'$. The final applications will use $S = S_\infty$. In a sense the whole point is that $S_\infty$ is almost never a 'larger' set, and all the work arises from the transition from $S'$ down to a smaller set $S$. We recall two standard items of notation: $E_S = \mathcal{O}_{K,S}^\times$ and $\Delta S$ is the kernel of the obvious sum map $\mathbb{Z}S \to \mathbb{Z}$. Both $E_S$ and $\Delta S$ come with a natural $\mathbb{Z}G$-module structure.

For duals we use the following convention: for any $\mathbb{Z}G$-module $M$, the $G$-action on $M^\vee = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is given by $(\gamma f)(x) = f(\gamma x)$ ($\gamma \in G, f \in M^\vee, x \in M$). This deviates from the standard convention, which has $f(\gamma^{-1}x)$ instead of $f(\gamma x)$, but it makes sense since $G$ is abelian throughout the paper, and it leads to the 'correct' final results.

Since we have to deal with a lot of fairly big diagrams of similar shape, we use the following *notational convention concerning suppression of zeros:* Without mention to the contrary, all rows or columns with three or four terms are supposed *to be exact*, and in all such rows and columns the *first* (respectively *last*) arrow is supposed to be *injective* (respectively *surjective*). The occasional exceptions to this convention (mainly bottom row of diagram D1 (see also (21)) and bottom row of the left part of (17)) will be clearly identified as such. In all cases, the obstruction to exactness is only a finite group.



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## 1. Tate sequences and refined Euler characteristics

Tate defined a canonical class $\tau = \tau_{S'} \in \mathrm{Ext}^2_{\mathbb{Z}G}(\Delta S', E_{S'})$ for 'larger' sets $S'$. Actually a slightly weaker hypothesis ($S'$ only 'large') is enough here, but we will not use this.

Burns [Bur01] establishes a four-term exact sequence of $\mathbb{Z}G$-modules

$$0 \to E_{S'} \to \Psi^0_{S'} \to \Psi^1_{S'} \to \Delta S' \to 0 \tag{1}$$

with the following properties: $\Psi^0_{S'}$ and $\Psi^1_{S'}$ are $G$-c.t. (short for 'cohomologically trivial'; equivalently, of finite projective dimension over $\mathbb{Z}G$), and the sequence represents the canonical class $\tau$ (see [Bur01], proof of Lemma 2.3.5, and [BF98], Theorem 3.2). Note that the latter property was *not used* in the construction of the sequence but had to be *proved* by a non-trivial argument. The construction uses derived functors, complexes and Verdier duality.

The existence of so-called Tate sequences

$$0 \to E_{S'} \to A \to B \to \Delta S' \to 0 \tag{2}$$

(that is, sequences having $A$ $G$-c.t., $B$ projective over $\mathbb{Z}G$, and representing $\tau$) has been known and used for a long time; see in particular the book [Wei96]. In [GRW98, GRW99] this sequence plus additional data is used to construct an element $\Omega_\varphi$ of the relative $K$-group $\mathrm{K}_0(\mathbb{Z}G, \mathbb{Q})$. The so-called lifted root number conjecture (LRNC) states that another element of that $K$-group, associated to $\Omega_\varphi$ by means of $L$-values and regulators, is zero. (Recall that we assume $G$ to be abelian; in general, root numbers intervene.) Some constructions of Ritter and Weiss will be extremely important for us, even though we will not refer to LRNC again and rather follow the setup and terminology of Burns in the sequel.

To do this, we have to review refined Euler characteristics. We stick to the setup in [Bur01] but will use some *ad hoc* terminology of our own. A 'metrised' complex either over $\mathbb{Z}G$ or over $R$ (context will tell) consists of two data: a complex in degrees 0 and 1,

$$A \to B, \tag{3}$$

together with an $\mathbb{R}G$-isomorphism,

$$\varphi : \mathbb{R} \otimes V \to \mathbb{R} \otimes U, \tag{4}$$

where both $A$ and $B$ are c.t. over $G$ and $U$ (respectively $V$) is the kernel (respectively cokernel) of $A \to B$. Alternatively one may write down an exact four-term sequence

$$0 \to U \to A \to B \to V \to 0$$

and consider the maps $U \to A$ (identifying $U$ with the kernel of $A \to B$) and $B \to V$ (identifying $V$ with the cokernel of $A \to B$) as part of the data. Since it is only the 'determinant' of $\varphi$ that counts, as shown by Burns, our terminology 'metrisation' may be justified since all we do is specify a measure (volume element) on $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R} \otimes V, \mathbb{R} \otimes U)$. But there will be no measure theory in our arguments.

The construction of [Bur01] associates to every metrised complex $E = (A \to B, \varphi)$ a refined Euler characteristic $\chi_{\mathrm{ref}}(E) \in \mathrm{K}_0(\mathbb{Z}G, \mathbb{R})$, which remains unchanged if $\varphi$ is changed by any automorphism of determinant 1 on either side. (The subscript in $\chi_{\mathrm{ref}}$ stands for 'refined' and is intended to leave unadorned $\chi$ free to be used for characters. We suppress the dependence on the base ring, $R$ or $\mathbb{Z}G$.) We will not review the complete construction of this, but the main idea is to 'transpose' $\varphi$ to an isomorphism $\tilde{\varphi} : \mathbb{R} \otimes B \to \mathbb{R} \otimes A$, and to put $\chi_{\mathrm{ref}}(A \to B, \varphi) = (B, \tilde{\varphi}, A)$ in the usual notation for explicit elements of $\mathrm{K}_0(\mathbb{Z}G, \mathbb{R})$. The work consists in describing the details of this translation process, which entails making some choices, and in showing that everything is well defined in the end. In any case, this is much simpler for abelian $G$ than in the general case done in [Bur01].

We need several properties of this construction. To get started, the most important one is the following (see [Bur01, Proposition 1.2.2(ii)]): Whenever $E = (A \to B, \varphi)$ and $E' = (A' \to B', \varphi)$ are metrised complexes fitting into a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & U & \longrightarrow & A & \longrightarrow & B & \longrightarrow & V & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & U & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & V & \longrightarrow & 0 \end{array} \tag{5}$$

and sharing the same $\varphi$, then $\chi_{\mathrm{ref}}(E) = \chi_{\mathrm{ref}}(E')$. This means that, if the metrisation ('trivialisation' in Burns's terminology) $\varphi : \mathbb{R} \otimes V \to \mathbb{R} \otimes U$ is given, we only have to know the class of $0 \to U \to A \to B \to V \to 0$ in $\mathrm{Ext}^2_{\mathbb{Z}G}(V, U)$ in order to know $\chi_{\mathrm{ref}}(A \to B, \varphi)$.

Now let $U = E_{S'}$, $V = \Delta S'$ and $\varphi^{-1} : \mathbb{R}E_{S'} \to \mathbb{R}\Delta S'$ the negative of the usual Dirichlet map, so $\varphi^{-1}(u) = -\sum_{v \in S'} \log |u|_v \cdot v$. Let $E$ be a metrised 4-sequence as follows. Take any 4-sequence with outer terms $E_{S'}$ (left) and $\Delta S'$ (right) and cohomologically trivial inner terms which represents the Tate canonical class $\tau$, and metrise it by $\varphi$. Then the ETN (equivariant Tamagawa number) attached to $h^0(K)$ is the following element of $\mathrm{K}_0(\mathbb{Z}G, \mathbb{R})$:

$$T\Omega(K/k, 0) := \psi^*_G(\chi_{\mathrm{ref}}(E) + \partial(L^*_{S'}(0)^\sharp)). \tag{6}$$

We repeat that we may take the Burns sequence (beginning of this section) or a Tate sequence as in [Wei96] (see above) as we please, it does not matter. But we have to explain some bits of notation. The map $\psi^*_G$ is a certain involution of $\mathrm{K}_0(\mathbb{Z}G, \mathbb{R})$, and we will forget about it at once since we are only interested in the nullity of $T\Omega(K/k, 0)$. The map $\partial$ is the usual connecting homomorphism $\mathbb{R}G^\times \to \mathrm{K}_0(\mathbb{Z}G, \mathbb{R})$ given by $\partial(x) = (\mathbb{Z}G, x, \mathbb{Z}G)$. Finally, the 'equivariant $L$-value at zero' $L^*_{S'}(0)^\sharp$ is, by definition, the unique element of $\mathbb{R}G$ which maps, under each character $\chi$ of $G$, to the leading coefficient

$$L^*_{S'}(0, \chi^{-1}) = \lim_{s \to 0} s^{-e(\chi)} L_{S'}(s, \chi^{-1}), \tag{7}$$

where $e(\chi)$ is the vanishing order of $L_{S'}(s, \chi^{-1})$ at $s = 0$. (This uniquely defines the equivariant $L$-value as an element of $\mathbb{C}G$, and it happens to lie in $\mathbb{R}G$.)

The equivariant Tamagawa number conjecture (ETNC) in this context simply states: *The element $T\Omega(K/k, 0)$ is zero.*

The following result, which can be seen as a starting point for this paper, was proved in [BG03a] with the exclusion of the 2-primary part. Flach [Fla02] extended the argument to cover the 2-primary part as well.

Theorem 1.1. *If $K$ is absolutely abelian, then $T\Omega(K/k, 0) = 0$.*

## 2. A variant of the Stickelberger ideal

We keep our assumptions concerning $K/k$. In this section, $\mathfrak{p}$ will always be a prime of $k$ that ramifies in $K/k$; in later sections, $\mathfrak{p}$ will lie in a fixed set $S'_k$ that contains all ramified primes. Here and later, we always fix a choice of a $K$-place $\mathfrak{P}$ above $\mathfrak{p}$.

We use the standard notation $G_\mathfrak{p}$, $G_{0,\mathfrak{p}}$ and $\bar{G}_\mathfrak{p} = G_\mathfrak{p}/G_{0,\mathfrak{p}}$ for, respectively, the decomposition group, inertia group, and residual group of $K/k$ at $\mathfrak{p}$. Write $F_\mathfrak{p}$ for a lift, chosen once for all times, of the (usual arithmetic) Frobenius $\mathrm{Frob}_\mathfrak{p} \in \bar{G}_\mathfrak{p}$ to $G_\mathfrak{p} \subset G$.

We will have to use idempotents of $\mathbb{Q}G_\mathfrak{p}$ defined as follows:

$$\begin{aligned} e'_\mathfrak{p} &= |G_{0,\mathfrak{p}}|^{-1}\, \mathrm{N}_{G_{0,\mathfrak{p}}}, \quad & e''_\mathfrak{p} &= 1 - e'_\mathfrak{p}; \\ \bar{e}_\mathfrak{p} &= |G_\mathfrak{p}|^{-1}\, \mathrm{N}_{G_\mathfrak{p}}, & \bar{\bar{e}}_\mathfrak{p} &= 1 - \bar{e}_\mathfrak{p}. \end{aligned} \tag{8}$$

Inspired by Sinnott [Sin80] we consider the element $\omega \in \mathbb{R}G$ given by

$$\chi(\omega) = L(0, \chi^{-1}) \quad \text{for all } \chi \in \hat{G}. \tag{9}$$

The primitive Artin $L$-function used here omits exactly the Euler factors that belong to primes dividing the conductor of $\chi$, considered as a character on the idele class group via reciprocity. The element $\omega$ is closely related to, but not identical with, the equivariant $L$-value discussed in § 1. From the work of Deligne and Ribet [DR80], Barsky [Bar77], and Cassou-Noguès [Cas79], we know that $\omega$ is actually in $\mathbb{Q}G$. (Note in this context that $\omega$ has plus part zero if $k \neq \mathbb{Q}$, and for $k = \mathbb{Q}$, the plus part of $\omega$ is essentially given by the value of the Riemann zeta function at 0, which is $-1/2$.)

From $\omega$ we now build a generalised version of Stickelberger's ideal, which is closely related to (but in general not equal to) Sinnott's ideal for $k = \mathbb{Q}$. For reasons that will become apparent soon, the ideal that is going to be constructed will be called the *Sinnott–Kurihara ideal*. To lighten notation a bit, let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ denote the primes of $k$ that ramify in $K$; shorten $G_{\mathfrak{p}_i}$ to $G_i$, $G_{0,\mathfrak{p}_i}$ to $G_{0,i}$, $e'_{\mathfrak{p}_i}$ to $e'_i$, $e''_{\mathfrak{p}_i}$ to $e''_i$, and $F_{\mathfrak{p}_i}$ to $F_i$.

We define the 'local' modules $U_{\mathfrak{p}_i} = U_i$ by

$$U_i = \langle \mathrm{N}_{G_{0,i}}, 1 - e'_i F_i^{-1} \rangle_{\mathbb{Z}G_i} \subset \mathbb{Q}G_i.$$

The 'global' module $U = U_{K/k}$ is defined by

$$U = U_1 \cdot \ldots \cdot U_s \cdot \mathbb{Z}G \subset \mathbb{Q}G. \tag{10}$$

The (fractional) Sinnott–Kurihara ideal is now defined by

$$\mathrm{SKu}'(K/k) = U\omega \subset \mathbb{Q}G$$

(compare p. 193 of [Sin80]). Note the following: the product defining $U$ runs over the ramified primes only, but one easily sees that $U_{\mathfrak{p}} = \mathbb{Z}G_{\mathfrak{p}}$, if $\mathfrak{p}$ is unramified, so nothing would change if some more primes went into the definition of $U$ and $\mathrm{SKu}'(K/k)$. We also remark that the plus part of $\mathrm{SKu}'(K/k)$ is not very interesting, just as in the case of Sinnott's ideal.

An important point will be to relate the modules $U_{\mathfrak{p}}$ and the modules $W_{\mathfrak{p}}^0$ used by Ritter and Weiss. The module $W_{\mathfrak{p}}$ (for details see below) is the so-called inertial lattice of [RW96], and $W_{\mathfrak{p}}^0$ is its $\mathbb{Z}$-dual. But we defer this, in order to explain the relation of $\mathrm{SKu}'(K/k)$ to earlier constructions, and in particular to the straightforward generalisation of Sinnott's construction to base fields other than $\mathbb{Q}$.

