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ON A PROBLEM OF CHEN AND LEV

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Abstract

For a given set $S \subset \mathbb{N}$, $R_S(n)$ is the number of solutions of the equation $n = s + s'$, $s < s'$, $s, s' \in S$. Suppose that m and r are integers with $m > r \geq 0$ and that A and B are sets with $A \cup B = \mathbb{N}$ and $A \cap B = \{r + mk : k \in \mathbb{N}\}$. We prove that if $R_A(n) = R_B(n)$ for all positive integers n, then there exists an integer $l \geq 1$ such that $r = 2^{2l} - 1$ and $m = 2^{2l+1} - 1$. This solves a problem of Chen and Lev ['Integer sets with identical representation functions', *Integers* **16** (2016), A36] under the condition $m > r$.

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1. Introduction

For a given set $S \subset \mathbb{N}$, the representation function $R_S(n)$ is the number of solutions of the equation $n = s + s'$ with $s < s'$, $s, s' \in S$. For a nonnegative integer a and a set of nonnegative integers S, we define the sumset $a + S = \{a + s : s \in S\}$. Define $R_{A,B}(n)$ to be the number of solutions of $a + b = n$ with $a \in A, b \in B$. The representation function was studied by Erdős, Sárközy and Sós in a series of papers many years ago (see [3–7]). Sárközy asked whether there exist two integer sets A and B with infinite symmetric difference and $R_A(n) = R_B(n)$ for all large enough integers n. In 2002, Dombi [2] proved that the set of positive integers can be partitioned into two subsets C and D such that $R_C(n) = R_D(n)$ for every positive integer n.

Let $\mathcal{A}$ be the set of those nonnegative integers that contain an even number of ones in their binary representation and $\mathcal{B} = \mathbb{N} \setminus \mathcal{A}$. Put $\mathcal{A}_l = \mathcal{A} \cap [0, 2^l - 1]$ and $\mathcal{B}_l = \mathcal{B} \cap [0, 2^l - 1]$. In 2017, Kiss and Sándor [8] gave the following extensions of Dombi's result.

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THEOREM 1.1 [8, Theorem 2]. *Let C and D be sets of nonnegative integers such that $C \cup D = \mathbb{N}$ and $C \cap D = \emptyset$, $0 \in C$. Then $R_C(n) = R_D(n)$ for every positive integer n if and only if $C = \mathcal{A}$ and $D = \mathcal{B}$.*

THEOREM 1.2 [8, Theorem 3]. *Let C and D be sets of nonnegative integers such that $C \cup D = [0, m]$ and $C \cap D = \emptyset$, $0 \in C$. Then $R_C(n) = R_D(n)$ for every positive integer n if and only if there exists a natural number l such that $C = \mathcal{A}_l$ and $D = \mathcal{B}_l$.*

In 2016, Chen and Lev [1] obtained the following result.

THEOREM 1.3 [1, Theorem 1]. *Let l be a positive integer. There exist sets C and D such that $\mathbb{N} = C \cup D$, $C \cap D = (2^{2l} - 1) + (2^{2l+1} - 1)\mathbb{N}$ and $R_C(n) = R_D(n)$ for every positive integer n.*

There are many investigations of partitions and their representation functions (see, for example, [9, 11–15]). In [1], Chen and Lev posed the following two problems.

PROBLEM 1.4. *Given that $R_C(n) = R_D(n)$ for every positive integer n, $C \cup D = \mathbb{N}$ and $C \cap D = r + m\mathbb{N}$ with integers $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2l} - 1, m = 2^{2l+1} - 1$?*

PROBLEM 1.5. *Given that $R_C(n) = R_D(n)$ for every positive integer n, $C \cup D = [0, m]$ and $C \cap D = \{r\}$ with integers $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2l} - 1, m = 2^{2l+1} - 2, C = \mathcal{A}_{2l} \cup (2^{2l} - 1 + \mathcal{B}_{2l})$ and $D = \mathcal{B}_{2l} \cup (2^{2l} - 1 + \mathcal{A}_{2l})$?*

In 2017, Kiss and Sándor solved Problem 1.5 affirmatively.

THEOREM 1.6 [8, Theorem 7]. *Let C and D be sets of nonnegative integers such that $C \cup D = [0, m]$ and $|C \cap D| = 1$, $0 \in C$. Then $R_C(n) = R_D(n)$ for every positive integer n if and only if there exists a natural number l such that $C = \mathcal{A}_{2l} \cup (2^{2l} - 1 + \mathcal{B}_{2l})$ and $D = \mathcal{B}_{2l} \cup (2^{2l} - 1 + \mathcal{A}_{2l})$.*

Recently, Li and the second author of this paper focused on Problem 1.4 and obtained the following result.

