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Uniform convergence and everywhere convergence of Fourier series. I

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Carleson and Hunt proved that the space of functions with almost everywhere convergent Fourier series contains L^p $(p > 1)$ as a subspace. We shall give two kinds of subspaces of the spaces of functions with everywhere convergent or uniformly convergent Fourier series.

1. Introduction

We consider only real valued functions of a real variable and periodic with period 2π, and the series of classes of such functions:

$$L \supset L^p \supset L^q \supset L^\infty \supset \left\{\begin{matrix} C \supset CBV \cup \mathrm{Lip}\alpha \\ \\ BV \supset CBV \end{matrix}\right\} \supset \mathrm{Lip}\ 1\ ,$$

where $1 < p < q < \infty$ and $0 < \alpha < 1$. By aec, ec or uc we denote the spaces of functions whose Fourier series converges almost everywhere, everywhere or uniformly. It is evident that

$$aec \supset ec \supset uc\ .$$

Carleson [2] and Hunt [5] proved that

THEOREM. *$aec \supset L^p$ for any $p > 1$.*

It is open to find good subspaces of the spaces ec and uc, where a good subspace means that:

(i) the subspace is sufficiently near to the whole space;

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(ii) the subspace contains known significant subspaces;

(iii) the subspace has a simple construction like L^p, L^∞, C, and so on.

In this direction, there are recent works of Goffman [4], Garsia and Sawyer [3] and Baernstein and Waterman [1].

We know that

$$ec \not\supset C\ ,\quad ec \supset BV \cup \mathrm{Lip}\alpha \quad (0 < \alpha \le 1)$$

and

$$C \supsetneqq uc \supset CBV \cup \mathrm{Lip}\alpha \quad (0 < \alpha \le 1)\ .$$

2. Theorems

For any integrable function f,

$$(1) \qquad \varphi_1(t) = \int_0^t \varphi_x(u)du = o(t) \quad \text{as} \quad t \to 0$$

for almost all x, where

$$\varphi_x(u) = f(x+u) + f(x-u) - 2f(x)\ .$$

We shall denote by LC the space of functions which satisfy condition (1) for all x.

Let the Fourier series of f be

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \equiv \sum_{n=0}^{\infty} A_n(x)\ .$$

By N^p we denote the space of functions f such that

$$(2) \qquad \sum_{m=n}^{\infty} \left(|a_m|^p + |b_m|^p\right) = o\left(1/n^{p-1}\right) \quad \text{as} \quad n \to \infty\ .$$

We denote by $f_1(u)$ the integral of f on the interval $(0, u)$ and

$$\Delta_{\pi/n} f_1(u) = f_1(u) - f_1(u-\pi/n)\ ,$$

$$\Delta^2_{\pi/n} f_1(u) = \Delta_{\pi/n}\left(\Delta_{\pi/n} f_1(u)\right) = f_1(u) - 2f_1(u-\pi/n) + f_1(u-2\pi/n)\ .$$

Let $p \geq 1$ and we denote by M^p the space of functions f such that

$$\int_{-\pi}^{\pi} \left|\Delta^2_{\pi/n} f_1(u)\right|^p du = o\left(1/n^{p+1}\right) \quad \text{as} \quad n \to \infty . \tag{3}$$

Thus we get the following theorems:

THEOREM 1.

(i) $ec \supset LC \cap N^p$, $(p > 1)$.

(ii) $ec \supset LC \cap M^p$, $(p \geq 1)$.

(iii) $uc \supset C \cap M^p$, $(p \geq 1)$.

(iv) $uc \supset C \cap N^p$, $(p > 1)$.

THEOREM 2.

(i) $M^p \supset \mathrm{Lip}(1/p, p)$, $(p \geq 1)$;

$M^p \supset \mathrm{Lip}(1/p', p') \cap C$, $(1 < p' < p)$;

$M^p \supset \mathrm{Lip}\alpha$, $(0 < \alpha \leq 1, \alpha p > 1)$.

(ii) $N^2 \supset CBV$;

$N^2 \supset \mathrm{Lip}\alpha$, $(1 \geq \alpha > 1/2)$.

(iii) $M^2 = N^2$;

$M^q \supset N^p$, $N^q \supset M^p$, $(1/p+1/q = 1, q > 2 > p)$.

For the proof of Theorem 1, we use the following theorem:

THEOREM 3. *(i) If condition* (1) *is satisfied, then*

$$s_n(x; f) - f(x) = \frac{n}{\pi} \int_{3\pi/n}^{\pi} \frac{\Delta^2_{\pi/n}\varphi_1(u)}{u} \cos nu du + o(1) \quad as \quad n \to \infty , \tag{4}$$

where $s_n(x; f)$ *denotes the nth partial sum of the Fourier series of* f *at the point* x , *and*

$$\Delta^2_{\pi/n}\varphi_1(u) = \varphi_1(u) - 2\varphi_1(u-\pi/n) + \varphi_1(u-2\pi/n) .$$

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(ii) If f *is continuous, then the term* $o(1)$ *in* (4) *holds uniformly in* x .