For any $I \subset \{1, \dots, s\}$ let $K_I$ be the largest intermediate field of $K/k$ which is ramified at most in primes $\mathfrak{p}_i$ with $i \in I$. Let $G(I) = \mathrm{Gal}(K_I/k)$. To any $I$ and any set $S_I$ of $k$-places that contains all primes ramifying in $K_I$, we attach a Brumer element as usual:

$$\theta(K_I/k, S_I) \in \mathbb{C}G(I), \quad \psi(\theta(K_I/k, S_I)) = L_{S_I}(0, \psi^{-1})$$

for all characters $\psi$ of $G(I)$. If $S_I$ is minimal, i.e., consists of exactly the primes that ramify in $K_I$, we omit it from the notation. Note that due to imprimitivity, the top level element $\theta(K/k, \{1, \dots, s\})$ is not quite the same as $\omega$. The natural generalisation of Sinnott's Stickelberger ideal now appears to be the following:

$$\mathrm{SSi}'(K/k) = \langle cor_{K/K_I} \theta(K_I/k, S(I)) \mid I \subset \{1, \dots, s\} \rangle_{\mathbb{Z}G} \subset \mathbb{C}G,$$

with $S(I) = \{\mathfrak{p}_i \mid i \in I\}$. Note that it is quite possible that some primes in $S(I)$ do *not* ramify in $K_I/k$. Again by [Bar77], [Cas79] and [DR80], $\mathrm{SSi}'(K/k)$ is a fractional $\mathbb{Z}G$-ideal in $\mathbb{Q}G$. To facilitate comparisons, we propose another definition:

$$\mathrm{SKu}'_1(K/k) = \langle a(I)\, cor_{K/K_I} \theta(K_I/k, S(I)) \mid I \subset \{1, \dots, s\} \rangle_{\mathbb{Z}G},$$



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where the positive integers $a(I)$ are given by

$$a(I) = \left(\prod_{j\notin I} |G_{0,i}|\right) \Big/ \left|\prod_{j\notin I} G_{0,i}\right|.$$

(Recall that $G_{0,i}$ is short for the inertia group $G_{0,\mathfrak{p}_i}$.) We have the obvious inclusion

$$\mathrm{SKu}_1'(K/k) \subset \mathrm{SSi}'(K/k).$$

We will see that this inclusion is proper in general, and that $\mathrm{SKu}_1'(K/k)$ agrees with $\mathrm{SKu}'(K/k)$. Before this, we have to discuss yet another construction, in order to explain the link to previous work of Kurihara [Kur03a]. This construction is *conditional* on the existence of an abelian extension $\tilde{K}/k$ containing $K$ such that $\tilde{G} = \mathrm{Gal}(\tilde{K}/k)$ is the direct product of all its inertia groups. If $k = \mathbb{Q}$, this is no restriction at all; the author thinks however that it is a fairly severe restriction in all other cases. Note that all indices $a(I)$ attached to $\tilde{K}$ are 1 (this is actually equivalent to the condition that all products of inertia groups are direct), and we may assume (shrinking $\tilde{K}$ suitably) that: each prime $\mathfrak{p}$ of $K$ has the same ramification index in $\tilde{K}$ as in $K$. We then put

$$\mathrm{SKu}_2'(K/k) = res_{\tilde{K}/k}\,\mathrm{SKu}_1'(\tilde{K}/k) = res_{\tilde{K}/k}\,\mathrm{SSi}'(\tilde{K}/k).$$

We want to show the following comparison results: over the base $k = \mathbb{Q}$, $\mathrm{SSi}'(K/k)$ essentially gives back Sinnott's original construction; in general $\mathrm{SKu}'(K/k) = \mathrm{SKu}_1'(K/k)$, and $\mathrm{SKu}'(K/k) = \mathrm{SKu}_2'(K/k)$, provided the latter is defined. We start with a lemma that simplifies the definition of $\mathrm{SSi}'$.

Lemma 2.1. *In the definition of $\mathrm{SSi}'(K/k)$ one may omit the sets $S(I)$ (that is, replace $S(I)$ by the set of primes that do ramify in $K_I/k$) without changing the outcome.*

*Proof.* It is clear by looking at Euler factors that the ideal defined with $S(I)$ absent contains the ideal defined with $S(I)$ present; we have to prove the other inclusion. In other words, we must show that $\mathrm{SSi}'(K/k)$ as it is defined already contains $cor_{K/K_I}\theta(K_I/k)$. Look at the subset $I' \subset I$ that consists of all $i$ such that $\mathfrak{p}_i$ does ramify in $K_I$. Then $K_{I'} = K_I$ and $G(I') = G(I)$. Therefore

$$\theta(K_I/k) = \theta(K_I/k, S(I')) = \theta(K_{I'}/k, S(I')) \in \mathbb{Q}G(I) = \mathbb{Q}G(I'),$$

and hence $cor_{K/K_I}\theta(K_I/k) = cor_{K/K_{I'}}\theta(K_{I'}/k, S(I'))$ is in $\mathrm{SSi}'(K/k)$. □

We can now show the following result.

Proposition 2.2. *If $k = \mathbb{Q}$ and $K/\mathbb{Q}$ is abelian, of conductor $f$, then $\mathrm{SSi}'(K/\mathbb{Q})$ is the same as the fractional ideal denoted $S'$ on p. 189 of [Sin80].*

*Proof.* Let $\{p_1, \ldots, p_s\}$ be the set of distinct prime divisors of $f$. For $d|f$ with $(d, f/d) = 1$, let $I(d)$ denote the set of prime divisors of $d$ and $K_d = K \cap \mathbb{Q}(\zeta_d)$. Then $K_d = K_{I(d)}$ in our notation. It is known that Sinnott's $S'$ is generated by the terms

$$cor_{K/K_d} res_{\mathbb{Q}(\zeta_d)/K_d}\theta_d(1),$$

where $d$ runs over all divisors of $f$ such that $(d, f/d) = 1$, and the notation $\theta_d(1)$ is again from Sinnott. The divisors $d$ correspond bijectively to the subsets of $\{1, \ldots, s\}$ via $d \mapsto I(d)$. On the other hand, $\theta_d(1)$ agrees with $\theta(\mathbb{Q}(\zeta_d), I(d))$, and so

$$res_{\mathbb{Q}(\zeta_d)/K_d}\theta_d(1) = \theta(K_d, I(d)).$$

Putting this together, we obtain the claimed equality. □

We proceed to compare the three variants of the Stickelberger–Kurihara ideal.

Proposition 2.3. *If* $\mathrm{SKu}'_2(K/k)$ *is defined (i.e., if the auxiliary extension* $\tilde{K}/k$ *having the required properties exists), then*

$$\mathrm{SKu}'_2(K/k) = \mathrm{SKu}'(K/k).$$

*Proof.* We assume that $\tilde{K}$ is minimal, that is, the canonical surjection $\tilde{G} \to G$ induces isomorphisms on inertia groups. In particular $\tilde{K}$ also ramifies exactly in $\mathfrak{p}_1, \dots, \mathfrak{p}_s$.

The ideals $\mathrm{SKu}'_2(K/k)$ (respectively $\mathrm{SKu}'(K/k)$) are generated by the elements $x_I$ (respectively $a(I)y_I$), where

$$x_I = \mathit{res}_{\tilde{K}/K}\,\mathit{cor}_{\tilde{K}/\tilde{K}_I}\theta(\tilde{K}_I/k, S(I)),$$
$$y_I = \mathit{cor}_{K/K_I}\theta(K_I/k, S(I)).$$

Here $I$ runs over all subsets of $\{1, \dots, s\}$ as above. We will show that $x_I = a(I)y_I$ for all $I$, which of course suffices. Begin by noting that $\theta(K_I/k, S(I)) = \mathit{res}_{\tilde{K}_I/K_I}\theta(\tilde{K}_I/k, S(I))$, and look at the four fields forming a 'square'.

$$\begin{array}{ccc} \tilde{K}_I & \xrightarrow{\;2\;} & \tilde{K} \\ {\scriptstyle 4}\uparrow & & \uparrow{\scriptstyle 1} \\ K_I & \xrightarrow{\;3\;} & K \end{array}$$

The four relevant inclusions have been labelled from 1 to 4. We need to relate $\mathit{res}_1\mathit{cor}_2$ and $\mathit{cor}_3\mathit{res}_4$ (hoping that this shorthand is self-explanatory). A direct calculation using the involved Galois groups yields:

$$\mathit{res}_1\mathit{cor}_2 = ([\tilde{K} : \tilde{K}_I]/[K : K_I]) \cdot \mathit{cor}_3\mathit{res}_4.$$

(This holds for every square of fields in which all fields are abelian over a given field $k$ and the bottom left field is the intersection of the bottom right and the top left field.) Now the numerator is exactly the order of the product over $j \notin I$ of the ramification groups of $\mathfrak{p}_j$ in $\tilde{K}$; by our assumptions on $\tilde{K}$, this is the same as the product over $j \notin I$ of the orders of the $G_{0,j}$. The denominator is the order of the product over $j \notin I$ of the $G_{0,j}$. Thus the quotient is exactly $a(I)$, and our claim follows. $\square$

Without further assumptions (that is, using just our blanket hypothesis that $K/k$ is abelian with group $G$) we can show the next result.

Proposition 2.4. *One has* $\mathrm{SKu}'(K/k) = \mathrm{SKu}'_1(K/k)$.

*Proof.* We let $b_i = 1 - e_iF_i^{-1}$ (see beginning of § 2), and for $I \subset \{1, \dots s\}$ we let

$$e(I) = \prod_{j \notin I} \mathrm{N}_{G_{0,j}} \cdot \prod_{i \in I} b_i.$$

Then by definition, $\mathrm{SKu}'(K/k)$ is the $\mathbb{Z}G$-span of all the elements $e(I)\omega$. It will suffice to show (for all $I$) that

$$e(I)\omega = a(I)\mathit{cor}_{K/K_I}\theta(K_I/k, S(I)). \qquad (*)$$

Let $H_I$ denote the product of the $G_{0,j}$, $j$ not in $I$. Then $\prod_{j \notin I} \mathrm{N}_{G_{0,j}} = a(I) \cdot \mathrm{N}_{H_I}$, essentially by definition of $a(I)$. So $(*)$ is equivalent to

$$\mathrm{N}_{H_I} \cdot \prod_{i \in I} b_i \cdot \omega = \mathit{cor}_{K/K_I}\theta(K_I/k, S(I)).$$

This is equivalent to the validity of the following equation for all characters $\chi$ of $G(I) = G/H_I = \mathrm{Gal}(K_I/k)$:

$$\chi\biggl(\prod_{i \in I} b_i\biggr)\chi(\omega) = \chi(\theta(K_I/k, S(I))). \qquad (**)$$

We look at the involved terms: $\chi(\omega) = L(0, \chi^{-1})$ (we repeat that we take the primitive $L$-function here); $\chi(b_i)$ is either 1 (if $\chi$ ramifies at $\mathfrak{p}_i$) or $1 - \chi^{-1}(F_i)$ (if $\chi$ does not ramify at $\mathfrak{p}_i$). Finally, $\chi(\theta(K_I/k, S(I))) = z \cdot L(0, \chi^{-1})$ where $z$ is the 'imprimitivity factor', that is, the product of the Euler factors belong to primes $\mathfrak{p}_i$, $i \in I$, for which $\chi$ is unramified. Hence $z$ is exactly the product of the $\chi(b_i)$. Therefore the equation ($**$), and hence ($*$), is true, and we are finished. $\square$

One issue still has to be addressed: the ideals $\mathrm{SSi}'(K/k)$ and $\mathrm{SKu}'(K/k)$ are fractional. It is not entirely clear what the integral version should be (except for $k = \mathbb{Q}$), since there are two options: intersecting with $\mathbb{Z}G$, or multiplying with $\mathcal{A} = \mathrm{Ann}_{\mathbb{Z}G}(\mu_K)$ (the annihilator of roots of unity), and we do not seem to know in general whether these lead to the same ideal. With a view towards the main results in this paper, we decide on the second option and define:

$$\mathrm{SSi}(K/k) = \mathcal{A} \cdot \mathrm{SSi}'(K/k), \quad \mathrm{SKu}(K/k) = \mathcal{A} \cdot \mathrm{SKu}'(K/k).$$

Yet again, the results in [Bar77], [Cas79] and [DR80] each imply that $\mathrm{SSi}(K/k)$ and hence also $\mathrm{SKu}(K/k)$ are contained in $\mathbb{Z}G$.

## 3. Outline of the construction

We start with a Tate sequence, which we will have to make much more explicit in the process:

$$0 \to E_{S'} \to A \to B_{S'} \to \Delta S' \to 0. \tag{11}$$

(We recall our simplifying notational convention: the bordering zeros will be omitted very soon.) To make things simpler, we introduce a small technical change with respect to § 1: for a given 4-sequence $0 \to U \to A \to B \to V \to 0$, we will consider metrisations ('trivialisations') going the other more natural way: $\varphi : \mathbb{R}U \to \mathbb{R}V$. The transposed map $\tilde{\varphi}$ will go, as a result, from $\mathbb{R}A$ to $\mathbb{R}B$. We change our definition and let $\chi_{\mathrm{ref}}(A \to B, \varphi) = (A, \tilde{\varphi}, B)$. This only results in a sign change. If $\varphi_{\mathrm{Dir}}$ denotes the negative of the Dirichlet map (cf. § 1) and $E_{\tau,\mathrm{Dir}}$ denotes the above 4-sequence metrised by $\varphi_{\mathrm{Dir}}$, ETNC now becomes the formula

$$\chi_{\mathrm{ref}}(E_{\tau,\mathrm{Dir}}) = \partial(L^*_{S'}(0)^\sharp), \tag{12}$$

with no minus sign on the right (cf. formula (6)).

Our principal tool will be the 'Tate sequence for small $S$' which was established in [RW96], and it will be unavoidable to go through the details of that construction. Therefore we prefer to begin with an outline of what we shall do.

Let $S'$ and $S$ be fixed until further notice (the final application will have $S = S_\infty$) and let $C$ be the free $\mathbb{Z}G$-module with basis elements $x_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over $S'_k \setminus S_k$. We will construct two diagrams. The first is as follows.

$$\begin{array}{ccccccc}
E_{S'} & \longrightarrow & A & \longrightarrow & B_{S'} & \longrightarrow & \Delta S' \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
C \oplus E_S & \longrightarrow & C \oplus A & \longrightarrow & B_S & \longrightarrow & \nabla \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
Z' & \longrightarrow & C & \longrightarrow & C & \longrightarrow & Z''
\end{array} \tag{D1}$$

The middle row is the Tate sequence for the 'small set' $S$, with $C$ added on in two places. We leave the maps unspecified for the moment, just mentioning that the middle bottom map $C \to C$ is in general far from being the identity. The second diagram will have the same middle row as the first.

$$\begin{array}{ccccccc}
E_S & \longrightarrow & A & \longrightarrow & \tilde{B} & \longrightarrow & \nabla/\delta(C) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
C \oplus E_S & \longrightarrow & C \oplus A & \longrightarrow & B_S & \longrightarrow & \nabla \\
\uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle\delta} \\
C & \xrightarrow{\text{id}} & C & \xrightarrow{0} & C & \xrightarrow{\text{id}} & C
\end{array} \qquad \text{(D2)}$$

Again, most maps will be specified later on. In both diagrams all columns are exact in the middle and the first (i.e., lower) vertical map is monic. The upper vertical maps will all be onto with the exception of $C \oplus E_S \to E_{S'}$, whose cokernel is the $S$-class group of $K$. The bottom row of (D1) is only exact after tensoring with $\mathbb{R}$ (or $\mathbb{Q}$) in general, but all other involved rows are exact.

All the occurring rows are 4-sequences, and they will be metrised in the next step. The point is that these metrisations are 'compatible' in the two diagrams, in a fairly straightforward sense which will be explained at the appropriate time. This will imply that, in both diagrams, the refined Euler characteristic of the middle row is the sum of the characteristics of the bottom and top rows. The input from ETNC gives the characteristic of the top row in (D1). The explicit nature of the other metrisations permits one to calculate the characteristic of the top row in (D2), and this is the main intermediate result. (We are a little imprecise here: for this we already need to set $S = S_\infty$ and to take minus parts throughout.)