THEOREM 1.7 [10, Theorem 1.2]. *Let $m > r \geq 0$ be integers. Let A and B be sets of nonnegative integers such that $A \cup B = \mathbb{N}$ and $A \cap B = \{r + mk : k \in \mathbb{N}\}$. If $R_A(n) = R_B(n)$ for every positive integer n, then there exists an integer $l \geq 1$ such that $r = 2^{2l} - 1$.*

In this paper, we solve Problem 1.4 affirmatively under the condition $m > r$.

THEOREM 1.8. *Let $m > r \geq 0$ be integers. Let A and B be sets such that $A \cup B = \mathbb{N}$ and $A \cap B = \{r + mk : k \in \mathbb{N}\}$. If $R_A(n) = R_B(n)$ for every positive integer n, then there exists an integer $l \geq 1$ such that $r = 2^{2l} - 1$ and $m = 2^{2l+1} - 1$.*

Throughout this paper, the characteristic function of the set C is denoted by

$$\chi_C(t) = \begin{cases} 0 & \text{if } t \notin C, \\ 1 & \text{if } t \in C, \end{cases}$$

and $C(x)$ denotes the set of integers in C that are less than or equal to x.

2. Lemmas

LEMMA 2.1 [8, Claim 1]. *Let $0 < r_1 < \cdots < r_s \le m$ be integers. Then there exists at most one pair of sets (C, D) such that $C \cup D = [0, m]$, $C \cap D = \{r_1, \ldots, r_s\}$, $0 \in C$ and $R_C(k) = R_D(k)$ for every $k \le m$.*

LEMMA 2.2. *Let $l \ge 1$ be a positive integer and let E, F be sets of nonnegative integers such that $E \cup F = [0, 3 \cdot 2^{2l} - 2]$, $0 \in E$ and $E \cap F = \{2^{2l} - 1\}$. Then $R_E(n) = R_F(n)$ for every positive integer $1 \le n \le 3 \cdot 2^{2l} - 2$ if and only if*

$$E := \mathcal{A}_{2l} \cup (2^{2l} - 1 + \mathcal{B}_{2l}) \cup (2^{2l+1} - 1 + (\mathcal{B}_{2l} \cap [0, 2^{2l} - 2])) \cup \{3 \cdot 2^{2l} - 2\}, \quad (2.1)$$

$$F := \mathcal{B}_{2l} \cup (2^{2l} - 1 + \mathcal{A}_{2l}) \cup (2^{2l+1} - 1 + (\mathcal{A}_{2l} \cap [0, 2^{2l} - 2])). \quad (2.2)$$

PROOF. *Sufficiency.* It is easy to verify that $E \cup F = [0, 3 \cdot 2^{2l} - 2]$, $0 \in E$ and $E \cap F = \{2^{2l} - 1\}$.

First, if $1 \le n \le 2^{2l+1} - 2$, then

$$R_E(n) = R_{\mathcal{A}_{2l}}(n) + R_{\mathcal{A}_{2l},\mathcal{B}_{2l}}(n - (2^{2l} - 1)), \quad R_F(n) = R_{\mathcal{B}_{2l}}(n) + R_{\mathcal{A}_{2l},\mathcal{B}_{2l}}(n - (2^{2l} - 1)).$$

By Theorem 1.2, $R_E(n) = R_F(n)$.

Next, if $2^{2l+1} - 2 < n \le 3 \cdot 2^{2l} - 3$, then

$$R_E(n) = R_{\mathcal{A}_{2l},\mathcal{B}_{2l}}(n - (2^{2l} - 1)) + R_{\mathcal{A}_{2l},\mathcal{B}_{2l}}(n - (2^{2l+1} - 1)) + R_{\mathcal{B}_{2l}}(n - 2(2^{2l} - 1)),$$
$$R_F(n) = R_{\mathcal{A}_{2l},\mathcal{B}_{2l}}(n - (2^{2l} - 1)) + R_{\mathcal{A}_{2l},\mathcal{B}_{2l}}(n - (2^{2l+1} - 1)) + R_{\mathcal{A}_{2l}}(n - 2(2^{2l} - 1)).$$

Again, by Theorem 1.2, $R_E(n) = R_F(n)$.