As another corollary of the theorem besides Theorem 1, we get

THEOREM 4. *(i) If condition* (1) *is satisfied and*

$$\int_{3\pi/n}^{\pi} \frac{\left|\Delta_{\pi/n}^{2}\varphi_1(u)\right|}{u}\,du = o(1/n) \quad \text{as} \quad n \to \infty, \tag{5}$$

then the Fourier series of f *converges to* $f(x)$ *at the point* x .

(ii) Part (i) is a generalization of Lebesgue's Convergence Criterion.

3. Proof of Theorem 3 *(i)*

We can suppose that n is an odd integer, $x = 0$, f is an even function and $\int_0^{\pi} f = 0$. Then $\varphi = f$ and $\varphi_1 = f_1$. Therefore, by condition (1),

$$\begin{aligned} s_n(0;\, f) &= \frac{1}{\pi}\int_0^{\pi} f(t)\,\frac{\sin nt}{t}\,dt + o(1) \\ &= \frac{1}{\pi}\int_{3\pi/n}^{\pi} f(t)\,\frac{\sin nt}{t}\,dt + o(1) \\ &= -\frac{1}{\pi}\int_{3\pi/n}^{\pi} f_1(t)\left[\frac{n\cos nt}{t} - \frac{\sin nt}{t^2}\right]dt + o(1) \\ &= -\frac{1}{\pi}\,(P-Q) + o(1)\ . \end{aligned}$$

We write

$$\begin{aligned} P &= \frac{n}{2}\int_{3\pi/n}^{\pi} \Delta_{\pi/n}\left(\frac{f_1(t)}{t}\right)\cos nt\,dt + o(1) \\ &= \frac{n}{2}\int_{3\pi/n}^{\pi} \frac{\Delta_{\pi/n}f_1(t)}{t}\cos nt\,dt - \frac{\pi}{2}\int_{3\pi/n}^{\pi} \frac{f_1(t)}{t(t-\pi/n)}\cos nt\,dt + o(1) \\ &= \frac{n}{4}\int_{3\pi/n}^{\pi} \frac{\Delta_{\pi/n}^{2}f_1(t)}{t}\cos nt\,dt - \frac{\pi}{4}\int_{3\pi/n}^{\pi} \frac{\Delta_{\pi/n}f_1(t)}{t(t-\pi/n)}\cos nt\,dt \end{aligned}$$

$$- \frac{\pi}{2} \int_{3\pi/n}^{\pi} \frac{f_1(t)}{t(t-\pi/n)} \cos ntdt + o(1)$$

$$= \frac{n}{4} \int_{3\pi/n}^{\pi} \frac{\Delta^2_{\pi/n} f_1(t)}{t} \cos ntdt + \frac{\pi}{4} \int_{3\pi/n}^{\pi} \frac{f_1(t-\pi/n)}{t(t-\pi/n)} \cos ntdt$$

$$- \frac{3\pi}{4} \int_{3\pi/n}^{\pi} \frac{f_1(t)}{t(t-\pi/n)} \cos ntdt + o(1)$$

$$= \frac{1}{4} R + \frac{\pi}{4} S - \frac{3\pi}{4} T + o(1) ,$$

and

$$Q = \int_{3\pi/n}^{\pi} f(t)dt \int_t^{t+\pi} \frac{\sin nu}{u^2} du + o(1) .$$

We have to prove that S, T and Q are $o(1)$ as $n \to \infty$. First of all, using the following formula:

$$\int_t^{t+\pi} \frac{\sin nu}{u^2} du = \int_t^{t+\pi/n} \sum_{k=0}^{(n-1)/2} \left\{\frac{1}{(u+2k\pi/n)^2} - \frac{1}{(u+(2k+1)\pi/n)^2}\right\} \sin nudu$$

$$= \frac{\pi}{n} \sum_{k=0}^{(n-1)/2} \int_t^{t+\pi/n} \frac{2u+(4k+1)\pi/n}{(u+2k\pi/n)^2 (u+(2k+1)\pi/n)^2} \sin nudu ,$$

we can write

$$Q = \frac{\pi}{n} \sum_{k=0}^{(n-1)/2} \int_{2\pi/n}^{\pi} \frac{\{2u+(4k+1)\pi/n\} \sin nu}{(u+2k\pi/n)^2 (u+(2k+1)\pi/n)^2} du \int_{u-\pi/n}^{u} f(t)dt + o(1)$$