The next step will be to calculate the $R$-Fitting ideal of $(\nabla/\delta(C))^-$. Actually this is fairly easy. If we assume $S = S_\infty$, and that the number of roots of unity in $K$ is a 2-power (equivalently $\mu(K)^- = 1$), and if we take minus parts, then the left term $E_S^-$ in the top row of (D2) vanishes. Also, the rightmost term $(\nabla/\delta(C))^-$ will be finite. The metrisation is therefore the only one possible: the unique isomorphism from the real vector space 0 to itself, written $\varphi_{\text{triv}}$. We need to discuss briefly the connecting map $\partial$ already introduced in § 1. It is trivial on $\mathbb{Z}G^\times$ (which in fact is the precise kernel of $\partial$), and hence gives a homomorphism from the group of free rank 1 $\mathbb{Z}G$-submodules of $\mathbb{R}G$ to $\mathrm{K}_0(\mathbb{Z}G, \mathbb{R})$, mapping $x\mathbb{Z}G$ to $(\mathbb{Z}G, x, \mathbb{Z}G) = (x\mathbb{Z}G, \text{id}, \mathbb{Z}G)$. So we can extend $\partial$ to the group of all projective rank 1 $\mathbb{Z}G$-submodules of $\mathbb{R}G$, by setting $\partial(P) = (P, \text{id}, \mathbb{Z}G)$. Then $\partial$ in the extended sense is still injective. (Write $\partial_p$ for the analogous map which arises when $\mathbb{Z}G$ is replaced by $\mathbb{Z}_{(p)}G$. Then all $\partial_p$ are injective, so if $\partial(P)$ is trivial, then $\mathbb{Z}_{(p)}P = \mathbb{Z}_{(p)}G$ for all primes $p$, hence $P = \mathbb{Z}G$.) Exactly the same construction can be done with $R$ in the place of $\mathbb{Z}G$. The next lemma will use $\partial$ in this extended sense. Note that we will state and prove a stronger form of this lemma as Lemma 8.9; the proof of that lemma needs more ingredients, so we begin with the weaker form here.

LEMMA 3.1. *If $0 \to A' \to B' \to Q \to 0$ is an exact sequence of $R$-modules, $A'$ and $B'$ are of finite cohomological dimension (so $Q$ is as well), and $Q$ is finite, then*

$$\chi_{\text{ref}}(A' \to B', \varphi_{\text{triv}}) = \partial(\mathrm{Fitt}_R(Q)).$$

*(Note that $\mathrm{Fitt}_R(Q)$ is projective of rank 1 and that the right-hand side determines the ideal $\mathrm{Fitt}_R(Q)$.)*

*Proof.* We will only use the case where $B'$ is without torsion and hence $R$-projective, so we will assume this. One verifies that the transpose of $\varphi_{\text{triv}}$ is exactly the given map $A' \to B'$, call it $\sigma$, tensored with $\mathbb{R}$. Note that this is an isomorphism. Then $\chi_{\text{ref}}(A' \to B', \varphi_{\text{triv}}) = (A', \sigma, B')$. By localisation we may replace $\mathbb{Z}G$ by $\mathbb{Z}_pG$ and assume that $A' = B' = (\mathbb{Z}_pG)^n$ are free of the same rank. Now $\mathrm{K}_1(\mathbb{R}G)$ is isomorphic to $\mathbb{R}G^\times$ via the determinant, so $\big((\mathbb{Z}_pG)^n, \sigma_p, (\mathbb{Z}_pG)^n\big)$ is equal to $((\mathbb{Z}_pG)^n, \det(\sigma_p) \oplus \text{id}_{n-1}, (\mathbb{Z}_pG)^n)$ in $\mathrm{K}_0(\mathbb{Z}_pG, \mathbb{R})$. From the definition of the Fitting ideal we get

$$\mathrm{Fitt}_{\mathbb{Z}_pG}(Q_p) = \mathrm{Fitt}_{\mathbb{Z}_pG}(\mathrm{coker}(\sigma_p)) = (\det(\sigma_p)),$$

and on the other hand

$$\partial(\det(\sigma_p)) = ((\mathbb{Z}_pG)^n, \sigma_p, (\mathbb{Z}_pG)^n) = \chi_{\mathrm{ref}}(A' \to B', \varphi_{\mathrm{triv}})_p.$$

Since this holds for all $p$, we are done. $\square$

The rest of the argument will extract the Fitting ideal of the dual of $cl_K^-$ from the Fitting ideal of $(\nabla/\delta(C))^-$, which is given to us by the preceding lemma. We will just have to use that $cl_K$ is the torsion part of $\nabla$, and that everything else is explicitly known.

## 4. Construction of the first main diagram

It is conspicuous that the processes leading to a Tate sequence on the one hand (cf. [Wei96, ch. 5]) and 'Tate sequences for small $S$' on the other (cf. [Wei96, ch. 14] and [RW96]) are similar, the latter one being more intricate. Basically our task is to make this similarity as solid and precise as possible, and it will not be a surprise to experts that this can be done.

We start by establishing a similar diagram in the local situation ($G$ being replaced by $G_{\mathfrak{p}}$). The global diagram will then arise as the result of three steps:

(i) inducing all local diagrams from $G_{\mathfrak{p}}$ to $G$;
(ii) taking the direct sum over $\mathfrak{p} \in S'_k$;
(iii) and finally (a technical nuisance which makes an already complicated picture even more complicated) taking, at every position, the kernel of certain maps which are only defined globally – one of them is the sum map $\mathbb{Z}S' \to \mathbb{Z}$, and the others are close relatives.

Let us abbreviate $\mathrm{ind}_{G_{\mathfrak{p}}}^G$ to $\mathrm{ind}_{\mathfrak{p}}$ and $\bigoplus_{\mathfrak{p}\in S'_k} \mathrm{ind}_{\mathfrak{p}} \ldots$ to $\widetilde{\bigoplus}_{\mathfrak{p}\in S'_k} \ldots$. Generally, we fix one $\mathfrak{P}$ above each $\mathfrak{p}$ and write $K_{\mathfrak{p}}$ instead of $K_{\mathfrak{P}}$ (so the former is never a semilocal completion, but a local field); $U_{\mathfrak{p}}$ is as usual the group of local units $\mathcal{O}_{K_{\mathfrak{p}}}^\times$.

In [RW96] one finds two diagrams for $\mathfrak{p} \in S'_k \setminus S_k$, which also can be fitted together horizontally, giving two 4-sequences on top of each other, as below.

$$\begin{array}{ccccc} K_{\mathfrak{p}}^\times & \longrightarrow & V_{\mathfrak{p}} & \longrightarrow & \Delta G_{\mathfrak{p}} \\ \uparrow & & \| & & \uparrow \\ U_{\mathfrak{p}} & \longrightarrow & V_{\mathfrak{p}} & \longrightarrow & W_{\mathfrak{p}} \end{array} \qquad \begin{array}{ccccc} \Delta G_{\mathfrak{p}} & \longrightarrow & \mathbb{Z}G_{\mathfrak{p}} & \longrightarrow & \mathbb{Z} \\ \uparrow & & \uparrow & & \uparrow \\ W_{\mathfrak{p}} & \longrightarrow & \mathbb{Z}G_{\mathfrak{p}} \oplus \mathbb{Z}G_{\mathfrak{p}} & \longrightarrow & W_{\mathfrak{p}}^0 \end{array} \tag{13}$$

We just quote the left-hand diagram (see *loc. cit.*, Proposition 3), but we will have to look at the right-hand diagram (see *loc. cit.*, Lemma 5(c)), called 'core diagram', in much detail; see § 5. We have similar, but simpler, diagrams for $\mathfrak{p} \in S_k$, as below.

$$\begin{array}{ccccc} K_{\mathfrak{p}}^\times & \longrightarrow & V_{\mathfrak{p}} & \longrightarrow & \Delta G_{\mathfrak{p}} \\ \| & & \| & & \| \\ K_{\mathfrak{p}}^\times & \longrightarrow & V_{\mathfrak{p}} & \longrightarrow & \Delta G_{\mathfrak{p}} \end{array} \qquad \begin{array}{ccccc} \Delta G_{\mathfrak{p}} & \longrightarrow & \mathbb{Z}G_{\mathfrak{p}} & \longrightarrow & \mathbb{Z} \\ \| & & \| & & \| \\ \Delta G_{\mathfrak{p}} & \longrightarrow & \mathbb{Z}G_{\mathfrak{p}} & \longrightarrow & \mathbb{Z} \end{array} \tag{14}$$

We now apply $\widetilde{\bigoplus}_{\mathfrak{p}\in S'_k}$ to these diagrams. Let

$$V_0 = \widetilde{\bigoplus_{\mathfrak{p}\in S'_k}} V_{\mathfrak{p}},$$

$$W_S = \widetilde{\bigoplus_{\mathfrak{p}\in S'_k\setminus S_k}} W_{\mathfrak{p}} \oplus \widetilde{\bigoplus_{\mathfrak{p}\in S_k}} \Delta G_{\mathfrak{p}}, \quad W_{S'} = \widetilde{\bigoplus_{\mathfrak{p}\in S'_k}} \Delta G_{\mathfrak{p}}$$

$$N_S = \widetilde{\bigoplus_{\mathfrak{p}\in S'_k\setminus S_k}} (\mathbb{Z}G_{\mathfrak{p}} \oplus \mathbb{Z}G_{\mathfrak{p}}) \oplus \widetilde{\bigoplus_{\mathfrak{p}\in S_k}} \mathbb{Z}G_{\mathfrak{p}}, \quad N_{S'} = \widetilde{\bigoplus_{\mathfrak{p}\in S'_k}} \mathbb{Z}G_{\mathfrak{p}},$$

$$W_S^0 = \widetilde{\bigoplus_{\mathfrak{p}\in S'_k\setminus S_k}} W_{\mathfrak{p}}^0 \oplus \widetilde{\bigoplus_{\mathfrak{p}\in S_k}} \mathbb{Z},$$

and note that $\widetilde{\bigoplus}_{\mathfrak{p}\in S'_k} \mathbb{Z}$ is identified with $\mathbb{Z}S'$, by the choice of $\mathfrak{P}|\mathfrak{p}$. We finally let $V = V_0 \oplus \widetilde{\bigoplus}_{\mathfrak{p}\notin S'_k} U_{\mathfrak{p}}$, that is, we add on the unit ideles outside $S'$ to $V_0$. Inducing up (13) and (14), taking direct sums and replacing $V_0$ by $V$ then yields the following two diagrams.

$$\begin{array}{ccccc} J_{K,S'} & \longrightarrow & V & \longrightarrow & W_{S'} \\ \uparrow & & \| & & \uparrow \\ J_{K,S} & \longrightarrow & V & \longrightarrow & W_S \end{array} \qquad \begin{array}{ccccc} W_{S'} & \longrightarrow & N_{S'} & \longrightarrow & \mathbb{Z}S' \\ \uparrow & & \uparrow & & \uparrow \\ W_S & \longrightarrow & N_S & \longrightarrow & W_S^0 \end{array} \tag{15}$$

There are two other short exact sequences, the one coming from class field theory (cf. [RW96]), the other obvious, which we write directly below the preceding diagrams.

$$C_K \longrightarrow \mathfrak{V} \longrightarrow \Delta G \qquad \Delta G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}. \tag{16}$$

Each term in (15) comes with a canonical map to the term in (16) directly below it, in such a way that a commutative 'three-dimensional' diagram results, which we think we need not write down; but let us discuss the maps. For the left half, this is in [RW96]. The map $J_{K,S'} \to C_K$ (and consequently $J_{K,S} \to C_K$ as well) is the canonical map from the idele group to the idele class group. The map $V \to \mathfrak{V}$ will never be used explicitly. The map $W_{S'} \to \Delta G$ comes from the surjection $W_{\mathfrak{p}} \to \Delta G_{\mathfrak{p}}$ (fourth column of (13)) and the inclusion $\Delta G_{\mathfrak{p}} \to \Delta G$. It is surjective as a consequence of the third condition in the definition of a 'larger' set. The map $N_{S'} \to \mathbb{Z}G$ comes from $\mathbb{Z}G_{\mathfrak{p}} \subset \mathbb{Z}G$, which becomes an equality on inducing $\mathbb{Z}G_{\mathfrak{p}}$ up to $\mathbb{Z}G$. The map $\mathbb{Z}S' \to \mathbb{Z}$ is the sum map. All of these vertical maps are onto, with the (possible) exception of the map to $C_K$, whose cokernel is the $S$-class group of $K$.

We now replace every single term in (15) by the kernel of the vertical map going from it to the term in (16) below it. We need to identify some of these kernels, and give names to the others. The kernel inside $J_{K,S}$ (respectively $J_{K,S'}$) is of course $E_S$ (respectively $E_{S'}$). The kernel of $V \to \mathfrak{V}$ is called $A$. (This letter will never mean anything else from now on.) The kernels inside $W_S$ (respectively $W_{S'}$) will not really matter in the sequel: call them $\tilde{W}_S$ (respectively $\tilde{W}_{S'}$). The kernels inside $N_S$ ($N_{S'}$) get the name $B_S$ ($B_{S'}$). The kernel from the map $\mathbb{Z}S' \to \mathbb{Z}$ is of course $\Delta S'$. Finally, the kernel of $W_S^0 \to \mathbb{Z}$ is, by definition, the module $\bar{\nabla}$ of Ritter and Weiss. Altogether we have the following diagrams.

$$\begin{array}{ccccc} E_{S'} & \longrightarrow & A & \longrightarrow & \tilde{W}_{S'} \\ \uparrow & & \| & & \uparrow \\ E_S & \longrightarrow & A & \longrightarrow & \tilde{W}_S \end{array} \qquad \begin{array}{ccccc} \tilde{W}_{S'} & \longrightarrow & B_{S'} & \longrightarrow & \Delta S' \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{W}_S & \longrightarrow & B_S & \longrightarrow & \bar{\nabla} \end{array} \tag{17}$$

All the short rows are short exact sequences (a consequence of the surjectivity of the maps from (15) to (16)), with one exception: the map $A \to \tilde{W}_S$ has cokernel $cl_{K,S}$, by the snake lemma, since $J_{K,S} \to C_K$ has that cokernel.

We now discuss the vertical maps in this diagram (not to be confused with the 'vertical' maps from (15) to (16) which have already dropped out of the game). The first one from the left is the obvious inclusion, the second is an equality. The map between the $\tilde{W}$-terms will not matter. A trivial argument shows that the two rightmost vertical maps in (17) are onto, with the same kernel as the corresponding maps in (15). In § 5 we shall explain that the middle vertical map in the core diagram (right part of (13)) is $F_{\mathfrak{p}} \cdot pr_2 : \mathbb{Z}G_{\mathfrak{p}} \oplus \mathbb{Z}G_{\mathfrak{p}} \to \mathbb{Z}G_{\mathfrak{p}}$. Consequently, the kernel of $N_S \to N_{S'}$ (or, what is the same, the kernel of $B_S \to B_{S'}$) is isomorphic to the direct sum of copies of $\mathbb{Z}G$, one for each $\mathfrak{p} \in S'_k \setminus S_k$. Still more explicitly: recall the definition of $C$ (free on $x_{\mathfrak{p}}$, $\mathfrak{p} \in S'_k \setminus S_k$) and map $C$ to $N_S$ by $x_{\mathfrak{p}} \mapsto 1 \otimes (1,0) \in \mathrm{ind}_{\mathfrak{p}}(\mathbb{Z}G_{\mathfrak{p}} \oplus \mathbb{Z}G_{\mathfrak{p}})$. This map then affords an isomorphism from $C$ to the kernel of $B_S \to B_{S'}$; the map $C \to B_S$ will be called $\gamma$.