Finally, suppose that $n = 3 \cdot 2^{2l} - 2$. Since 0 and $3 \cdot 2^{2l} - 2 \in E$, $2^{2l} - 1$ and $2^{2l+1} - 1 \in F$,

$$R_E(3 \cdot 2^{2l} - 2) = 1 + R_{\mathcal{A}_{2l},\mathcal{B}_{2l}}(2^{2l} - 1) + R_{\mathcal{B}_{2l}}(2^{2l}),$$
$$R_F(3 \cdot 2^{2l} - 2) = 1 + R_{\mathcal{A}_{2l},\mathcal{B}_{2l}}(2^{2l} - 1) + R_{\mathcal{A}_{2l}}(2^{2l}).$$

By Theorem 1.2, $R_E(3 \cdot 2^{2l} - 2) = R_F(3 \cdot 2^{2l} - 2)$.

Necessity. The necessity follows from Lemma 2.1 and the sufficiency.

This completes the proof of Lemma 2.2. □

LEMMA 2.3. *Let (S_1, S_2) and (A_1, A_2) be two pairs of finite sets such that $S_1 \subseteq A_1$, $S_2 \subseteq A_2$, $S_1 \cap S_2 = A_1 \cap A_2$ and $S_1 \cup S_2 = A_1 \cup A_2$. Then $S_1 = A_1$ and $S_2 = A_2$.*

PROOF. Noting that, for any two finite sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$, we have $|S_1| + |S_2| = |A_1| + |A_2|$. It follows from $S_1 \subseteq A_1$ and $S_2 \subseteq A_2$ that $|S_1| = |A_1|$ and $|S_2| = |A_2|$, and then $S_1 = A_1$ and $S_2 = A_2$. □

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3. Proof of Theorem 1.8

By Theorem 1.7, there exists an integer $l \geq 1$ such that $r = 2^{2l} - 1$. Let E and F be as in (2.1) and (2.2). If $m \geq 2^{2l+1}$ and $0 \in A$, then

$$A(3 \cdot 2^{2l} - 2) \cup B(3 \cdot 2^{2l} - 2) = [0, 3 \cdot 2^{2l} - 2],$$
$$A(3 \cdot 2^{2l} - 2) \cap B(3 \cdot 2^{2l} - 2) = \{2^{2l} - 1\}.$$

Moreover, for $1 \leq n \leq 3 \cdot 2^{2l} - 2$,

$$R_{A(3\cdot 2^{2l}-2)}(n) = R_A(n) = R_B(n) = R_{B(3\cdot 2^{2l}-2)}(n).$$

By Lemma 2.2,

$$A(3 \cdot 2^{2l} - 2) = E, \quad B(3 \cdot 2^{2l} - 2) = F.$$

Noting that $0 \in A$, $1 \in \mathcal{B}_{2l}$, $2^{2l} - 1 \in \mathcal{A}_{2l}$ and $3 \cdot 2^{2l} - 2 \in A$,

$$R_A(3 \cdot 2^{2l} - 1) = R_{\mathcal{A}_{2l},\mathcal{B}_{2l}}(2^{2l}) + R_{\mathcal{B}_{2l}}(2^{2l} + 1) + \chi_A(3 \cdot 2^{2l} - 1).$$
$$R_B(3 \cdot 2^{2l} - 1) = R_{\mathcal{A}_{2l},\mathcal{B}_{2l}}(2^{2l}) + R_{\mathcal{B}_{2l}}(2^{2l} + 1) - 1.$$

By Theorem 1.2, $R_A(3 \cdot 2^{2l} - 1) > R_B(3 \cdot 2^{2l} - 1)$, which is a impossible. Hence $m < 2^{2l+1}$. It is sufficient to prove that if $2^{2l} - 1 < m < 2^{2l+1} - 1$, then $R_A(n) = R_B(n)$ cannot hold for all positive integers n.

Now we assume that $2^{2l} \leq m \leq 2^{2l+1} - 2$ and $0 \in A$. Let

$$M = 2^{2l} - 1 + m.$$

Since $2^{2l+1} - 1 \leq M \leq 3 \cdot 2^{2l} - 3$, by Lemma 2.2,

$$E(M) \cup F(M) = [0, M], \quad E(M) \cap F(M) = \{2^{2l} - 1\}, \tag{3.1}$$
$$R_{E(M)}(n) = R_E(n) = R_F(n) = R_{F(M)}(n) \quad \text{for } 1 \leq n \leq M. \tag{3.2}$$

Moreover,

$$A(M) \cup B(M - 1) = [0, M], \quad A(M) \cap B(M - 1) = \{2^{2l} - 1\}. \tag{3.3}$$

Since $R_A(n) = R_B(n)$ for every positive integer n and $0 \notin B$, for $1 \leq n \leq M$,

$$R_{A(M)}(n) = R_A(n) = R_B(n) = R_{B(M-1)}(n). \tag{3.4}$$

By (3.1)–(3.4) and Lemma 2.1,

$$A(M) = E(M), \quad B(M - 1) = F(M). \tag{3.5}$$

Hence, $\chi_F(M) = 0$.