$$= \pi n \sum_{k=0}^{(n-1)/2} \int_{2\pi}^{n\pi} \frac{\{2v+(4k+1)\pi\} \sin v}{(v+2k\pi)^2 (v+(2k+1)\pi)^2} dv \int_{v/n-\pi/n}^{v/n} f(t)dt + o(1)$$

$$= \pi n \sum_{k=0}^{(n-1)/2} \sum_{j=2}^{n-1} (-1)^j \int_0^{\pi} \frac{\{2j\pi+(4k+1)\pi+2v\} \sin v}{(j\pi+2k\pi+v)^2 (j\pi+(2k+1)\pi+v)^2} dv$$

$$\cdot \int_{((j-1)\pi+v)/n}^{(j\pi+v)/n} f(t)dt + o(1)$$

$$= n \sum_{j=2}^{n-1} (-1)^j \int_0^{\pi} \sin v \cdot \Delta_{\pi/n} f_1((j\pi+v)/n)$$

$$\cdot \sum_{k=0}^{(n-1)/2} \left\{\frac{1}{(j\pi+2k\pi+v)^2} - \frac{1}{(j\pi+(2k+1)\pi+v)^2}\right\} dv + o(1)$$

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by condition (1). Using the function

$$J(w) = [w] - w + 1/2 \sim \sum_{k=1}^{\infty} \frac{\sin 2\pi k w}{\pi w},$$

the inner summation can be written as follows:

$$\int_0^{(n-1)/2} \left\{\frac{1}{(j\pi+2t\pi+v)^2} - \frac{1}{(j\pi+\pi+2t\pi+v)^2}\right\}(dt+dJ(t))$$

$$= \frac{1}{4\pi}\left[\left\{\frac{1}{j\pi+v} - \frac{1}{j\pi+(n-1)\pi+v}\right\} - \left\{\frac{1}{j\pi+\pi+v} - \frac{1}{j\pi+n\pi+v}\right\}\right]$$

$$+ 4\pi\int_0^{(n-1)/2}\left\{\frac{1}{(j\pi+2t\pi+v)^3} - \frac{1}{(j\pi+\pi+2t\pi+v)^3}\right\}J(t)dt$$

$$= \frac{1}{4}\left\{\frac{1}{\{(j+1)\pi+v\}(j\pi+v)} - \frac{1}{\{(n+j-1)\pi+v\}\{(n+j)\pi+v\}}\right\} + O(1/j^3) .$$

Thus we get, by using condition (1),

$$Q = \frac{n}{4}\sum_{j=2}^{n-1}(-1)^j\int_0^{\pi}\sin v \cdot \Delta_{\pi/n}f_1((j\pi+v)/n)$$

$$\cdot\left\{\frac{1}{\{(j+1)\pi+v\}(j\pi+v)} - \frac{1}{\{(n+j-1)\pi+v\}\{(n+j)\pi+v\}}\right\}dv + o(1)$$

$$= \frac{1}{4}(U-V) + o(1) .$$

Further we use $J(w)$ again,

$$U = n\sum_{j=1}^{(n-1)/2}\int_0^{\pi}\sin v\,\frac{\Delta^2_{\pi/n}f_1(\{(2j+1)\pi+v\}/n)}{\{(2j-1)\pi+v\}(2j\pi+v)}\,dv$$

$$= n\int_0^{\pi}\sin v dv\int_{1/2}^{n/2}\frac{\Delta^2_{\pi/n}f_1((2\pi w+\pi+v)/n)}{(2\pi w-\pi+v)(2\pi w+v)}\,(dw+dJ(w))$$

$$= W + X ,$$

where

$$W = \frac{n}{2\pi}\int_0^{\pi/n}\sin nv dv\int_{v+2\pi/n}^{v+\pi+\pi/n}\frac{\Delta^2_{\pi/n}f_1(w)}{(w-2\pi/n)(w-\pi/n)}\,dw$$

$$= n\int_0^{\pi/n}\sin nv dv\int_{v+2\pi/n}^{v+\pi+\pi/n}\frac{dw}{(w-2\pi/n)(w-\pi/n)}\int_{w-\pi/n}^{w}(f(y)-f(y-\pi/n))dy$$

$$= n \int_0^{\pi/n} \sin nv\,dv \int_{v+2\pi/n}^{v+\pi+\pi/n} \big(f(y)-f(y-\pi/n)\big)dy \int_y^{y+\pi/n} \frac{dw}{(w-2\pi/n)(w-\pi/n)} + o(1)$$