We next discuss the kernel of $\bar{\nabla} \to \Delta S'$ (right column of (17)). Let $Z''_{\mathfrak{p}}$ be the kernel of $W^0_{\mathfrak{p}} \to \mathbb{Z}$ (rightmost column of (13)); this will be made explicit in § 5. Let $Z_0 = \bigoplus_{\mathfrak{p} \in S'_k \setminus S_k} Z''_{\mathfrak{p}}$. Then $Z_0$ identifies with the kernel of $\bar{\nabla} \to \Delta S'$, and we get the diagram below.

$$\begin{array}{ccccc} \tilde{W}_{S'} & \longrightarrow & B_{S'} & \longrightarrow & \Delta S' \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{W}_S & \longrightarrow & B_S & \longrightarrow & \bar{\nabla} \\ & & \uparrow{\scriptstyle \gamma} & & \uparrow \\ & & C & \longrightarrow & Z_0 \end{array} \tag{18}$$

We intend to compose this with the left part of (17); the problem is, however, that the right bottom arrow $A \to \tilde{W}_S$ is not surjective. To remedy this, let $\tilde{W}'_S$ denote temporarily the image of that arrow (so we know that $\tilde{W}_S/\tilde{W}'_S \cong cl_{K,S}$), and, following Ritter and Weiss, we *define* $\nabla$ as $B_S/\tilde{W}'_S$. Then one has a natural sequence $0 \to cl_{K,S} \to \nabla \to \bar{\nabla} \to 0$, and one still has an upward map $\nabla \to \Delta_{S'}$. If $Z''$ is its kernel, we again find $0 \to cl_{K,S} \to Z'' \to Z_0 \to 0$, and the last diagram turns into the following one.

$$\begin{array}{ccccc} \tilde{W}_{S'} & \longrightarrow & B_{S'} & \longrightarrow & \Delta S' \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{W}'_S & \longrightarrow & B_S & \longrightarrow & \nabla \\ & & \uparrow{\scriptstyle \gamma} & & \uparrow \\ & & C & \longrightarrow & Z'' \end{array} \tag{19}$$

Note that after tensoring with $\mathbb{R}$, (18) and (19) become the same. Diagram (19) can now be amalgamated with the left part of (17) to produce a diagram in which the first two rows are exact 4-sequences.

$$\begin{array}{ccccccc} E_{S'} & \longrightarrow & A & \longrightarrow & B_{S'} & \longrightarrow & \Delta S' \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ E_S & \longrightarrow & A & \longrightarrow & B_S & \longrightarrow & \nabla \\ & & & & \uparrow & & \uparrow \\ & & & & C & \longrightarrow & Z'' \end{array} \tag{20}$$

We will show in § 5 that the map $C \to Z''$ is onto modulo torsion and that its kernel is generated by the elements $\mathrm{N}_{G_{\mathfrak{p}}}\, x_{\mathfrak{p}}$. Let therefore $\nu : C \to C$ be the $\mathbb{Z}G$-endomorphism defined by $x_{\mathfrak{p}} \mapsto \mathrm{N}_{G_{\mathfrak{p}}}\, x_{\mathfrak{p}}$ for all $\mathfrak{p} \in S'_k \setminus S_k$, and let $Z' = \ker(\nu)$. We now can complete diagram (20) so as to arrive at diagram (D1) at last. We give some maps a name: thus $\iota : E_{S'} \to A$ (a monomorphism) and $\alpha : A \to B_{S'}$. We change and complete (20) as follows. First we write down the new diagram, and then we explain the new and the changed maps. Note that the map $\nu : C \to C$ was replaced by $h\nu$ (the auxiliary integer $h$ will be explained promptly).

$$\begin{array}{ccccccc}
E_{S'} & \xrightarrow{\iota} & A & \xrightarrow{\alpha} & B_{S'} & \longrightarrow & \Delta S' \\
\uparrow{\scriptstyle (\beta,\mathrm{inc.})} & & \uparrow{\scriptstyle (\iota\beta,\mathrm{id}_A)} & & \uparrow & & \uparrow \\
C \oplus E_S & \xrightarrow{(\mathrm{id}_C \oplus \iota)} & C \oplus A & \xrightarrow{(0,\alpha)} & B_S & \longrightarrow & \nabla \\
\uparrow{\scriptstyle (-\mathrm{id}_C,\beta)} & & \uparrow{\scriptstyle (-\mathrm{id}_C,\iota\beta)} & & \uparrow & & \uparrow \\
Z' & \longrightarrow & C & \xrightarrow{h\nu} & C & \longrightarrow & Z''
\end{array} \tag{21}$$

We start out by defining $\beta : C \to E_{S'}$ (which one should think of as a kind of partial inverse to the Dirichlet map). Choose a positive multiple $h$ of $h_K$ once and for all. For every $\mathfrak{p} \in S'_k \setminus S_k$, there then exists an $S'$-unit $u_{\mathfrak{p}}$, whose valuation at the chosen prime $\mathfrak{P}$ over $\mathfrak{p}$ is $h$, and 0 at every other finite prime. We define $\beta$ by

$$\beta : C \to E_{S'}, \quad x_{\mathfrak{p}} \mapsto u_{\mathfrak{p}}.$$

This is a $\mathbb{Z}G$-linear injection.

It is no problem to check that the first and second columns of (21) are short exact. Likewise, the commutativity of the two leftmost squares is a direct consequence of the definitions. The upper middle square commutes since $\iota\beta(C) \subset \iota(E_{S'})$ goes to zero under $\alpha$, by exactness of the top row.

Proposition 4.1. *The lower middle square of (21) commutes, and hence the whole diagram (21) commutes.*

We defer the proof to the next section. Note that the factor $h$ at the map $\nu$ has a consequence which appears quite unpleasant at first sight: the bottom row is no longer exact. But this will not cause any problems, because it will suffice to have exactness after tensoring with $\mathbb{R}$. (Or else, one could replace $Z''$ by the cokernel of $h\nu$, which is an extension of $Z''$ by a finite module; then the rightmost column would have a finite kernel in its first map. Again this will not matter at the crucial point.)

Modulo various technical results which will be proved in the next section, this finishes the construction of diagram (D1).

## 5. The core diagram

In this section we discuss in detail the objects and maps which form the right part of (13), largely following the work of Ritter and Weiss, and prove some important auxiliary results.

The exponent $^0$, which has already occurred in names of objects, generally denotes $\mathbb{Z}$-duals. For any group $H$ and any $\mathbb{Z}H$-module $M$, $M^0$ is $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, with the $H$-action formula $(\eta f)(m) = \eta f(\eta^{-1}m) = f(\eta^{-1}m)$ for $\eta \in H$, $f \in M^0$, $m \in M$, the last equality holding since $\mathbb{Z}$ is a trivial $H$-module. This formula makes $M^0$ again into a $\mathbb{Z}H$-module, even if $H$ is non-commutative.

For $M = \mathbb{Z}H$ there is a standard identification which will be regarded as an equality in the sequel:

$$\mathbb{Z}H \xrightarrow{\sim} \mathbb{Z}H^0, \quad \eta \mapsto \delta_\eta. \tag{22}$$

The map $\delta_\eta$ is of course given by $\delta_\eta(\eta') = \delta_{\eta,\eta'}$ $(\eta, \eta' \in H)$, where the right-hand $\delta$ is Kronecker's. One checks that this identification is indeed $H$-linear.

If $f \in \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}H, \mathbb{Z}H)$ is given as multiplication with a central element $\gamma \in H$, then $f^0$ is given, in the above identification, as multiplication by $\gamma^{-1}$ on $\mathbb{Z}H$, as one easily verifies.

Under the above identification, $(\Delta H)^0$ becomes $\mathbb{Z}H/(\mathrm{N}_H)$. For every quotient $\bar{H} = H/H_0$, one has a completely analogous identification of $\mathbb{Z}H$-modules $(\mathbb{Z}\bar{H})^0 = \mathbb{Z}\bar{H}$. The dual of the canonical epimorphism $\mathbb{Z}H \to \mathbb{Z}\bar{H}$ is, under our identifications, the map $\mathbb{Z}\bar{H} \to \mathbb{Z}H$ that sends 1 to $\mathrm{N}_{H_0}$. In particular for the extreme case $H = H_0$, the augmentation map $\epsilon_H : \mathbb{Z}H \to \mathbb{Z}$ is dual to the map $1 \mapsto \mathrm{N}_H$ from $\mathbb{Z}$ to $\mathbb{Z}H$. All this will be used throughout our calculations.

We now come to the local part of this section, dealing with the inertial lattice $W_{\mathfrak{p}}$ of Ritter and Weiss, and its dual. To streamline notation, let $H = G_{\mathfrak{p}}$, $H_0 = G_{0,\mathfrak{p}}$ (the inertia group) and $\bar{H} = H/H_0$. We will omit the index $\mathfrak{p}$ at $W$ and $W^0$ now and reinstate it later when we go global again. All we do in this section is essentially due to Gruenberg, Ritter and Weiss. We recall that $F \in H$ (index again omitted) is a lift of $\mathrm{Frob}_{\mathfrak{p}} \in \bar{H}$. We now give the definition of $W$. In categorical language it is a difference kernel, and $W^0$ will be a difference cokernel:

$$W := \{(x, y) \in \Delta H \times \mathbb{Z}\bar{H} \mid \bar{x} = (F-1)y\}. \tag{23}$$

Here the overbar denotes the canonical map $\mathbb{Z}H \to \mathbb{Z}\bar{H}$. The natural projection from $W$ to $\Delta H$ will be written $pr_x$. There is another canonical map $\tau : \mathbb{Z} \to W$ which sends 1 to $(0, \mathrm{N}_{\bar{H}})$. The latter element is indeed in $W$ since $(F-1)\,\mathrm{N}_{\bar{H}} = 0$, so the map $\tau$ makes sense.

More categorically, $W = \ker(\Delta H \times \mathbb{Z}\bar{H} \to \mathbb{Z}\bar{H})$, the map being overbar on the first component and $1 - F$ on the second. This permits, by functoriality, the identification $W^0 = \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z})$ as follows, using the identifications concerning $\mathbb{Z}$-duals explained above:

$$W^0 = \mathrm{coker}\left(\mathbb{Z}\bar{H} \to \frac{\mathbb{Z}H}{(\mathrm{N}_H)} \times \mathbb{Z}\bar{H}\right), \tag{24}$$

the map being given by $1 \mapsto (\mathrm{N}_{H_0}, 1 - F^{-1})$. The module $W^0$ comes with two natural maps: firstly $i_x$, the dual of $pr_x$, in other words the natural map $\mathbb{Z}H/(\mathrm{N}_H) \to W^0$ coming from the first injection into the product; and secondly the dual $W^0 \to \mathbb{Z}^0 = \mathbb{Z}$ of the map $\tau : \mathbb{Z} \to W$. By our previous remarks, this map is given by $(0, \epsilon_{\bar{H}})$. (We will not need the following fact, but let us state that this map is also given by evaluation in $w_1$ in the notation of [GW96].) Let $\kappa$ denote the canonical epimorphism from $\mathbb{Z}H \oplus \mathbb{Z}H$ to $W^0$. We get the following diagram in which all arrows except the horizontal one starting at $W$ are determined, and the two eastern squares are easily seen to commute.

$$\begin{array}{ccccc}
\Delta H & \longrightarrow & \mathbb{Z}H & \xrightarrow{\ \epsilon_H\ } & \mathbb{Z} \\
\uparrow{\scriptstyle pr_x} & & \uparrow{\scriptstyle F\,pr_2} & & \uparrow{\scriptstyle (0,\epsilon_{\bar{H}})} \\
W & \longrightarrow & \mathbb{Z}H \oplus \mathbb{Z}H & \xrightarrow{\ \kappa\ } & W^0 \\
\uparrow{\scriptstyle \tau} & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_x} \\
\mathbb{Z} & \xrightarrow{\ \cdot \mathrm{N}_H\ } & \mathbb{Z}H & \longrightarrow & \mathbb{Z}H/(\mathrm{N}_H)
\end{array} \tag{25}$$

The north-east square would commute just as well without the factor $F$ in the map $F\,pr_2$, but that factor will be needed for the completion of the diagram. To do that, we discuss the kernel of the map $\kappa$. Let $\Delta(H, \bar{H}) = \ker(\mathbb{Z}H \to \mathbb{Z}\bar{H})$. Then

$$\ker(\kappa) = \langle(\mathrm{N}_H, 0)\rangle + 0 \times \Delta(H, \bar{H}) + \langle z\rangle$$

with $z := (\mathrm{N}_{H_0}, 1 - F^{-1})$. The first generator may be omitted for the following reason: $(1 + F + \cdots + F^{|\bar{H}|-1}) \cdot z = (\mathrm{N}_H, z')$ with some $z'$ that goes to 0 in $\mathbb{Z}\bar{H}$ and hence is in $\Delta(H, \bar{H})$; therefore $(\mathrm{N}_H, 0)$ can be generated from the other two summands in the above formula for $\ker(\kappa)$.

We construct an isomorphism $q : W \to \ker(\kappa)$ by letting

$$q(x, y) = (\mathrm{N}_{H_0}\, y, F^{-1}x) \quad \text{for } (x, y) \in W \subset \Delta H \times \mathbb{Z}\bar{H}.$$



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(The product $\mathrm{N}_{H_0}\, y$ makes sense as an element of $\mathbb{Z}H$ even though $y$ is from $\mathbb{Z}\bar{H}$ and not from $\mathbb{Z}H$: one just chooses a lift of $y$, and $\mathrm{N}_{H_0}\, y$ does not depend on the way of lifting.) There are a few things to show.

(a) First, $q$ takes values in $\ker(\kappa)$. We have $\bar{x} = (F-1)y$, so the two terms $q(x,y) = (\mathrm{N}_{H_0}\, y, F^{-1}x)$ and $yz = (\mathrm{N}_{H_0}\, y, (1-F^{-1})y)$ agree modulo $0 \times \Delta(H,\bar{H})$, which is contained in $\ker(\kappa)$. But $yz$ is in $\ker(\kappa)$ as well.

(b) The isomorphism $q : W \to \ker(\kappa)$ is surjective. On letting $y = 0$ and $x$ run through $\Delta(H,\bar{H})$, one obtains that $0 \times \Delta(H,\bar{H})$ is contained in the image of $q$. On the other hand one sees that $q(F-1,\bar{1}) = z$ is also in the image. This suffices.

(c) It is straightforward to check that $W$, and hence $W_0$, have $\mathbb{Z}$-rank $|H|$. (Indeed one has $\mathbb{Q} \otimes_{\mathbb{Z}} W \cong \mathbb{Q}H$.) Therefore $\ker(\kappa)$ has rank $|H|$ too, and hence $q$ must be an isomorphism.

We now insert the map $q : W \to \ker(\kappa) \subset \mathbb{Z}H \oplus \mathbb{Z}H$ in (25). It remains to show commutativity of the western squares. For the northern one, we chase $(x,y) \in W$ both ways. Upward then to the right gives $x$. To the right gives $(0, F^{-1}x)$, and this then goes to $x$. For the southern square we chase $1 \in \mathbb{Z}$. Upward then to the right gives $q(0, \mathrm{N}_{H_0}) = (\mathrm{N}_H, 0)$. The other way produces the same.

We revert to the notation $G_{\mathfrak{p}}$ for $H$. We now have established the core diagram, that is, the right half of (13). More than that, we have identified the kernel (which we called $Z''_{\mathfrak{p}}$) of the map $W^0_{\mathfrak{p}} \to \mathbb{Z}$ with $\mathbb{Z}G_{\mathfrak{p}}/(\mathrm{N}_{G_{\mathfrak{p}}})$; the map $\mathbb{Z}G_{\mathfrak{p}} \to Z''_{\mathfrak{p}}$ arising from the vertical kernels in (13) is the canonical one, and consequently $Z_0$ (see (18)) becomes identified with $\widetilde{\bigoplus}_{\mathfrak{p} \in S'_k \setminus S_k} \mathbb{Z}G_{\mathfrak{p}}/(\mathrm{N}_{G_{\mathfrak{p}}})$; the map $C \to Z_0$ in (18) is again the canonical one, $x_{\mathfrak{p}}$ mapping to the vector in $Z_0$ which has $\bar{1}$ (more precisely, $1 \otimes \bar{1}$, because of the induction performed) in position $\mathfrak{p}$, and zeros elsewhere.

We now prove Proposition 4.1.