Let t be be an arbitrary nonnegative integer such that the conditions $M \leq M + t$ and $M + t + 1 \leq 3 \cdot 2^{2l} - 2$ both hold. Then $0 \leq t \leq 2^{2l} - 2$. Write

$$S_1 := (E \cap A)(M + t) \cup (F(M + t) \setminus B(M + t)),$$
$$S_2 := (F \cap B)(M + t) \cup (E(M + t) \setminus A(M + t)).$$

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Noting that

$$E(M+t) \cup F(M+t) = [0, M+t] = A(M+t) \cup (B(M+t) \setminus \{M\}),$$

we find

$$S_1 \subseteq A(M+t), \quad S_2 \subseteq B(M+t) \setminus \{M\},$$
$$S_1 \cup S_2 = E(M+t) \cup F(M+t) = A(M+t) \cup (B(M+t) \setminus \{M\}).$$
$$S_1 \cap S_2 = A(M+t) \cap (B(M+t) \setminus \{M\}) = \{2^{2l} - 1\}.$$

By Lemma 2.3,

$$S_1 = A(M+t), \quad S_2 = B(M+t) \setminus \{M\}. \tag{3.6}$$

For $M + t \le n \le 3 \cdot 2^{2l} - 2$, write

$$T_1(t,n) = R_{E(2^{2l}-2), E(M+t)\setminus A(M+t)}(n),$$
$$T_2(t,n) = R_{F(2^{2l}-2), E(M+t)\setminus A(M+t)}(n),$$
$$T_3(t,n) = R_{E(2^{2l}-2), F(M+t)\setminus B(M+t)}(n),$$
$$T_4(t,n) = R_{F(2^{2l}-2), F(M+t)\setminus B(M+t)}(n).$$

Then

$$|E(M+t) \setminus A(M+t)| = T_1(t,n) + T_2(t,n), \tag{3.7}$$
$$|F(M+t) \setminus B(M+t)| = T_3(t,n) + T_4(t,n). \tag{3.8}$$

In fact, if $E(M+t) \setminus A(M+t) = \emptyset$, then $T_1(t,n) = T_2(t,n) = 0$ and (3.7) holds. If $E(M+t) \setminus A(M+t) \neq \emptyset$, then write

$$E(M+t) \setminus A(M+t) = \{e_1, \dots, e_h\}.$$

By (3.5), $e_1, \dots, e_h \ge M + 1$, and it follows that

$$0 \le n - e_i \le 3 \cdot 2^{2l} - 2 - (M+1) \le 2^{2l} - 2 \quad \text{for } i = 1, \dots, h.$$

Noting that

$$E(2^{2l}-2) \cup F(2^{2l}-2) = [0, 2^{2l}-2], \quad E(2^{2l}-2) \cap F(2^{2l}-2) = \emptyset,$$

we see that

$$T_1(t,n) + T_2(t,n) = \sum_{i=1}^{h} \chi_{E(2^{2l}-2)}(n - e_i) + \sum_{i=1}^{h} \chi_{F(2^{2l}-2)}(n - e_i) = h.$$

Hence, (3.7) holds. Similarly, we can obtain (3.8).

Since $M + t < 3 \cdot 2^{2l} - 2 < 2^{2l+2} \le 2M + 2$,

$$R_{E(M+t)}(n) = R_{(E\cap A)(M+t)}(n) + R_{E(2^{2l}-2), E(M+t)\setminus A(M+t)}(n)$$
$$= R_{(E\cap A)(M+t)}(n) + T_1(t,n).$$

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Thus, by (3.6),

$$\begin{aligned} R_{A(M+t)}(n) &= R_{(E\cap A)(M+t)}(n) + T_3(t,n) \\ &= R_{E(M+t)}(n) - T_1(t,n) + T_3(t,n). \end{aligned} \tag{3.9}$$

Similarly, by (3.6),

$$\begin{aligned} R_{B(M+t)}(n) &= R_{B(M+t)\setminus\{M\}}(n) + \chi_F(n-M) \\ &= R_{(F\cap B)(M+t)}(n) + T_2(t,n) + \chi_F(n-M) \\ &= R_{F(M+t)}(n) - T_4(t,n) + T_2(t,n) + \chi_F(n-M). \end{aligned} \tag{3.10}$$

