Canad. Math. Bull. Vol. **18** (1), 1975

ON THE EXTENSION OF UNIFORMLY CONTINUOUS FUNCTIONS[1]

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ABSTRACT. A theorem of M. Katětov asserts that a bounded uniformly continuous function f on a subspace Q of a uniform space P has a bounded uniformly continuous extension to all of P. In this note we give new proofs of two special cases of this theorem: (i) Q is totally bounded, and (ii) P is a locally compact group and Q is a subgroup, both P and Q having the left uniformity.

In this note we prove some results concerning the extension of uniformly continuous functions on a subspace Q of a uniform space P to all of P. Katětov's theorem (quoted above) is Theorem 3 in [4]. All spaces are assumed Hausdorff and all functions are assumed to be real valued. The reader can refer to [5] and [8] for the general theory of uniform spaces. Of the considerable sequel to [4] in the literature, we mention only [2], for its succinct proof of Katětov's theorem.

Our reasons for presenting proofs of the two special cases are essentially opposite: in the totally bounded case (Theorem 1), the proof is interesting because of its novelty, whereas in the group case (Theorem 2), interest derives from the routineness of the proof and the resulting explicitness of the result.

I. The extension theorems

THEOREM 1. *Let f be a bounded uniformly continuous function on a totally bounded subspace Q of a uniform space P. Then f has a bounded uniformly continuous extension F to all of P.*

Proof. By a theorem in Kelley [5; Theorem 16, p. 188], we may regard P as a subset of a product of metric spaces. Then, by a result in Simmons [7; problem 5, p. 84], we may regard P as a subset of a product $E=\pi_\alpha E_\alpha$ of real Banach spaces, where E has the product topology, which is the topology of the product of the metric uniformities.

We now take the closure $\bar{P}$ of P in E and note that $\bar{Q}\subset\bar{P}$ is compact. For each continuous linear form λ on E, let $m_\lambda=\min\{\lambda(x) \mid x\in\bar{Q}\}$, $M_\lambda=\max\{\lambda(x) \mid x\in\bar{Q}\}$,

AMS 1970 *subject classifications*. Primary 46J10, 54E15; Secondary 43A15.

Key words and phrases. Uniform space, uniformly continuous function, extension, locally compact group.

[1] This research was supported in part by NRC grants A4006 and A7857.

and define $\Lambda \in C(E)$ by $\Lambda(x)=(m_\lambda \vee \lambda(x)) \wedge M_\lambda$. Each such λ is uniformly continuous on E and hence the uniformly closed algebra A generated by such Λ's and the constant function 1 consists entirely of uniformly continuous functions. Since the functions Λ separate the points of $\bar{Q}$, the Stone-Weierstrass theorem implies that $A|_{\bar{Q}}=C(\bar{Q})$, from which the assertion of the theorem follows directly.

THEOREM 2. *Let f be a bounded left uniformly continuous function on a closed subgroup Q of a locally compact topological group P. Then f has a bounded left uniformly continuous extension F to all of P.*

Proof. As a clarifying remark, we observe first that the left uniformity of Q, which Q has by virtue of being a topological group, is the same as the uniformity Q inherits by being a subspace of P furnished with its left uniformity. We also recall that a function f on a *topological group Q* is *left uniformly continuous* i.e., *uniformly continuous with respect to the left uniformity*, if and only if, given $\varepsilon>0$, $\exists$ a neighbourhood V of $e \in Q$ such that $|f(s)-f(t)|<\varepsilon$ whenever $st^{-1} \in V$. We denote the class of bounded left uniformly continuous functions on Q by $LUC(Q)$.

As a first step in the proof now, we note that the restriction homomorphism $\psi: LUC(P) \to LUC(Q)$, $\psi(v)=v|_Q$ for $v \in LUC(P)$, has closed image [1; 1.8.3. Corollaire]. Hence, we need only show that, given $f \in LUC(Q)$ and $\varepsilon>0$, $\exists v \in LUC(P)$ such that $\|v|_Q-f\|_\infty=\sup_{s\in Q}|v(s)-f(s)|<\varepsilon$. To this end, $\exists$ open $V \subset Q$ such that $e \in V$ and $|f(s)-f(t)|<\varepsilon$ whenever $st^{-1} \in V$. Choose symmetric open $W \subset P$, $W \neq \varnothing$, such that $W \cap Q \subset V$ and symmetric open $W_1 \subset P$, $W_1 \neq \varnothing$, such that $W_1^2 \subset W$. Finally, choose $g \in C(P)$ such that $g \geq 0$, g has compact support $K \subset W_1$ and $\int_Q g \underset{P}{*} g(t)\,dt=\int_Q (\int_P g(x)g(x^{-1}t)\,dx)\,dt=1$. (Here dx and dt are left Haar measures on P and Q, respectively.)

If we now define a function v_1 on P by $v_1(x)=\int_Q f(t^{-1})g(xt)\,dt$, then $v_1 \in C(P)$ (using (15.21) Theorem of [3] to show that $\|v_1\|_\infty<\infty$) and hence $v=g \underset{P}{*} v_1 \in LUC(P)$ [3; (20.16) Theorem (The reader will note that the notation used by the present authors is quite different from that used in this reference.)]. v is the function we are after; for if $s \in Q$, then

$$\begin{aligned}|v(s)-f(s)| &= \left|\int_P g(y)\left(\int_Q f(t^{-1})g(y^{-1}st)\,dt\right)dy-f(s)\right| \\ &= \left|\int_Q f(t^{-1})g \underset{P}{*} g(st)\,dt-f(s)\right| \\ &\leq \int_Q |f(t^{-1}s)-f(s)|g \underset{P}{*} g(t)\,dt \leq \varepsilon\end{aligned}$$

as required.

The following corollary is a consequence of fundamental results of Weil [8; pp. 22, 32] and Theorem 2. We omit the details of a proof.

https://doi.org/10.4153/CMB-1975-027-2 Published online by Cambridge University Press

COROLLARY 3. *The hypotheses of Theorem 2 can be weakened in two ways without losing the conclusion:*

(i) *Q need not be closed.*

(ii) *Instead of being locally compact, P need only have a totally bounded neighbourhood of the identity.*

REMARKS. (i) The conclusion of Katětov's theorem can fail, even in the setting of Theorem 2, if the function f is allowed to be unbounded. The well-known example of the function $f(n)=n^2$ defined on the integers is uniformly continuous, but obviously has no uniformly continuous extension to the reals.

(ii) In [6], an example is given of a bounded function that is (left and right) uniformly continuous on a closed normal abelian subgroup Q of a locally compact topological group P, but which has no left and right uniformly continuous extension to P. Thus the left and right uniformly continuous functions on a topological group are *not* the class of functions uniformly continuous with respect to any uniformity that behaves properly with respect to the taking of subgroups. (The "proper behaviour" of the left uniformity is asserted immediately after the statement of Theorem 2.)

Note added in proof: The computation in the proof of Theorem 2 is unnecessarily complicated. With $\psi=g \overset{*}{_P} g$, $v(x)=\int_Q f(t^{-1})\psi(xt)\,dt$. Using only that ψ is continuous with support $\subset W$ and $\int_Q \psi(t)\,dt=1$, we have by inspection that $v \in LUC(P)$, and $\|v|_Q-f\|_\infty<\varepsilon$. This should be compared to *e.g.* R. G. Burkel, *Weakly almost periodic functions*, Gordon and Breach 1970, page 49, which can be traced, in essence, back to H. Reiter, J. London Math. Soc. **32** (1957) 477–483.

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