*Proof of Proposition 4.1.* The map $\gamma : C \to B_S$ (see (18)) arises from the vertical injections $i_1 : \mathbb{Z}G_{\mathfrak{p}} \to \mathbb{Z}G_{\mathfrak{p}} \oplus \mathbb{Z}G_{\mathfrak{p}}$ in the core diagram by induction and summing, so $\gamma(x_{\mathfrak{p}}) = 1 \otimes (1,0) \in \mathrm{ind}_{\mathfrak{p}}(\mathbb{Z}G_{\mathfrak{p}} \oplus \mathbb{Z}G_{\mathfrak{p}})$. We have to show that

$$\alpha\iota\beta(x_{\mathfrak{p}}) = h\gamma(\mathrm{N}_{G_{\mathfrak{p}}}\, x_{\mathfrak{p}}).$$

The left-hand term is $\alpha\iota(u_{\mathfrak{p}})$ (the $S'$-unit $u_{\mathfrak{p}}$ was introduced right after diagram (21)). The map $\iota : E_{S'} \to A$ is the restriction of a map, call it $\iota'$, from $J_{K,S'}$ to $V$; see diagram (15). This in turn comes from (14) via inducing up and summing; write $\iota_{\mathfrak{p}} : K_{\mathfrak{p}} \to V_{\mathfrak{p}}$ for the corresponding local map. Let $\alpha_{\mathfrak{p}} : V_{\mathfrak{p}} \to W_{\mathfrak{p}}$ be the map from the left part of (13). We now consider $u_{\mathfrak{p}}$ as a (principal) idele, that is, as an element of $J_{K,S'}$, and our task is to chase it through (15) as follows: to $V$, then continue in the lower row (using the equality sign between the two $V$) and go by $\alpha$ from $V$ to $N_S$ (via $W_S$). All this can be done locally, that is, looking at (14). Any local unit will go to 1 under this process, by exactness of the lower rows in (14). So if we let $\alpha_{\mathfrak{p}} : V_{\mathfrak{p}} \to \mathbb{Z}G_{\mathfrak{p}} \oplus \mathbb{Z}G_{\mathfrak{p}}$ (via $W_{\mathfrak{p}}$) be the local counterpart of $\alpha$, then $\alpha_{\mathfrak{p}}\iota_{\mathfrak{p}}(y)$ only depends on the valuation of $y \in K^{\times}_{\mathfrak{P}}$. Let $\pi$ be a parameter of $K_{\mathfrak{P}}$. Because of the choice of $u_{\mathfrak{p}}$ we then have that $u_{\mathfrak{p}}$ agrees, up to a unit idele, with the idele which has $1 \otimes \pi^h$ over $\mathfrak{p}$ and 1s over all $\mathfrak{q} \neq \mathfrak{p}$. (Note: This is the only time that we need to remember that $\mathrm{ind}_{\mathfrak{p}} K^{\times}_{\mathfrak{P}}$ is identified with the product of all $K^{\times}_{\mathfrak{P}'}$, where $\mathfrak{P}'$ runs over all places of $K$ above $\mathfrak{p}$, and $\mathfrak{P}$ is, as always in this paper, a fixed choice among them.)

Thus it remains to chase the element $\pi^h \in K^*_{\mathfrak{p}}$ through (13): the outcome, after inducing up again, will be the left-hand side of the equality we want to prove. For convenience we reproduce the



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left part of (13) below.

$$\begin{array}{ccccc} K_{\mathfrak{p}}^{\times} & \xrightarrow{\iota_{\mathfrak{p}}} & V & \longrightarrow & \Delta G_{\mathfrak{p}} \\ \uparrow & & \| & & \uparrow \\ U_{\mathfrak{p}} & \longrightarrow & V & \longrightarrow & W_{\mathfrak{p}} \end{array}$$

Now by Proposition 2.2 and the following Remark in [RW96], this diagram as a whole is obtained by trivial rearrangement, including writing $V$ twice, from the following diagram.

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\tau_{\mathfrak{p}}} & W_{\mathfrak{p}} & \longrightarrow & \Delta G_{\mathfrak{p}} \\ \uparrow & & \uparrow{\scriptstyle \alpha'_{\mathfrak{p}}} & & \| \\ K_{\mathfrak{p}}^{\times} & \xrightarrow{\iota_{\mathfrak{p}}} & V & \longrightarrow & \Delta G_{\mathfrak{p}} \\ \uparrow & & \uparrow & & \\ U & = & U & & \end{array}$$

Here the top row is the same as the leftmost column in (25) (in slightly differing notation), and the map $K_{\mathfrak{p}}^{\times} \to \mathbb{Z}$ is just the normalised valuation. The map $\alpha'_{\mathfrak{p}}$ gives $\alpha_{\mathfrak{p}}$ when followed by the map $W_{\mathfrak{p}} \to \mathbb{Z}G_{\mathfrak{p}} \oplus \mathbb{Z}G_{\mathfrak{p}}$ from (13), which we momentarily call $q_{\mathfrak{p}}$; this is also the map $q$ from (25).

Then one sees directly that $\alpha'_{\mathfrak{p}}\iota_{\mathfrak{p}}(\pi^h) = \tau_{\mathfrak{p}}(h) = h \cdot (0, 1 + F_{\mathfrak{p}} + \cdots + F_{\mathfrak{p}}^{f-1})$ with $f := |G_{\mathfrak{p}}/G_{0,\mathfrak{p}}|$. When we apply $q_{\mathfrak{p}}$ to this expression, we get

$$\alpha_{\mathfrak{p}}\iota_{\mathfrak{p}}(\pi^h) = h \cdot q_{\mathfrak{p}}(0, 1 + F_{\mathfrak{p}} + \cdots + F_{\mathfrak{p}}^{f-1}) = h \cdot (\mathrm{N}_{G_{\mathfrak{p}}}, 0).$$

By definition of $\gamma$, this is the same as $h \cdot \gamma(\mathrm{N}_{G_{\mathfrak{p}}}\, x_{\mathfrak{p}})$, and we are done. □

## 6. Construction of the second main diagram

The middle row of (D1) constitutes an important step, but our goal is to find a 4-sequence with finite outer terms, in order to make the metrisations disappear. So another modification is called for; it seems to partially undo previous constructions, but we could not find a more direct approach to the final 4-sequence.

The construction does what it is supposed to do only if we assume that $S = S_\infty$ and take minus parts, but it can be performed without this. We will shortly define a $\mathbb{Z}G$-monomorphism $\delta : C \to \nabla$. Let $\delta' : C \to B_S$ be any lift of $\delta$ through the epimorphism $B_S \to \nabla$. On setting $\tilde{B} = \mathrm{coker}(\delta')$, this gives the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc} E_S & \xrightarrow{\iota} & A & \longrightarrow & \tilde{B} & \longrightarrow & \nabla/\delta(C) \\ \uparrow{\scriptstyle pr_2} & & \uparrow{\scriptstyle pr_2} & & \uparrow & & \uparrow \\ C \oplus E_S & \longrightarrow & C \oplus A & \longrightarrow & B_S & \longrightarrow & \nabla \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle \delta'} & & \uparrow{\scriptstyle \delta} \\ C & = & C & \xrightarrow{0} & C & = & C \end{array} \qquad \text{(D2)}$$

It is known from [RW96] (and will also become clear in the next section) that the outer terms of the middle row become isomorphic over $\mathbb{Q}$. Hence the same holds for the outer terms of the top row. Now if $S$ is set equal to $S_\infty$, that is, $E_S$ is just the global unit group of $K$, then $E_S^-$ (the group of roots of unity) is finite, and hence $(\nabla/\delta(C))^-$ is finite as well. It therefore remains for us to define $\delta$. We will make choices there (as before), but all choices will drop out in the final calculation.

Recall that $W^0_{\mathfrak{p}}$ was described as a certain quotient of $\mathbb{Z}G_{\mathfrak{p}} \oplus \mathbb{Z}G_{\mathfrak{p}}$ and that the canonical epimorphism in this context is denoted $\kappa$. We define $\delta_{\mathfrak{p}} : \mathbb{Z}G_{\mathfrak{p}} \cdot x_{\mathfrak{p}} \to W^0_{\mathfrak{p}}$ by

$$\delta_{\mathfrak{p}}(x_{\mathfrak{p}}) := \kappa((-1, 1)). \tag{26}$$

From this we obtain by applying $\widetilde{\bigoplus}_{\mathfrak{p} \in S'_k \setminus S_k}$ a map $\delta_0$ from $C$ to $W^0_S$. But we want a map to $\bar{\nabla}$, which is the kernel of the map $W^0_S \to \mathbb{Z}$ in the core diagram; that map takes $\kappa\big((-1,1)\big)$ to 1. We choose an infinite place $w_\infty$ of $K$ and define the modified map $\delta_1$ by $\delta_1(x_{\mathfrak{p}}) = \delta_0(x_{\mathfrak{p}}) - d_{w_\infty}$. (This modification will be irrelevant later when we take minus parts; it will suffice to know $\delta_0$.) Finally let $\delta : C \to \nabla$ be any lift of $\delta_1$.

Lemma 6.1. *The element $\delta_{\mathfrak{p}}(x_{\mathfrak{p}})$ is a $\mathbb{Q}G_{\mathfrak{p}}$-generator of $\mathbb{Q}W^0_{\mathfrak{p}}$; $\delta_{\mathfrak{p}}$ and $\delta$ are injective.*

*Proof.* Let $d_{\mathfrak{p}} := \delta_{\mathfrak{p}}(x_{\mathfrak{p}})$. Using that $(\mathrm{N}_{G_{0,\mathfrak{p}}}, 1 - F_{\mathfrak{p}}^{-1})$ is in $\ker(\kappa)$, we find

$$\mathrm{N}_{G_{0,\mathfrak{p}}}\, d_{\mathfrak{p}} = \kappa((-\mathrm{N}_{G_{0,\mathfrak{p}}}, |G_{0,\mathfrak{p}}|)) = \kappa((0, g_{\mathfrak{p}})),$$

with $g_{\mathfrak{p}} := |G_{0,\mathfrak{p}}| + 1 - F_{\mathfrak{p}}^{-1}$. It is easy to show, for example using characters, that $g_{\mathfrak{p}}$ maps to a non-zero divisor in $\mathbb{Z}[G_{\mathfrak{p}}/G_{0,\mathfrak{p}}]$. Let (by abuse of notation) $g_{\mathfrak{p}}^{-1}$ denote any element of $\mathbb{Q}[G_{\mathfrak{p}}]$ such that $g_{\mathfrak{p}} g_{\mathfrak{p}}^{-1}$ goes to $1 \in \mathbb{Q}[G_{\mathfrak{p}}/G_{0,\mathfrak{p}}]$. Then the element $g_{\mathfrak{p}}^{-1}\,\mathrm{N}_{G_{0,\mathfrak{p}}}$ is uniquely defined, and we may write $\kappa\big((0,1)\big) = g_{\mathfrak{p}}^{-1}\,\mathrm{N}_{G_{0,\mathfrak{p}}}\, d_{\mathfrak{p}} \in \mathbb{Q}G_{\mathfrak{p}} d_{\mathfrak{p}}$. Since $W^0_{\mathfrak{p}}$ is generated by the two elements $d_{\mathfrak{p}}$ and $\kappa\big((0,1)\big)$, the first statement of the lemma follows. The rest is a consequence of the (known and easily reproved) fact that $\mathbb{Q}W^0_{\mathfrak{p}} \cong \mathbb{Q}G_{\mathfrak{p}}$. (We will reuse the element $g_{\mathfrak{p}}$ later.) □

## 7. Endowing the diagrams with metrisations

We recall that a metrisation of a complex $A \xrightarrow{f} B$ of cohomologically trivial modules (over $\mathbb{Z}G$ or over $R$) is an $\mathbb{R}G$-isomorphism $\varphi : \ker(\mathbb{R} \otimes f) \to \mathrm{coker}(\mathbb{R} \otimes f)$. In other words, if $0 \to U \to A \xrightarrow{f} B \to V \to 0$ is exact ($A$ and $B$ as above), then every $\mathbb{R}G$-isomorphism $\varphi : \mathbb{R} \otimes U \to \mathbb{R} \otimes V$ is a metrisation. To each metrised pair $E = (A \to B, \varphi)$ we attached a refined Euler characteristic $\chi_{\mathrm{ref}}(E)$ lying in $\mathrm{K}_0(\mathbb{Z}G, \mathbb{R})$ or $\mathrm{K}_0(R, \mathbb{R})$. We will now give metrisations to the rows in (D1) and (D2) in a 'compatible' way, starting from the standard Dirichlet metrisation of the top row of (D1). So we first have to explain what kind of compatibility is meant here. The crucial technical result is due to Burns [Bur01, Proposition 1.2.3 and Remark 1.2.4], and goes as follows. (We state it over $\mathbb{Z}G$, but it holds over $R$ just as well. It should be said here that Burns proves quite a bit more.)

Proposition 7.1. *Assume that a diagram of c.t. $\mathbb{Z}G$-modules is given as follows.*

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ \downarrow & & \downarrow \\ A_2 & \xrightarrow{f_2} & B_2 \\ \downarrow & & \downarrow \\ A_3 & \xrightarrow{f_3} & B_3 \end{array}$$

*Suppose that the columns of this are short exact sequences, and that the resulting snake map $\ker(\mathbb{R} \otimes f_3) \to \mathrm{coker}(\mathbb{R} \otimes f_1)$ is zero. (Expressed more directly: the resulting column of kernels, or of cokernels, becomes short exact on tensoring with $\mathbb{R}$.) Suppose finally that the three horizontal complexes making up the above diagram are all metrised, via $\varphi_i : \ker(\mathbb{R} \otimes f_i) \to \mathrm{coker}(\mathbb{R} \otimes f_i)$, such*



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*that the obviously arising ladder*

$$\begin{array}{ccc} \ker(\mathbb{R}\otimes f_1) & \xrightarrow{\varphi_1} & \operatorname{coker}(\mathbb{R}\otimes f_1) \\ \downarrow & & \downarrow \\ \ker(\mathbb{R}\otimes f_2) & \xrightarrow{\varphi_2} & \operatorname{coker}(\mathbb{R}\otimes f_2) \\ \downarrow & & \downarrow \\ \ker(\mathbb{R}\otimes f_3) & \xrightarrow{\varphi_3} & \operatorname{coker}(\mathbb{R}\otimes f_3) \end{array}$$

*is commutative. (Let us call such a setup a 'well-metrised short exact sequence of complexes'.) Then*

$$\chi_{\mathrm{ref}}(A_2 \to B_2, \varphi_2) = \chi_{\mathrm{ref}}(A_1 \to B_1, \varphi_1) + \chi_{\mathrm{ref}}(A_3 \to B_3, \varphi_2).$$

*To coin a phrase, 'the refined Euler characteristic is additive on well-metrised short exact sequences of complexes'.*

*Remark* 7.2. This applies to both diagrams (D1) and (D2), since in both of them the $\mathbb{R}$-tensored snake map is zero (equivalently, the border columns become exact on tensoring with $\mathbb{R}$). For (D2) this is obvious. In (D1), $Z'$ was constructed to be the exact kernel in the leftmost column, and the cokernel of the upper vertical map is $cl_{K,S}$, which is finite and disappears under $\mathbb{R}\otimes -$. We recall that the top and middle row of (D1) are exact, and the bottom row becomes exact after $\mathbb{R}\otimes -$. As a consequence, the rightmost column also becomes exact after $\mathbb{R}\otimes -$ since this is true for the leftmost column.