$$= n \int_0^{\pi/n} \sin nv\,dv$$

$$\cdot \int_{v+2\pi/n}^{v+\pi+\pi/n} f(y)\left[\int_y^{y+\pi/n} \frac{dw}{(w-2\pi/n)(w-\pi/n)} - \int_{y+\pi/n}^{y+2\pi/n} \frac{dw}{(w-2\pi/n)(w-\pi/n)}\right]dy$$

$$+ o(1)$$

$$= \pi \int_0^{\pi/n} \sin nv\,dv \int_{v+2\pi/n}^{v+\pi+\pi/n} f(y)dy \int_{y+\pi/n}^{y+2\pi/n} \frac{dw}{(w-2\pi/n)(w-\pi/n)} + o(1)$$

$$= \pi \int_0^{\pi/n} \sin nv\,dv \int_{v+2\pi/n}^{v+\pi+\pi/n} |f_1(y)|dy \cdot O\left(\frac{\pi}{n} \cdot \frac{1}{y^4}\right) + o(1)$$

$$= O\left(\frac{1}{n} \int_0^{\pi/n} \sin nv\,dv \cdot n^2\right) + o(1)$$

$$= o(1) \ , \text{ as } \ n \to \infty \ ,$$

by condition (1), and

$$X = n \int_0^{\pi} \sin v\,dv \int_{1/2}^{n/2} \frac{\Delta^2_{\pi/n} f_1\big((2\pi w+\pi+v)/n\big)}{(2\pi w-\pi)2\pi w}\, dJ(w) + o(1)$$

$$= - \int_0^{\pi} \sin v\,dv \int_{1/2}^{n/2} \frac{\Delta^2_{\pi/n} f\big((2\pi w+\pi+v)/n\big)}{(2\pi w-\pi)w}\, J(w)dw + o(1)$$

$$= - \int_{1/2}^{n/2} \frac{J(w)}{(2\pi w-\pi)w}\, dw \int_0^{\pi} \Delta^2_{\pi/n} f\big((2\pi w+\pi+v)/n\big) \cdot \sin v\,dv + o(1)$$

$$= \int_{1/2}^{n/2} \frac{J(w)}{(2\pi w-\pi)w}\, dw \cdot n^2 \int_0^{\pi/n} \Delta^2_{\pi/n} f_1\big((2\pi w+\pi)/n+v\big) \cdot \cos nv\,dv + o(1) \ .$$

By the transformations $w = nw'$ and $2\pi w' + v = v'$,

$$X = \frac{n}{2\pi} \int_{1/2n}^{1/2} \frac{J(nw')\cos 2nw'}{(w'-1/2n)w'}\, dw' \int_{2\pi w'}^{2\pi w'+\pi/n} \Delta^2_{\pi/n} f_1(v'+\pi/n) \cdot \cos nv'dv'$$

$$+ \frac{n}{2\pi} \int_{1/2n}^{1/2} \frac{J(nw')\sin 2\pi nw'}{(w'-1/2n)w'}\, dw' \int_{2\pi w'}^{2\pi w'+\pi/n} \Delta^2_{\pi/n} f_1(v'+\pi/n) \cdot \sin nv'dv' + o(1)$$

$$= Y + Z + o(1) \ ,$$

where

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$$Y = \frac{n}{2\pi}\int_{2\pi/n}^{\pi} \Delta^2_{\pi/n} f_1(v'+\pi/n)\cdot \cos nv'dv' \int_{v'/2\pi-1/2n}^{v'/2\pi} \frac{J(nw')\cos 2\pi nw'}{(w'-1/2n)w'}dw'$$

$$= \frac{n}{2\pi}\int_{2\pi/n}^{\pi} \Delta^2_{\pi/n} f_1(v'+\pi/n)\frac{\cos nv'}{(v'/2\pi-1/2n)v'/2\pi}dv'$$

$$\cdot \int_{v'/2\pi-1/2n}^{v'/2\pi} J(nw')\cos 2\pi nw'dw' + o(1)$$

$$= 2n\sum_{l=1}^{\infty}\frac{1}{l}\int_{2\pi/n}^{\pi}\frac{\Delta^2_{\pi/n} f_1(v'+\pi/n)}{(v'-\pi/n)v'}\cos nv'dv'$$

$$\cdot \int_{v'/2\pi-1/2n}^{v'/2\pi} \sin 2\pi lnw'\cdot \cos 2\pi nw'dw' + o(1)$$

$$= \frac{1}{2\pi}\sum_{\substack{l=2\\ l\ \text{even}}}^{\infty}\frac{1}{l(l+1)}\int_{2\pi/n}^{\pi}\Delta^2_{\pi/n} f_1(v'+\pi/n)\frac{\cos nlv'+\cos n(l+2)v'}{(v'-\pi/n)v'}dv'$$