By (3.9) and (3.10),

$$\begin{aligned} R_{A(M+t+1)}(M+t+1) &= R_{A(M+t)}(M+t+1) + \chi_A(M+t+1) \\ &= R_{E(M+t+1)}(M+t+1) - \chi_E(M+t+1) - T_1(t,M+t+1) \\ &\quad + T_3(t,M+t+1) + \chi_A(M+t+1), \end{aligned} \tag{3.11}$$

$$\begin{aligned} R_{B(M+t+1)}(M+t+1) &= R_{B(M+t)}(M+t+1) \\ &= R_{F(M+t+1)}(M+t+1) - T_4(t,M+t+1) \\ &\quad + T_2(t,M+t+1) + \chi_F(t+1). \end{aligned} \tag{3.12}$$

Noting that

$$R_{A(n)}(n) = R_{B(n)}(n), \quad R_{E(n)}(n) = R_{F(n)}(n),$$

by (3.9) and (3.10),

$$T_1(t,M+t) + T_2(t,M+t) + \chi_F(t) = T_3(t,M+t) + T_4(t,M+t),$$

and by (3.11) and (3.12),

$$\begin{aligned} &T_3(t,M+t+1) + T_4(t,M+t+1) + \chi_A(M+t+1) \\ &\quad = T_1(t,M+t+1) + T_2(t,M+t+1) + \chi_E(M+t+1) + \chi_F(t+1). \end{aligned}$$

By (3.7) and (3.8),

$$|E(M+t) \setminus A(M+t)| + \chi_F(t) = |F(M+t) \setminus B(M+t)|. \tag{3.13}$$

$$\begin{aligned} &|F(M+t) \setminus B(M+t)| + \chi_A(M+t+1) \\ &\quad = |E(M+t) \setminus A(M+t)| + \chi_E(M+t+1) + \chi_F(t+1). \end{aligned} \tag{3.14}$$

By (3.13) and (3.14),

$$\chi_F(t) + \chi_A(M+t+1) = \chi_E(M+t+1) + \chi_F(t+1). \tag{3.15}$$

If M is odd, we can write

$$M = 2^{2l+1} - 1 + \sum_{i=1}^{2l-1} b_i 2^i,$$

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where $b_i \in \{0, 1\}$. Since $\chi_F(M) = 0$, it follows that $\chi_{\mathcal{B}_{2l}}(\sum_{i=1}^{2l-1} b_i 2^i) = 1$ and so

$$\chi_F(3 \cdot 2^{2l} - 2 - M) = \chi_{\mathcal{B}_{2l}}\Big(\sum_{i=1}^{2l-1}(1 - b_i)2^i + 1\Big) = 1,$$

$$\chi_F(3 \cdot 2^{2l} - 3 - M) = \chi_{\mathcal{B}_{2l}}\Big(\sum_{i=1}^{2l-1}(1 - b_i)2^i\Big) = 0.$$

But then by (3.15), $\chi_A(3 \cdot 2^{2l} - 2) = 2$, which is impossible.

If M is even, then there exists an integer $f \in \{0, 1, \ldots, 2l - 2\}$ such that

$$M = 2^{2l+1} - 1 + \sum_{i=0}^{f} 2^i + \sum_{i=f+2}^{2l-1} b_i 2^i,$$

where $b_i \in \{0, 1\}$. Since $\chi_F(0) = 0, \chi_F(1) = 1$, it follows from (3.15) that $\chi_E(M + 1) = 0$ and so

$$\chi_{\mathcal{A}_{2l}}\Big(2^{f+1} + \sum_{i=f+2}^{2l-1} b_i 2^i\Big) = 1. \tag{3.16}$$

Moreover, $\chi_F(M) = 0$, so that

$$\chi_{\mathcal{B}_{2l}}\Big(\sum_{i=0}^{f} 2^i + \sum_{i=f+2}^{2l-1} b_i 2^i\Big) = 1. \tag{3.17}$$

By (3.16) and (3.17), it follows that f is odd, so that

$$\chi_F(3 \cdot 2^{2l} - 2 - M) = \chi_{\mathcal{B}_{2l}}\Big(\sum_{i=f+2}^{2l-1} (1 - b_i)2^i + 2^{f+1}\Big) = 1,$$

$$\chi_F(3 \cdot 2^{2l} - 3 - M) = \chi_{\mathcal{B}_{2l}}\Big(\sum_{i=f+2}^{2l-1} (1 - b_i)2^i + 2^{f+1} - 1\Big) = 0.$$

By (3.15), $\chi_A(3 \cdot 2^{2l} - 2) = 2$, which is impossible.

This completes the proof of Theorem 1.8.

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