We choose a metrisation $\varphi_1$ for the bottom row in (D1) as follows. Both $Z'$ and $Z''/tors = Z_0$ are obtained by taking $\widetilde{\bigoplus}_{\mathfrak{p}\in S'_k\setminus S_k}$ of local terms, and the corresponding sequence of local terms is

$$\Delta G_{\mathfrak{p}}\cdot x_{\mathfrak{p}} \to \mathbb{Z}G_{\mathfrak{p}}\cdot x_{\mathfrak{p}} \xrightarrow{h\,\mathrm{N}_{G_{\mathfrak{p}}}} \mathbb{Z}G_{\mathfrak{p}}\cdot x_{\mathfrak{p}} \to \mathbb{Z}G_{\mathfrak{p}}/(\mathrm{N}_{G_{\mathfrak{p}}})\cdot x_{\mathfrak{p}}.$$

Note that this sequence becomes exact upon tensoring with $\mathbb{R}$ (or $\mathbb{Q}$). We declare $\varphi_1 : \mathbb{R}\otimes\Delta G_{\mathfrak{p}} \to \mathbb{R}G_{\mathfrak{p}}/(\mathrm{N}_{G_{\mathfrak{p}}})$ to be the map that is induced by the identity map on $\mathbb{R}G_{\mathfrak{p}}$. (It is straightforward that this map $\varphi_1$ is an isomorphism; over $\mathbb{Z}$ we would only get an injection.)

For $\varphi_3 : \mathbb{R}\otimes E_{S'} \to \mathbb{R}\otimes\Delta S'$ we take the negative of the Dirichlet map exactly as in § 1. (This was called $\varphi_{\mathrm{Dir}}$.)

We now define $\varphi_2 : \mathbb{R}\otimes(C\oplus E_S) \to \mathbb{R}\nabla$. On $\mathbb{R}E_S$ we take any lift of $\varphi_3|\mathbb{R}E_S$ through the epimorphism $\mathbb{R}\nabla \to \mathbb{R}\Delta S'$. (This is possible since $\mathbb{R}G$ is semisimple.) We recall the element $d_{\mathfrak{p}} = \kappa((-1,1)) \in W^0_{\mathfrak{p}}$ defined in the proof of Lemma 6.1; we define another element $d'_{\mathfrak{p}} \in W^0_{\mathfrak{p}}$ to be $\kappa((1,0))$, and we define (recalling that $\bar{e}$ and $\bar{\bar{e}}$ were defined in formula (8)):

$$\varphi_2(x_{\mathfrak{p}}) = \bar{e}_{\mathfrak{p}}\biggl(h\cdot\log(\mathrm{N}\,\mathfrak{P})d_{\mathfrak{p}} - \sum_{w|\infty}\log|u_{\mathfrak{p}}|_w d_w\biggr) - \bar{\bar{e}}_{\mathfrak{p}}d'_{\mathfrak{p}}.$$

We suppress the map $\mathrm{ind}_{\mathfrak{p}}$ here. The correction term $\sum_{w|\infty}\dots$ is chosen in such a way that $\varphi_2(x_{\mathfrak{p}}) \in \bar{\nabla}$ (note that the map $W^0_{S'} \to \mathbb{Z}$ takes $d_{\mathfrak{p}}$ and $d_w$ to 1 and use the product formula, recalling $u_{\mathfrak{p}}O_K = \mathfrak{P}^h$), and will be irrelevant in the end.

Again, an adjustment (which uses some infinite place and will disappear in the minus part) is needed to put $\varphi_2(x_{\mathfrak{p}})$ into $\mathbb{R}\nabla$.

In order to prove the next result, we have to take an irrevocable step: we let $S = S_\infty$ throughout, and we will work in the minus part only. Recall that we use a non-standard convention: $M^- = R\otimes_{\mathbb{Z}G} M$ with $R = \mathbb{Z}[\frac{1}{2}][G]/(1+c)$, so inverting 2 is included in taking minus parts. For example, $E_S^- = \mu_{\mathrm{odd}}(K)$, the odd part of the group of roots of unity in $K$.



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LEMMA 7.3. *The resulting diagram*

$$\begin{array}{ccc} \mathbb{R}\otimes E_{S'} & \xrightarrow{\varphi_3} & \mathbb{R}\otimes \Delta S' \\ \uparrow{\scriptstyle(\beta,\mathrm{inc})} & & \uparrow \\ \mathbb{R}\otimes (C\oplus E_S) & \xrightarrow{\varphi_2} & \mathbb{R}\otimes \nabla \\ \uparrow & & \uparrow \\ \mathbb{R}\otimes Z' & \xrightarrow{\varphi_1} & \mathbb{R}\otimes Z'' \end{array}$$

*commutes in the minus part, so in particular $\varphi_2^-$ is an isomorphism as well and (D1)$^-$ is now well metrised. (All vertical maps are directly from the diagram (D1).)*

*Proof.* We start with the lower square. We have to unravel the map $Z'' \to \bar{\nabla}$. It suffices to deal with the local terms. There $Z''_{\mathfrak{p}} = \mathbb{Z}G_{\mathfrak{p}}/(\mathrm{N}_{G_{\mathfrak{p}}}) \cdot x_{\mathfrak{p}}$, and looking at (19) and the core diagram we see that $\bar{1} \cdot x_{\mathfrak{p}} \in Z''_{\mathfrak{p}}$ maps to $d'_{\mathfrak{p}} = \kappa((1,0))$. (See diagrams (21) and (25).) Moreover $Z'_{\mathfrak{p}} = \Delta G_{\mathfrak{p}} \cdot x_{\mathfrak{p}}$. We are now ready to chase a typical element $\alpha x_{\mathfrak{p}}$ (with $\alpha \in \Delta G_{\mathfrak{p}}$). Under $\varphi_1$ this becomes $\bar{\alpha} \cdot x_{\mathfrak{p}}$. This goes to $\alpha d'_{\mathfrak{p}} \in W^0_{\mathfrak{p}}$. The other way round, $\alpha x_{\mathfrak{p}}$ maps upward to $(-\alpha x_{\mathfrak{p}}, \alpha\beta(x_{\mathfrak{p}})$ (see (21)). We want to see where this is sent to by $\varphi_2$. But $\mathbb{R}E_S$ becomes zero in the minus part, so we may replace the second component $\alpha\beta(x_{\mathfrak{p}})$ by zero. Now $\bar{e}_{\mathfrak{p}}\alpha = 0$ and $\bar{\bar{e}}_{\mathfrak{p}}\alpha = \alpha$, so in evaluating $\varphi_2((-\alpha x_{\mathfrak{p}}, 0))$ only the $\bar{\bar{e}}_{\mathfrak{p}}$ part matters, and the definition gives $\bar{\bar{e}}_{\mathfrak{p}}\alpha d'_{\mathfrak{p}}$, so the lower square commutes.

For the upper square, $\varphi_2$ is defined just in the right way that chasing $u \in E_S$ either way gives the same result. So it only remains to chase $(x_{\mathfrak{p}}, 0)$. From the core diagram we know that $d_{\mathfrak{p}}$ (respectively $d'_{\mathfrak{p}}$) goes to 1 (respectively 0) under the map $W^0_{\mathfrak{p}} \to \mathbb{Z}$ (which gives rise to $\nabla \to \Delta S'$). From this and from $\beta(x_{\mathfrak{p}}) = u_{\mathfrak{p}}$), one can see that $(x_{\mathfrak{p}}, 0)$ goes to $h \log(\mathrm{N}\,\mathfrak{P}) \cdot \mathfrak{P} - \sum_{w|\infty} |u_{\mathfrak{p}}|_w \cdot w \in \Delta S'$ via both routes in the square. (Here we did not need to take minus parts.) $\square$

In order to metrise (D2) we *retain the assumption* that $S = S_\infty$, and we continue to take minus parts of all modules involved. We recall that $E_S^-$ is now finite. From the construction or from other arguments it follows that $(\nabla/\delta(C))^-$ is finite as well. Thus one has just the obvious metrisation $0 \xrightarrow{\sim} 0$ for the minus part of the top row of (D2), and we call it $\varphi_{\mathrm{triv}}$. For the middle row of (D2), which agrees with the middle row of (D1), we take the metrisation $\varphi_2$. For the bottom row of (D2), which looks very simple-minded, we cannot take the most obvious metrisation which would be $\mathrm{id} : \mathbb{R}C \to \mathbb{R}C$. To make the next lemma work, we have to choose

$$\psi = \delta^{-1}\, \varphi_2 | \mathbb{R}C : \mathbb{R}C \to \mathbb{R}C.$$

LEMMA 7.4. *With these three metrisations $\varphi_{\mathrm{triv}}, \varphi_2, \psi$, the minus part of diagram (D2) is well metrised.*

*Proof.* As in the last lemma, we have to show that two squares are commutative. For the square involving $\varphi_{\mathrm{triv}}$ (which is the zero map between two zero modules), this is trivial. So we have to show that the following diagram commutes (and this works without taking minus parts).

$$\begin{array}{ccc} \mathbb{R}\otimes (C\oplus E_S) & \xrightarrow{\varphi_2} & \mathbb{R}\otimes \nabla \\ \uparrow & & \uparrow{\scriptstyle\delta} \\ \mathbb{R}\otimes C & \xrightarrow{\psi} & \mathbb{R}\otimes C \end{array}$$

But this is obvious, according to the definition of $\psi$. $\square$

One row can be dealt with quickly (see following remark).

*Remark* 7.5. Because of the zero in the middle of the bottom row of (D2), the transpose of $\psi$ is $\psi$ itself, and thus the refined Euler characteristic of that row metrised with $\psi$ is $\partial(\det_{\mathbb{R}G} \psi)$.

In the following definitions we mention various rows of diagrams; of course we mean these rows together with the metrisations as just explained. Let

$$\begin{aligned} X_{S'} &= \chi_{\mathrm{ref}}(\text{top row of (D1)}),\\ X_1 &= \chi_{\mathrm{ref}}(\text{middle row of (D1)}),\\ X_C &= \chi_{\mathrm{ref}}(\text{bottom row of (D1)}),\\ X_\infty^- &= \chi_{\mathrm{ref}}(\text{bottom row of (D1)}^- \text{ with } S = S_\infty).\end{aligned}$$

(The notation (D1)$^-$ should be self-explanatory: take minus parts of all terms. Note that 'minus part' is an exact functor.)

By Proposition 7.1, Lemmas 7.3 and 7.4 and Remark 7.5 we conclude that

$$X_1 = X_{S'} + X_C \quad \text{in } \mathrm{K}_0(\mathbb{Z}G, \mathbb{R}) \tag{27}$$

and

$$X_\infty^- = X_1^- - \partial(\det_{\mathbb{R}G}(\psi))^- \in \mathrm{K}_0(R, \mathbb{R}). \tag{28}$$

Note here that we write the natural map $\mathrm{K}_0(\mathbb{Z}G, \mathbb{R}) \to \mathrm{K}_0(R, \mathbb{R})$ also simply by an exponent minus.

Therefore our target object $X_\infty^-$ can be calculated from the three quantities $X_{S'}$, $X_C$, $\partial(\det_{\mathbb{R}G}(\psi))$. The first of these three is given by ETNC, and the two others are explicitly calculable. We do one of them now.

Lemma 7.6. *Let $v_\mathfrak{p} = h \cdot |G_\mathfrak{p}|\bar{e}_\mathfrak{p} + \bar{\bar{e}}_\mathfrak{p} \in \mathbb{Q}G_\mathfrak{p}$. Then*

$$X_C = \sum_{\mathfrak{p} \in S'_k \setminus S_k} \partial(v_\mathfrak{p}).$$

*Proof.* Both $\chi_{\mathrm{ref}}$ and $\delta$ are compatible with induction, so we may work locally and calculate the refined Euler characteristic of

$$\mathbb{Z}G_\mathfrak{p} \xrightarrow{h \cdot \mathrm{N}_{G_\mathfrak{p}}} \mathbb{Z}G_\mathfrak{p}$$

with 'identity' $\mathbb{R}\Delta G_\mathfrak{p} \to \mathbb{R}G_\mathfrak{p}/(\mathrm{N}_{G_\mathfrak{p}})$ as metrisation. We tensor with $\mathbb{R}$ and decompose along the characters $\chi$ of $G_\mathfrak{p}$. (By the way, it is allowed to tensor with $\mathbb{C}$, since the natural map $\mathrm{K}_0(\mathbb{Z}G_\mathfrak{p}, \mathbb{R}) \to \mathrm{K}_0(\mathbb{Z}G_\mathfrak{p}, \mathbb{C})$ is injective.) The map in the above complex is zero, respectively an isomorphism, in the $\chi$ part, according to whether $\chi$ is non-trivial or trivial. In the $\chi$ parts for non-trivial $\chi$ therefore, the transpose of the metrisation isomorphism is the metrisation isomorphism itself, that is, identity. For trivial $\chi$, the metrisation is zero, and its transpose is the given map of the complex, which is multiplication by $h|G_\mathfrak{p}|$. Hence the transposed isomorphism is multiplication by $v_\mathfrak{p}$, and this gives $\partial(v_\mathfrak{p})$ as $\chi_{\mathrm{ref}}$ of the above metrised sequence. This proves the lemma. □

In preparation for our final calculations we put Lemma 7.6, (27) and (28) together for reference.

Proposition 7.7. *One has $X_\infty^- = X_{S'}^- + \sum_{\mathfrak{p} \in S'_k \setminus S_k} \partial(v_\mathfrak{p})^- - \partial(\det_{\mathbb{R}G}(\psi))^-$.*

## 8. Refined Euler characteristics and Fitting ideals

We recall that $S = S_\infty$. We will calculate $X_\infty^-$, the refined Euler characteristic of the top row of diagram (D2) in the minus part, and as a consequence we will obtain information on Fitting ideals.

For practical reasons we introduce a slightly generalised notation. (We only treat $R$-modules, the case of $\mathbb{Z}G$-modules being completely analogous). Suppose $P$ is $R$-free of rank $n$ and $L$ is any

$R$-lattice in $\mathbb{Q}P$ (that is, any finitely generated $R$-submodule of $\mathbb{Q}P$ whose $\mathbb{Q}$-span is all of $\mathbb{Q}P$). We define

$$\mathrm{Fitt}_R(P; L) := x^{-n} \cdot \mathrm{Fitt}_R(P/xL),$$

where $x \in R$ is any non-zero divisor such that $xL \subset P$. It is helpful in this context to remember that $\mathrm{Fitt}_R(P/xP) = Rx^n$.

Lemma 8.1. $\mathrm{Fitt}_R(P; L)$ *is well defined.*

*Proof.* Assume that $y \in R$ is another non-zero divisor such that $yL \subset P$. We show that the right-hand side in the above definition does not change if $x$ is replaced by $xy$ (and this suffices by symmetry). If $z_1, \dots, z_m$ is a set of generators of $L$, then $\mathrm{Fitt}(P/xL)$ is generated by the $n \times n$ minors of the matrix whose rows are the $xz_i$, written as coordinate vectors according to a chosen basis of $P$. It is then clear that the Fitting ideal gets multiplied by $y^n$ when $xL$ is replaced by $xyL$; this exactly cancels against the $y^{-n}$ which comes from the first factor. ◻

Let now $f \in \mathbb{Q}R^\times$ be such that $\partial(f) = X_\infty^-$. Then $f$ is unique modulo $R^\times$, but we just make a once-for-all choice. Before we start putting everything together, we explain how one gets from knowledge of $f$ to the Fitting ideal of $cl_K^{\vee -}$. (The Pontryagin dual comes in unavoidably.) We recall the sequence which gave rise to $X_\infty^-$, making the *final extra assumption* that $\mu_{\mathrm{odd}}(K)$ is trivial:

$$0 \to A^- \to \tilde{B}^- \to (\nabla/\delta(C))^- \to 0. \tag{29}$$

All terms in it are now c.t. over $R$, and Lemma 3.1 tells us that

$$\mathrm{Fitt}_R((\nabla/\delta(C))^-) = (f) \tag{30}$$

as principal ideals of $R$. (The *a priori* fractional ideal $(f)$ now turns out actually to be an ideal.) The transition to an explicit description is done by the following result.