$$+ \frac{1}{2\pi}\sum_{\substack{l=2\\ l\ \text{even}}}^{\infty}\frac{1}{l(l+1)}\int_{2\pi/n}^{\pi}\Delta^2_{\pi/n} f_1(v'+\pi/n)\frac{\cos nlv'+\cos n(l-2)v'}{(v'-\pi/n)v'}dv' + o(1)$$

$$= \frac{1}{2\pi}\sum_{\substack{l=2\\ l\ \text{even}}}^{\infty}\frac{1}{l(l+1)}\int_{2\pi/n}^{\pi} f_1(v')\left(\cos nlv'+\cos n(l+2)v'\right)$$

$$\cdot \Delta^2_{\pi/n}\left(\frac{1}{v'(v'+\pi/n)}\right)dv' + \frac{1}{4\pi}\sum_{\substack{l=2\\ l\ \text{even}}}^{\infty}\frac{1}{l(l-1)}\int_{2\pi/n}^{\pi} f_1(v')$$

$$\cdot \left(\cos nlv'+\cos n(l-2)v'\right)\cdot \Delta^2_{\pi/n}\left(\frac{1}{v'(v'+\pi/n)}\right)dv' + o(1)$$

$$= o(1)\ , \text{ as } n\to\infty .$$

Similarly $Z = o(1)$ and then $U = o(1)$. Thus we have proved that $Q = o(1)$ as $n\to\infty$. S and T are also $o(1)$ as $n\to\infty$ by the same way of estimation. Summing up above, we get the required result.

4. Proof of Theorem 3 *(ii)*

If f is continuous, then condition (1) holds uniformly for all x, so that we can prove the theorem easily, along the lines of the proof of Theorem 3 *(i)*.

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5. Proof of Theorem 1 *(i)*

By Theorem 3 *(i)* and the estimation of T in Section 2, it is sufficient to prove that

$$P = n\int_{3\pi/n}^{\pi} \frac{\Delta^2_{\pi/n}\varphi_1(u+\pi/n)}{u}\cos nu\,du = o(1) \quad \text{as} \quad n \to \infty .$$

Now,

$$\varphi_x(t) \sim 2\sum_{l=1}^{\infty} A_l(x)\cos lt - 2f(x)$$

and then

$$\Delta^2_{\pi/n}\varphi_1(u+\pi/n) = 8\sum_{l=1}^{\infty} \frac{A_l(x)}{l}\sin^2\frac{l\pi}{2n}\sin lu .$$

Therefore

$$\begin{aligned}
P &= 8n\sum_{l=1}^{\infty}\frac{A_l(x)}{l}\sin^2\frac{l\pi}{2n}\int_{3\pi/n}^{\pi}\frac{\sin lu\cos nu}{u}\,du \\
&= 4n\sum_{l=1}^{\infty}\frac{A_l(x)}{l}\sin^2\frac{l\pi}{2n}\int_{3\pi/n}^{\pi}\frac{\sin(l+n)u+\sin(l-n)u}{u}\,du \\
&= 2\pi\sum_{l<n}A_l(x)\frac{\sin^2 l\pi/2n}{l\pi/2n}\left\{\int_{3\pi(l+n)/n}^{(l+n)}\frac{\sin v}{v}\,dv - \int_{3\pi(n-l)/n}^{(n-l)}\frac{\sin v}{v}\,dv\right\} \\
&\quad + 4A_n(x)\int_{6\pi}^{2n\pi}\frac{\sin v}{v}\,dv \\
&\quad + 2\pi\sum_{l<n}A_l(x)\frac{\sin^2 l\pi/2n}{l\pi/2n}\left\{\int_{3\pi(l+n)/n}^{(l+n)}\frac{\sin v}{v}\,dv - \int_{3\pi(l-n)/n}^{(l-n)\pi}\frac{\sin v}{v}\,dv\right\} \\
&= 2\pi Q + o(1) + 2\pi Q' \quad ,
\end{aligned}$$

and further we write

$$\begin{aligned}
Q &= \sum_{l<n}A_l(s)\frac{\sin^2 l\pi/2n}{l\pi/2n}\left\{-\int_{3\pi(1-l/n)}^{3\pi(1+l/n)}\frac{\sin v}{v}\,dv + \int_{(n-l)\pi}^{(n+l)\pi}\frac{\sin v}{v}\,dv\right\} \\
&= -Q_1 + Q_2 .
\end{aligned}$$

Now we shall define $A(l,\,x)$ for all $l \in (0,\,\infty)$ such that

$$A(l,\,x) = a(l)\cos lx + b(l)\sin lx ,$$

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where $a(l)$ and $b(l)$ are defined in $(0, \infty)$ such that

(1°) $a(l) = a_l$, $b(l) = b_l$ for all $l = 0, 1, 2, \ldots$,

(2°) $a(l)$ and $b(l)$ are linear for non-integral l ,

(3°) they are continuous in the whole interval.