Lemma 8.2. *One has* $\mathrm{Fitt}_R(cl_K^{\vee -}) = f \cdot \mathrm{Fitt}_R(\delta(C)^-; \bar{\nabla}^-)$.

*Proof.* Just for this proof we agree to omit all minus exponents. Choose a non-zero divisor $x \in R$ such that $x\nabla \subset \delta(C)$ (possible since $\nabla/\delta(C)$ is finite). There is the crucial short exact sequence $0 \to cl_K \to \nabla \to \bar{\nabla} \to 0$. By abuse of notation we also use $\delta$ for the map $C \to \bar{\nabla}$ and note that this is still injective since $C$ is free. This produces two short exact sequences

$$0 \to cl_K \to \frac{\nabla}{\delta(C)} \to \frac{\bar{\nabla}}{\delta(C)} \to 0, \quad 0 \to \frac{\bar{\nabla}}{\delta(C)} \to \frac{x^{-1}\delta(C)}{\delta(C)} \to \frac{x^{-1}\delta(C)}{\bar{\nabla}} \to 0.$$

These combine into the four-term sequence

$$0 \to cl_K \to \frac{\nabla}{\delta(C)} \to \frac{x^{-1}\delta(C)}{\delta(C)} \to \frac{x^{-1}\delta(C)}{\bar{\nabla}} \to 0.$$

From Lemma 5 in [BG03b] we obtain

$$\begin{aligned}\mathrm{Fitt}_R(cl_K^\vee) &= \mathrm{Fitt}_R\left(\frac{\nabla}{\delta(C)}\right) \cdot \mathrm{Fitt}_R^{-1}\left(\frac{x^{-1}\delta(C)}{\delta(C)}\right) \cdot \mathrm{Fitt}_R\left(\frac{x^{-1}\delta(C)}{\bar{\nabla}}\right) \\ &= f \cdot \mathrm{Fitt}_R(\delta(C); \bar{\nabla}).\end{aligned}$$

For safety we repeat that all modules in the preceding argument silently carry an exponent minus. ◻

We next calculate the Fitting ideal on the right in the statement of Lemma 8.2, to give an idea of the final result. To begin with, we note that in the minus part, $\bar{\nabla}$ agrees with $\widetilde{\bigoplus}_{\mathfrak{p} \in S_k' \setminus S_k} W_{\mathfrak{p}}^0$. Thus everything decomposes: we just need to calculate $\mathrm{Fitt}_R(R\delta(x_{\mathfrak{p}}); W_{\mathfrak{p}}^{0-})$ and take the product over all $\mathfrak{p} \in S'_k \setminus S_\infty$. Recall the description of $W^0_{\mathfrak{p}}$ from § 5. The element $d_{\mathfrak{p}} = \delta(x_{\mathfrak{p}})$ was defined as $\kappa((-1,1))$ in § 6, and from the proof of Lemma 6.1 we easily extract

$$W^0_{\mathfrak{p}} = d_{\mathfrak{p}} \cdot \langle 1, g_{\mathfrak{p}}^{-1} \mathrm{N}_{G_{0,\mathfrak{p}}} \rangle. \tag{31}$$

We recall from § 6 that $g_{\mathfrak{p}} = |G_{0,\mathfrak{p}}| + 1 - F_{\mathfrak{p}}^{-1}$ maps to a non-zero divisor $\bar{g}$ of $\mathbb{Z}_p[G_{\mathfrak{p}}/G_{0,\mathfrak{p}}]$, and $g_{\mathfrak{p}}^{-1}$ stands for any lift of $\bar{g}^{-1}$ to $\mathbb{Q}_p[G_{\mathfrak{p}}]$. So we have

$$\mathrm{Fitt}_R(R\delta(x_{\mathfrak{p}}); W^0_{\mathfrak{p}}) = \langle 1, g_{\mathfrak{p}}^{-1} \mathrm{N}_{G_{0,\mathfrak{p}}} \rangle, \tag{32}$$

and $\mathrm{Fitt}_R(\delta(C)^-; \bar{\nabla}^-)$ is the product of all these fractional ideals (extended to ideals of $R$). We now identify the individual ideals.

Lemma 8.3. *If we let $h_{\mathfrak{p}} = e'_{\mathfrak{p}} g_{\mathfrak{p}} + e''_{\mathfrak{p}}$, then*

$$\langle 1, g_{\mathfrak{p}}^{-1} \mathrm{N}_{G_{0,\mathfrak{p}}} \rangle = h_{\mathfrak{p}}^{-1} U_{\mathfrak{p}}$$

*as fractional ideals of $\mathbb{Z}G_{\mathfrak{p}}$.*

*Proof.* We omit all indices $\mathfrak{p}$ in this proof (the $h$ that results has nothing to do with the integer factor $h$ used before), and we write $N$ for $\mathrm{N}_{G_{0,\mathfrak{p}}}$. We note that $h$ is a non-zero divisor in $\mathbb{Q}[G_{\mathfrak{p}}]$ to start with. Recall from § 2 that $U = U_{\mathfrak{p}} = \langle 1 - e'F^{-1}, N \rangle$. We multiply the left-hand side in the lemma by $h$; this gives $\langle h, N \rangle$. Now $h = e'(N + 1 - F^{-1}) + e'' = 1 - e'F^{-1} + N$. Thus $\langle h, N \rangle = \langle 1 - e'F^{-1} + N, N \rangle = U$, as had to be shown. □

If we assemble Lemmas 8.2 and 8.3 with (32) and the sentence following it, we obtain a non-explicit version of our main result. Let $h_{\mathrm{glob}} = \prod_{\mathfrak{p} \in S'_k \setminus S_\infty} h_{\mathfrak{p}}$, and recall that $U$ is the product of all $U_{\mathfrak{p}}$ with $\mathfrak{p} \in S'_k \setminus S_\infty$.

Corollary 8.4. *One has* $\mathrm{Fitt}_R\, cl_K^{\vee -} = f h_{\mathrm{glob}}^{-1} U^-$.

The explicit form of our main result runs as follows (for the sake of clarity we repeat even the most standard assumptions we are making; the quantity $\omega$ was defined in § 2).

Theorem 8.5. *Assume $K/k$ is abelian with group $G$, $K$ is CM and $k$ totally real. Assume that ETNC holds for the motive $h^0(K)$ with coefficients in $\mathbb{Z}G$. Then*

$$f h_{\mathrm{glob}}^{-1} R = \omega R$$

*and consequently, if the number of roots of unity in $K$ is a power of 2,*

$$\mathrm{Fitt}_R(cl_K^{\vee -}) = \omega U^- = \mathrm{SKu}'(K/k)^-$$

*(see § 2).*

*Remark.* Since all relevant exact sequences are base-changed from $\mathbb{Z}[G]$ to $R$, it is even enough to suppose the validity of $\mathrm{ETNC}(h^0(K), R)$ instead of $\mathrm{ETNC}(h^0(K), \mathbb{Z}[G])$.

*Proof of Theorem 8.5.* It suffices to prove the first formula, since the second is a consequence via Corollary 8.4.

We take Proposition 7.7 and give values to the right-hand side. For $X_{S'}$ we put $\partial(L^*_{S'}(0)^\sharp)$, according to ETNC. Therefore $f$ equals, up to a factor in $R^*$, the product

$$f_1 := L^*_{S'}(0)^\sharp \cdot \prod_{\mathfrak{p} \in S'_k \setminus S_\infty} (h|G_{\mathfrak{p}}|\bar{e}_{\mathfrak{p}} + \bar{\bar{e}}_{\mathfrak{p}}) \cdot \det_{\mathbb{R}G}(\psi)^{-1}. \tag{33}$$

(Of course this is meant as an equation in $\mathbb{R} \otimes R$; we suppress the canonical epimorphism $\mathbb{R}[G] \to \mathbb{R} \otimes R$.) The terms on the right-hand side all split up as products over $\mathfrak{p}$. For the middle term this is obvious, and it is well known for the $L$-value; we will see the details very soon. The determinant also factors since $\psi = \widetilde{\bigoplus}_{\mathfrak{p}\in S'_k\setminus S_k} \psi_{\mathfrak{p}}$ for local maps $\psi_{\mathfrak{p}}$. We will show that, for every odd character $\chi$ of $G$, the values $\chi(f_1 h_{\mathrm{glob}}^{-1})$ and $\chi(\omega)$ agree. This will prove $f_1 h_{\mathrm{glob}}^{-1} = \omega$, and we will be done. (Note that $h_{\mathrm{glob}}$ is likewise a product of local factors.)

For the rest of the argument we fix one character $\chi$ of $G$. Tacitly, $\chi$ will also be considered as a character of $G_{\mathfrak{p}}$ by restriction. We write the set $S'_k \setminus S_{\infty,k}$ as the disjoint union of three subsets $T_1$, $T_2$ and $T_3$ (depending on $\chi$) as follows:

1. $\mathfrak{p} \in T_1$ if and only if $\chi$ is trivial on $G_{\mathfrak{p}}$, equivalently if and only if $\chi(\bar{e}_{\mathfrak{p}}) = 1$;
2. $\mathfrak{p} \in T_2$ if and only if $\chi$ is non-trivial on $G_{\mathfrak{p}}$ but trivial on $G_{0,\mathfrak{p}}$, equivalently if and only if $\chi(e'_{\mathfrak{p}}) = 1$ but $\chi(\bar{e}_{\mathfrak{p}}) = 0$;
3. $\mathfrak{p} \in T_3$ if and only if $\chi$ is non-trivial on $G_{0,\mathfrak{p}}$, i.e. $\chi(e'_{\mathfrak{p}}) = 0$.

We will call $\mathfrak{p}$ of type 1, 2 or 3 respectively. This is equivalent to $\mathfrak{p}$ being split (type 1), unramified non-split (type 2), and ramified (type 3), respectively, in the subfield of $K$ cut out by $\chi$. This division into types also corresponds to the decomposition of 1 into orthogonal idempotents

$$1 = \bar{e}_{\mathfrak{p}} + \bar{\bar{e}}_{\mathfrak{p}} e'_{\mathfrak{p}} + \bar{\bar{e}}_{\mathfrak{p}} e''_{\mathfrak{p}},$$

where type $i$ corresponds to $\chi$ sending the $i$th of the right-hand summands to 1 and the other two to 0, for $i = 1, 2, 3$.

Lemma 8.6. *In the minus part, $\psi = \widetilde{\bigoplus}_{\mathfrak{p}\in S'_k\setminus S_k} \psi_{\mathfrak{p}}$, where each $\psi_{\mathfrak{p}}$ is the endomorphism of $x_{\mathfrak{p}}\mathbb{R}G_{\mathfrak{p}}$ which is given via multiplication by the following element $t_{\mathfrak{p}}$:*

$$t_{\mathfrak{p}} = h \log \mathrm{N}\,\mathfrak{P} \cdot \bar{e}_{\mathfrak{p}} + \frac{1 - F_{\mathfrak{p}}^{-1}}{g_{\mathfrak{p}}} \cdot \bar{\bar{e}}_{\mathfrak{p}} e'_{\mathfrak{p}} + \bar{\bar{e}}_{\mathfrak{p}} e''_{\mathfrak{p}}.$$

*The denominator $g_{\mathfrak{p}}$ of the second summand may also be changed to $h_{\mathfrak{p}}$.*

*Proof.* By definition, $\psi$ is $\delta^{-1}\varphi_2$. In the minus part, the adjustment terms at infinity used in the definition of $\delta$ and $\varphi_2$ simply disappear. So we may calculate with $\delta_0$ instead of $\delta$, and with the map $\varphi'_2$ defined exactly as $\varphi_2$ with the term involving $\sum_{w|\infty}$ omitted. Since $\delta_0$ maps $x_{\mathfrak{p}}$ to $d_{\mathfrak{p}}$, we are reduced to checking the equality

$$\varphi'_2(x_{\mathfrak{p}}) = t_{\mathfrak{p}} d_{\mathfrak{p}}.$$

The $\bar{e}_{\mathfrak{p}}$ component of this equality is a direct consequence of the definitions. Thus it suffices to show that our formula is correct in the $\bar{\bar{e}}_{\mathfrak{p}}$ component. By definition, $\bar{\bar{e}}_{\mathfrak{p}}\varphi'_2(x_{\mathfrak{p}}) = -\bar{\bar{e}}_{\mathfrak{p}}\kappa((1,0))$. Using Lemma 6.1 and its proof we get

$$\begin{aligned}\kappa((1,0)) &= -\kappa((-1,1)) + \kappa((0,1)) \\ &= (-1 + g_{\mathfrak{p}}^{-1}\,\mathrm{N}_{G_{0,\mathfrak{p}}})d_{\mathfrak{p}}.\end{aligned}$$

On multiplying this expression by $\bar{\bar{e}}_{\mathfrak{p}} e'_{\mathfrak{p}}$, the right-hand term becomes

$$\bar{\bar{e}}_{\mathfrak{p}} e'_{\mathfrak{p}} g_{\mathfrak{p}}^{-1}(-|G_{0,\mathfrak{p}}| - 1 + F_{\mathfrak{p}}^{-1} + \mathrm{N}_{G_{0,\mathfrak{p}}})d_{\mathfrak{p}}.$$

But $e'_{\mathfrak{p}}(-|G_{0,\mathfrak{p}}| + \mathrm{N}_{G_{0,\mathfrak{p}}}) = 0$, so we obtain $\bar{\bar{e}}_{\mathfrak{p}} e'_{\mathfrak{p}}\kappa((1,0)) = \bar{\bar{e}}_{\mathfrak{p}} e'_{\mathfrak{p}} g_{\mathfrak{p}}^{-1}(F_{\mathfrak{p}}^{-1} - 1)d_{\mathfrak{p}}$, which means that

$$\bar{\bar{e}}_{\mathfrak{p}} e'_{\mathfrak{p}} \varphi'_2(x_{\mathfrak{p}}) = \bar{\bar{e}}_{\mathfrak{p}} e'_{\mathfrak{p}} t_{\mathfrak{p}} d_{\mathfrak{p}}.$$



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When we multiply the equation $\kappa((1,0)) = (-1+g_{\mathfrak{p}}^{-1}\,\mathrm{N}_{G_{0,\mathfrak{p}}})d_{\mathfrak{p}}$ by $\bar{\bar{e}}_{\mathfrak{p}}e''_{\mathfrak{p}}$ (note that this is equal to $e''_{\mathfrak{p}}$, but we retain $\bar{\bar{e}}$ for clarity), then the term $g_{\mathfrak{p}}^{-1}\,\mathrm{N}_{G_{0,\mathfrak{p}}}$ drops out. So we are left with $\bar{\bar{e}}_{\mathfrak{p}}e''_{\mathfrak{p}}\kappa((1,0)) = -d_{\mathfrak{p}}$, which means that

$$\bar{\bar{e}}_{\mathfrak{p}}e''_{\mathfrak{p}}\varphi'_2(x_{\mathfrak{p}}) = \bar{\bar{e}}_{\mathfrak{p}}e''_{\mathfrak{p}}t_{\mathfrak{p}}d_{\mathfrak{p}},$$

and we are done.

The concluding statement of the lemma is an easy consequence of the definition of $h_{\mathfrak{p}}$, see Lemma 8.3. □

We now continue in the proof of Theorem 8.5. Recall that we have to show that $\chi(f_1 h_{\mathrm{glob}}^{-1}) = \chi(\omega)$, and that $f_1$ is given in (33) as a product of three terms; we process these, and then $h_{\mathrm{glob}}^{-1}$, one after another.

The $L$-function $L_{S'}(s,\chi)$ arises from the standard function $L(s,\chi)$, which omits Euler factors exactly at the primes in $T_3$ ('ramified'), by removing the Euler factors attached to primes in $T_1$ and $T_2$. Among these, precisely the inverse Euler factor for primes in $T_1$ have a (simple) zero in $s=0$, with leading term (first derivative at $s=0$) $\log \mathrm{N}\,\mathfrak{p}$. This gives

$$\begin{aligned}\chi(L^*_{S'}(0)^\sharp) &= L^*_{S'}(0,\chi^{-1})\\ &= \prod_{\mathfrak{p}\in T_1} \log \mathrm{N}\,\mathfrak{p}\cdot L_{T_2\cup T_3}(0,\chi^{-1})\\ &= \prod_{\mathfrak{p}\in T_1} \log \mathrm{N}\,\mathfrak{p}\cdot \prod_{\mathfrak{p}\in T_2}\chi^{-1}(1-F_{\mathfrak{p}})\cdot L(0,\chi^{-1}).\end{aligned}\tag{34}$$

Note that the last factor already equals $\chi(\omega)$, so the point will be that the remaining factors cancel against the terms that are yet to be calculated.

The second factor in (33) yields

$$\chi\biggl(\prod_{\mathfrak{p}\in S'_k\setminus S_{\infty,k}} (h|G_{\mathfrak{p}}|\bar{e}_{\mathfrak{p}} + \bar{\bar{e}}_{\mathfrak{p}})\biggr) = \prod_{\mathfrak{p}\in T_1}(h|G_{\mathfrak{p}}|),\tag{35}$$

since the factors for $\mathfrak{p}$ of type 2 or 3 just evaluate to 1.

From Lemma 8.6 we deduce that

$$\chi(\det(\psi)) = \prod_{\mathfrak{p}\in T_1} h\log \mathrm{N}\,\mathfrak{P}\cdot\prod_{\mathfrak{p}\in T_2}\chi(h_{\mathfrak{p}}^{-1}(1-F_{\mathfrak{p}}^{-1})).\tag{36}$$

Finally, it follows from the definitions that $\chi(h_{\mathfrak{p}}) = 1$ for $\mathfrak{p}$ of type 3 and that $\chi(h_{\mathfrak{p}}) = |G_{0,\mathfrak{p}}|$ for $\chi$ of type 1. This gives

$$\chi(h_{\mathrm{glob}}) = \prod_{\mathfrak{p}\in T_1}|G_{0,\mathfrak{p}}|\cdot\prod_{\mathfrak{p}\in T_2}\chi(h_{\mathfrak{p}}).\tag{37}$$

Now for the conclusion. The desired value $\chi(f_1 h_{\mathrm{glob}}^{-1})$ is the product of the left-hand sides of (34) and (35), divided by the left-hand sides of (36) and (37). We thus have to look at the corresponding right-hand sides. It is directly visible that the type 2 products cancel out. The type 1 products cancel out likewise, since $\log \mathrm{N}\,\mathfrak{P}$ equals $[G_{\mathfrak{p}} : G_{0,\mathfrak{p}}]\log \mathrm{N}\,\mathfrak{p}$. Only $L(0,\chi^{-1})$ remains, and as already said this means that we are done. □

*Remark.* Under our hypotheses, the minus part of $\mathcal{A}$ (the annihilators of roots of unity) coincides with $R$, and $\mathrm{SKu}'(K/k)^- = \mathrm{SKu}(K/k)^-$ in the notation of § 2.

From Theorem 8.5 and the main result of [BG03a] we obtain a corollary.

COROLLARY 8.7. *If $K$ is absolutely abelian (and $K/k$ abelian, $K$ imaginary, $k$ real as always), and if the number of roots of unity in $K$ is a 2-power, then*

$$\mathrm{Fitt}_R(cl_K^{\vee -}) = \mathrm{SKu}'(K/k)^-.$$

We can go a little further with little extra effort. We can also capture the case where $\mu(K)^-$ has projective dimension at most 1 over $R$ (instead of being zero), and all it takes is sharpening a lemma.

THEOREM 8.8. *If $K/k$ is $G$-abelian, $K$ is CM and $k$ totally real, if $\mu(K)^-$ is $R$-cohomologically trivial and ETNC holds for the motive $h^0(K)$ with coefficients in $\mathbb{Z}G$, then*

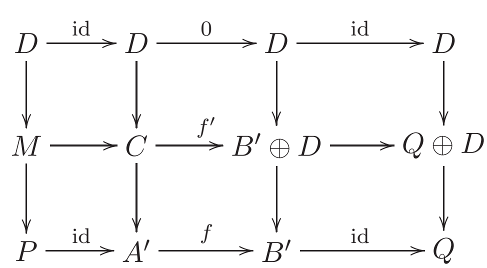
$$\mathrm{Fitt}_R(cl_K^{\vee -}) = \mathrm{SKu}(K/k)^-.$$

*Proof.* Let $\mathcal{A} = \mathrm{Fitt}_R(\mu(K)^-)$. Then our assumption implies that this $R$-ideal is projective; $\mathcal{A}$ is also the same as the $R$-annihilator of $\mu(K)^-$. The sequence (29) picks up an extra term $\mu(K)^-$ on the left. From Lemma 8.9 below (which is a generalisation of Lemma 3.1) we obtain that each of (30), Lemma 8.2 and Corollary 8.4 continue to hold if an extra factor $\mathcal{A}$ is inserted in the right-hand side. The claim of Theorem 8.8 then follows from the definition of $\mathrm{SKu}(K/k)$ (see § 2). We are done (modulo the lemma). □

LEMMA 8.9. *If $0 \to P \to A' \to B' \to Q \to 0$ is an exact sequence of $R$-modules of projective dimension at most 1, and $P$ and $Q$ are both finite, then*

$$\chi_{\mathrm{ref}}(A' \to B', \varphi_{\mathrm{triv}}) = \partial(\mathrm{Fitt}_R(Q)) - \partial(\mathrm{Fitt}_R(P)).$$

*Proof.* We will only need the case where $B'$ is torsion-free (hence projective), so we will assume this. Pick a projective resolution $0 \to D \to C \to A' \to 0$; let $f : A' \to B'$ be the given map and $f' : C \to B'$ be the obvious composition. We use our convention of omitting zeros in big diagrams. $M$ in the following diagram is defined by exactness; it is again of projective dimension at most 1.




The top row has the canonical trivialisation, and its $\chi_{\mathrm{ref}}$ is obviously zero. The bottom row has the canonical trivialisation. To make the above diagram into a well-metrised short exact sequence of complexes, we have to metrise it via $\varphi : \mathbb{R}M \to \mathbb{R}(Q \oplus D) = \mathbb{R}D$, arising as the inverse of the injective map $j : D \to M$ given by the diagram. Then we have to determine the transpose $\tilde{\varphi}$. It is easy to see (and enough for us) that $\tilde{\varphi} = f' + j^*$, where $j^*$ is any extension of $\mathbb{R} \otimes j^{-1} : \mathbb{R}M \to \mathbb{R}D$ to $\mathbb{R}C$. Then one has, by the standard rules for calculating in relative $K_0$-groups:

$$(C, \tilde{\varphi}, B' \oplus D) = (M, j^{-1}, D) + (C/M, f', B').$$

Now $P$ is the cokernel of $j : D \to M$, and $Q$ is the cokernel of $f' : C/M \to B'$. From this it follows just as in the proof of Lemma 3.1 that

$$\delta(\mathrm{Fitt}_R(P)) = -(M, j^{-1}, D), \quad \delta(\mathrm{Fitt}_R(Q)) = (C/M, f', B').$$

Since $\chi_{\mathrm{ref}}(A' \to B', \varphi_{\mathrm{triv}}) = \chi_{\mathrm{ref}}(C \to B' \oplus D, \varphi) = (C, \tilde{\varphi}, B' \oplus D)$ by multiplicativity of $\chi_{\mathrm{ref}}$ (Proposition 7.1), we obtain exactly the formula stated in the lemma. □

*Remarks.* (1) As already pointed out in § 2, the right-hand side in Theorem 8.8 lies in $\mathrm{SKu}'(K/k)^- \cap R$, but we cannot prove equality. The point is that $\mathrm{SKu}'(K/k)$ might be 'too integral'.

(2) Obviously there is an 'absolute' corollary ($k = \mathbb{Q}$) for Theorem 8.8, in analogy with Corollary 8.7 for Theorem 8.5.

As a consequence of our main results we can also say something about annihilators. We start by pointing out that the modules $cl_K^{\vee -}$ and $cl_K^-$ have the same $R$-annihilator, and we recall that the annihilator of any $R$-module $M$ contains the Fitting ideal of $M$. Therefore we know that, under the hypotheses of Theorem 8.8, the ideal $\mathrm{SKu}(K/k)$ annihilates $cl_K^-$. Actually slightly more is true, as in the following theorem.

Theorem 8.10. *Under the hypotheses of Theorem 8.8, the generalised Sinnott ideal* $\mathrm{SSi}(K/k)^-$ *annihilates* $cl_K^-$.

*Proof.* Recall that $\mathrm{SSi}(K/k)^- = \mathcal{A}\,\mathrm{SSi}'(K/k)^-$ (where $\mathcal{A} = \mathrm{Fitt}_R(\mu(K)^-)$ as above), and that $\mathrm{SSi}'(K/k)$ is generated by $x_I = cor_{K/K_I}\theta(K_I/k, S(I))$ with $I$ running through all subsets of $\{1, \ldots, s\}$; for further notation we refer to § 2. Recall likewise that $\mathrm{SKu}'(K/k)$ is generated by all the elements $a(I)x_I$. It is obvious from the definition that $a(I) = 1$ if $I$ is the maximal set $\{1, \ldots, s\}$. In particular $\mathcal{A}x_{\{1,\ldots,s\}} = \mathcal{A}\theta(K/k)$ annihilates $cl_K^-$ since it is contained in the Fitting ideal of $cl_K^{\vee -}$. The point is to see that the other terms $\mathcal{A}x_I$ annihilate $cl_K^-$ as well. But (as follows from the definition of corestriction) it is sufficient for this that $\mathcal{A}\theta(K_I/k, S(I))$ annihilates $cl_{K_I}^-$. We claim that already $\mathcal{A}\theta(K_I/k)$ has this annihilation property; to show this, simply note again that $\theta(K_I/k) \in \mathrm{SKu}'(K_I/k)$ and invoke Theorem 8.8 for $K_I/k$ instead of $K/k$. (One has to check two things to make this work: $\mu(K_I)$ is again of projective dimension at most 1 over $R_I := \mathbb{Z}G(I)^-$; and $\mathrm{Fitt}_R(\mu(K)^-)\theta(K_I/k) \subset \mathrm{Fitt}_{R_I}(\mu(K_I)^-)\theta(K_I/k)$. Both of these are easy; the first property uses that $\mu(K_I)$ can be obtained from $\mu(K)$ by taking $\mathrm{Gal}(K/K_I)$-invariants; and the second can be seen via replacing Fitting ideals by annihilators.) ☐

As said in the previous proof, $\mathcal{A}\theta(K/k)$ annihilates $cl_K^-$ under the hypotheses of Theorem 8.8. But this can be rephrased differently, as follows.

Corollary 8.11. *Under the hypotheses of Theorem 8.8, the Brumer conjecture for $K/k$ is true outside the 2-part. (Note that the Brumer conjecture is true in the plus part for trivial reasons.)*

While we do not have any simple-to-state result at hand, it is not too difficult (even for $k = \mathbb{Q}$) to find examples for which $\mathrm{SKu}(K/k)$ is strictly contained in $\mathrm{SSi}(K/k)$. We do not give details and just point out that whenever this happens and the hypotheses of Theorem 8.8 are satisfied, we are assured by Theorem 8.10 that the Fitting ideal of $cl_K^{\vee -}$ is strictly smaller than its annihilator.

We conclude this paper by discussing the necessity of taking the Pontryagin dual in our main results and comparing our results to Kurihara's (see [Kur03a]). It is well known that for any cyclic group $G$ and any finite $\mathbb{Z}G$-module $M$ the $\mathbb{Z}G$-Fitting ideals of $M$ and $M^\vee$ are the same. The analogous statement for $R = \mathbb{Z}G^-$-modules (still $G$-cyclic) is an easy consequence via base change, so one can replace $cl_K^{\vee -}$ by $cl_K^-$ in Theorems 8.5 and 8.8 and Corollary 8.7 in that case. For general abelian groups, this is definitely not true. The question remains whether there are reasonable conditions which would ensure $\mathrm{Fitt}_R(cl_K^{\vee -}) = \mathrm{Fitt}_R(cl_K^-)$, or, more generally, which would permit one to calculate $\mathrm{Fitt}_R(cl_K^-)$. For now, it is quite interesting that Kurihara [Kur03a] does obtain a formula for $\mathrm{Fitt}_R(cl_K^-)$ (not the dual) in certain cases, and the right-hand side in his formula is the same as ours.

We now go through the cases treated in [Kur03a], noting first that in that paper it is assumed that $k = \mathbb{Q}$, and the results are stated over the $p$-completion of $R$, with $p$ being an arbitrary prime. (Our result over $R$ is of course equivalent to the conjunction of all its $p$-adified versions, with $p \neq 2$.) The first case (Theorem 0.4 in [Kur03a]) deals with extensions $K/k$ such that no $p$-adic prime splits in $K^+/K$. If we make the extra assumption that $K/k$ is 'nice' (see [Gre00]; this includes cohomological triviality of $\mu_K^-$), then it follows that $cl_K^-$ is $R$-cohomologically trivial, and then its Fitting ideal does not change upon dualising, so our result in Theorem 8.5 and the quoted theorem of [Kur03a] are 'equivalent'. Compare also [Gre00]. The second case (Theorem 0.5 in [Kur03a]) assumes that $K$ is the $n$th layer of the $p$-cyclotomic extension over a field $K_0$ whose degree over $\mathbb{Q}$ is prime to $p$. Here the $p$-part of $G$ is cyclic, so in the $p$-part, dualising again does not change the Fitting ideal. The most interesting case in the present context is Kurihara's Theorem 0.6, where he assumes that $\zeta_p \notin K$ and that $p$ is at most tamely ramified. Here the $p$-part of $G$ need not be cyclic. If we impose the slightly stronger hypothesis that $\mu(K)$ is of 2-power order, then [Kur03a] and our Theorem 8.5 show in particular that $cl_K^-$ and its dual have the same $R$-Fitting ideal, which now is a genuine restriction on the module structures. It would be very nice to understand just why this happens.

Joint work with Kurihara (in preparation) shows the existence of examples $K/k$ for which the top level element $\theta(K/k)$ is not in $\mathrm{Fitt}_R(cl_K^-)$. Since these examples all have $k \neq \mathbb{Q}$, we do not know the validity of ETNC, so in general we have no definite result on $\mathrm{Fitt}_R(cl_K^{\vee -})$ yet, even if $K/k$ satisfies the assumptions in Theorem 8.8. But from this joint work and [Kur03b] one can deduce that there are unconditional examples with $\mathrm{Fitt}_R(cl_K^{\vee -}) = \mathrm{SKu}(K/k)$ and $\theta(K/k) \notin \mathrm{Fitt}_R(cl_K^-)$.

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