In particular $A(l, x) = A_l(x)$ for all $l = 0, 1, 2, \ldots$. Then using the function $J(l) = [l] - l + 1/2$ for all $l \in (0, \infty)$, we can write

$$Q_1 = \int_{1/2}^{n-1/2} A(l, x) \frac{\sin^2 l\pi/2n}{l\pi/2n} \left(dl + dJ(l)\right) \int_{3\pi-3\pi l/n}^{3\pi+3\pi l/n} \frac{\sin v}{v} dv$$
$$= R_1 + R_2 ,$$

where

$$R_1 = -\int_{1/2}^{n-1/2} A(l, x) \frac{\sin^2 l\pi/2n}{l\pi/2n} dl \int_{-3\pi l/n}^{3\pi l/n} \frac{\sin v}{v+3\pi} dv$$
$$= -\int_{-3\pi+3\pi/2n}^{-3\pi/2n} dv \int_{-nv/3\pi}^{n-1/2} dl + \int_{-3\pi/2n}^{3\pi/2n} dv \int_{1/2}^{n-1/2} dl + $$
$$+ \int_{3\pi/2n}^{3\pi-3\pi/2n} dv \int_{nv/3\pi}^{n-1/2} dl$$
$$= o(1) ,$$

since

$$\int_n^\infty |A(l, x)|^p dx \le A \sum_{m=n}^\infty \left(|a_m|^p + |b_m|^p\right) = o\left(1/n^{p-1}\right)$$

by (2), and

$$\int_{3\pi/2n}^{3\pi-3\pi/2n} \left|\frac{\sin v}{v+3\pi}\right| dv \int_{nv/3\pi}^{n-1/2} |A(l, x)| \frac{\sin^2 l\pi/2n}{l\pi/2n} dl$$
$$\le A \int_{3\pi/2n}^{3\pi-3\pi/2n} v dv \left[\int_{nv/3\pi}^{n-1/2} |A(l, x)|^p dx\right]^{1/p} \left[\int_{nv/3\pi}^{n-1/2} \left(\frac{\sin^2 l\pi/2n}{l\pi/2n}\right)^q dl\right]^{1/q}$$
$$= A \int_{3\pi/2n}^{3\pi} v \cdot o\left[\frac{1}{(nv)^{p-1}}\right]^{1/p} n^{1/q} dv \quad (1/p + 1/q = 1)$$

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$$= o\left(\int_{3\pi/2n}^{3\pi} v^{1/p} dv\right) = o(1) ,$$

by using Hölder's inequality; and the further remaining terms can be estimated similarly. Since

$$|A'(l, x)| \le \left(|a_n|+|a_{n+1}|+|b_n|+|b_{n+1}|\right) \quad \text{for} \quad n \le l \le n+1 ,$$

n being any positive integer, using integration by parts and Hölder's inequality, we get

$$R_2 = -\int_{1/2}^{n-1/2} J(l)A'(l, x) \frac{\sin^2 l\pi/2n}{l\pi/2n} dl \int_{-3\pi l/n}^{3\pi l/n} \frac{\sin v}{v+3\pi} dv$$

$$- \int_{1/2}^{n-1/2} J(l)A(l, x)\left(-\frac{\sin^2 l\pi/2n}{l^2\pi/2n} + \frac{2\sin l\pi/2n\cdot\cos l\pi/2n}{l}\right) dl \int_{-3\pi l/n}^{3\pi l/n} \frac{\sin v}{v+3\pi} dv$$

$$- \int_{1/2}^{n-1/2} J(l)A(l, x) \frac{\sin^2 l\pi/2n}{l\pi/2n} \left(\frac{3\pi}{n} \frac{\sin 3\pi l/n}{3\pi+3\pi l/n} + \frac{3\pi}{n} \frac{\sin 3\pi l/n}{3\pi-3\pi l/n}\right) dl$$

$$= o(1) .$$

Therefore $Q_1 = o(1)$. Similarly Q_2 is also $o(1)$ and then $Q = o(1)$ as $n \to \infty$. Finally,

$$Q' \le A \sum_{l<n} \left|A_l(x) \frac{\sin l\pi/2n}{l\pi/2n}\right|$$

$$\le A\left(\sum_{l<n} |A_l(x)|^p\right)^{1/p} \left(\sum_{l<n} \left(\frac{\sin l\pi/2n}{l\pi/2n}\right)^q\right)^{1/q} = o(1) .$$

Thus $P = o(1)$ and then the Fourier series of the functions belonging to $LC \cap N^p$ are everywhere convergent.

6. Proof of Theorem 1 *(ii)*

Let $1/p + 1/q = 1$, then by Hölder's inequality and condition (3),

$$\left|n \int_{3\pi/n}^{\pi} \frac{\Delta^2_{\pi/n}\varphi_1(u)}{u} \cos nu du\right| \le n\left(\int_{3\pi/n}^{\pi} \left|\Delta^2_{\pi/n}\varphi_1(u)\right|^p du\right)^{1/p} \left(\int_{3\pi/n}^{\pi} u^{-q} du\right)^{1/q}$$

$$\le A\left(\int_{-\pi}^{\pi} \left|\Delta^2_{\pi/n} f_1(u)\right|^p du\right)^{1/p} n^{1+1/p} = o(1) .$$

Thus the Fourier series of the functions belonging to $LC \cap M^p$ are everywhere convergent by Theorem 3 *(i)*.

7. Proof of Theorem 1 *(iii)* and *(iv)*

We can prove these similarly as in Sections 5 and 6. For if $f \in C$, then (1) holds uniformly in x and then the term $o(1)$ in (4) holds uniformly; and further the integral on the right side of (4) tends to 0 uniformly in x by the conditions (2) or (3).

8. Proof of Theorem 2 *(i)*

The case $p = 1$ is evident. Suppose that $p > 1$ and $f \in \mathrm{Lip}(1/p, p)$. Then, by Hölder's inequality,

$$\int_{-\pi}^{\pi} \left|\Delta^2_{\pi/n} f_1(u)\right|^p du = \int_{-\pi}^{\pi} \left|\int_{u-\pi/n}^{u} \Delta_{\pi/n} f(v)dv\right|^p du$$

$$\leq \int_{-\pi}^{\pi} du \int_{u-\pi/n}^{u} |\Delta_{\pi/n} f(v)|^p dv \left(\int_{u-\pi/n}^{u} du\right)^{p/q}$$

$$\leq An^{-p/q} \int_{-\pi}^{\pi} du \int_{u-\pi/n}^{u} |\Delta_{\pi/n} f(v)|^p dv$$

$$\leq An^{-p/q-1} \int_{-\pi}^{\pi} |\Delta_{\pi/n} f(v)|^p dv = o\left(1/n^{p+1}\right) .$$

Therefore $f \in M^p$, that is, $\mathrm{Lip}(1/p, p) \subset M^p$.

Now, let $0 < \varepsilon < 1$, $p' = (1-\varepsilon)p$ and $1/p + 1/q = 1$. If $f \in C \cap \mathrm{Lip}(1/p', p')$, then

$$\int_{-\pi}^{\pi} \left|\int_{u-\pi/n}^{u} \Delta_{\pi/n} f(v)dv\right|^p du$$

$$\leq \int_{-\pi}^{\pi} \left(\int_{u-\pi/n}^{u} |\Delta_{\pi/n} f(v)|^{(1-\varepsilon)p} dv\right)\left(\int_{u-\pi/n}^{u} |\Delta_{\pi/n} f(v)|^{\varepsilon q} dv\right)^{p/q} du$$

$$= o\left(n^{-p/q}\right) \int_{-\pi}^{\pi} du \int_{u-\pi/n}^{u} |\Delta_{\pi/n} f(v)|^{p'} dv$$

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$$= o\left(n^{-1-p/q} \int_{-\pi}^{\pi} |\Delta_{\pi/n} f(v)|^{p'} dv\right)$$

$$= o\left(n^{-2-p/q}\right) = o\left(1/n^{p+1}\right) .$$

Therefore $M^p \supset C \cap \mathrm{Lip}(1/p', p')$.

Finally, if $0 < \alpha < 1$, $\alpha p > 1$, $p > 1$ and $f \in \mathrm{Lip}\alpha$, then

$$\int_{-\pi}^{\pi} \left|\Delta^2_{\pi/n} f_1(u)\right|^p du = \int_{-\pi}^{\pi} \left|\int_{u-\pi/n}^{u} \Delta_{\pi/n} f(u) dv\right|^p du$$

$$\le A/n^{(\alpha+1)p} = o\left(1/n^{p+1}\right) .$$

Thus we get $f \in M^p$, that is, $\mathrm{Lip}\alpha \subset M^p$.

9. Proof of Theorem 2 *(ii)*

If $f \in CBV$, then the Fourier coefficients of f satisfy condition (3) for $p = 1$ by Wiener's Theorem. Therefore $N^2 \supset CBV$.

If $f \in \mathrm{Lip}\alpha$ $(1 \ge \alpha > 1/2)$, then

$$s_n(x;\ f) - f(x) = O\left(1/n^{\alpha}\right)$$

and then

$$\sum_{k=n}^{\infty} \left(a_k^2+b_k^2\right) = \frac{1}{\pi} \int_0^{2\pi} \left(s_n(x;\ f)-f(x)\right)^2 dx$$

$$= O\left(1/n^{2\alpha}\right) = o(1/n)$$

for $\alpha > 1/2$. Therefore $N^2 \supset \mathrm{Lip}\alpha$ $(1 \ge \alpha > 1/2)$.

10. Proof of Theorem 2 *(iii)*

Since

$$\Delta^2_{\pi/n} f_1(u+\pi/n) = \sum_{k=1}^{\infty} \frac{a_k}{k} \sin^2 \frac{k\pi}{2n} \sin ku + \sum_{k=1}^{\infty} \frac{b_k}{k} \sin^2 \frac{k\pi}{2n} \cos ku ,$$

by Parseval's Formula, we get

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$$\int_{-\pi}^{\pi} \left|\Delta^2_{\pi/n} f_1(u)\right|^2 du = \sum_{k=1}^{\infty} \left(\frac{a_k}{k}\right)^2 \sin^4 \frac{k\pi}{2n} + \sum_{k=1}^{\infty} \left(\frac{b_k}{k}\right)^2 \sin^4 \frac{k\pi}{2n} ,$$

which is $o\left(1/n^3\right)$ if condition (3) holds for $p = 2$, because

$$\sum_{k=1}^{\infty} \left(\frac{a_k}{k}\right)^2 \sin^4 \frac{k\pi}{2n} = \sum_{k=1}^{n} + \sum_{k=n+1}^{\infty}$$

$$\leq \frac{1}{n^4}\left\{\sum_{k=1}^{n-1} k \sum_{i=k}^{\infty} a_k^2 + n^2 \sum_{k=n}^{\infty} a_k^2\right\} + \frac{1}{n^2} \sum_{k=n}^{\infty} a_k^2$$

$$= o\left(1/n^3\right) .$$

Thus $N^2 \subset M^2$. On the other hand, if $f \in M^2$,

$$\sum_{k=1}^{\infty} \left(\frac{a_k}{k}\right)^2 \sin^4 \frac{k\pi}{2n} = o\left(1/n^3\right)$$

holds and then

$$\sum_{k=1}^{n} k^2 a_k^2 = o(n) .$$

Thus we can get

$$\sum_{k=n}^{\infty} a_k^2 = \sum_{k=n}^{\infty} k^2 a_k \cdot \frac{1}{k^2} = \frac{1}{n^2} \sum_{k=1}^{n} k^2 a_k^2 + \sum_{k=n}^{\infty} \frac{1}{k^3} \sum_{i=1}^{k} i^2 a_i^2$$

$$= o(1/n) ,$$

that is $M^2 \subset N^2$.

We shall prove the other part of Theorem 2 *(iii)*. For the sake of simplicity, we suppose that f is even. Then, by the Hausdorff-Young inequality,

$$\left\{\sum_{k=1}^{\infty} \left(\frac{a_k}{k}\right)^q \sin^{2q} \frac{k\pi}{2n}\right\}^{1/q} \leq \left\{\int_{-\pi}^{\pi} \left|\Delta^2_{\pi/n} f_1(u)\right|^p du\right\}^{1/p} ,$$

$$\left\{\int_{-\pi}^{\pi} \left|\Delta^2_{\pi/n} f_1(u)\right|^q du\right\}^{1/q} \leq \left\{\sum_{k=1}^{\infty} \left(\frac{a_k}{k}\right)^p \sin^{2p} \frac{k\pi}{2n}\right\}^{1/p} ,$$

where $1 < p < 2 < q < \infty$ and $1/p + 1/q = 1$. If $f \in M^p$ $(1 < p < 2)$, then the right side of the first inequality is $o\left(1/n^{1+1/p}\right)$, and thus

$$\sum_{k=1}^{n} k^q a_k^q = o(n) ,$$

and then

$$\begin{aligned}\sum_{k=n}^{\infty} a_k^q &= \sum_{k=n}^{\infty} k^q a_k^q \cdot k^{-q} \\ &= n^{-q} \sum_{k=1}^{n} k^q a_k^q + \sum_{k=n}^{\infty} k^{-q} \sum_{i=1}^{k} i^q a_i^q \\ &= o\left(n^{-q-1}\right) .\end{aligned}$$

That is, $f \in N^q$ $(1/p+1/q = 1)$ and then $N^q \supset M^p$. Similarly, we can see that $N^p \supset M^q$ from the second inequality.

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