

Equivalent criterion for the grand Riemann hypothesis associated with Maass cusp forms

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In this article, we obtain transformation formulas analogous to the identity of Ramanujan, Hardy and Littlewood in the setting of primitive Maass cusp form over the congruence subgroup $\Gamma_0(N)$ and also provide an equivalent criterion of the grand Riemann hypothesis for the L -function associated with the primitive Maass cusp form over $\Gamma_0(N)$.

Keywords: Maass cusp form; grand Riemann hypothesis; L -function associated with Maass cusp form; equivalent criterion; Bessel function

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1. Introduction

Let f be any Maass cusp form (cf. [6]) over the congruence subgroup $\Gamma_0(N) \subset \Gamma = SL_2(\mathbb{Z})$ with $\Delta f = \left(\frac{1}{4} + \nu^2\right) f$, where Δ is the non-Euclidean Laplacian operator

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} \right).$$

Since the Laplacian is self-adjoint, any eigenvalue associated with Δ must be a positive real number, and therefore ν is either real or of the form it with t real and $-1/2 < t < 1/2$.

We denote the Fourier coefficients of f at cusp infinity by $\lambda_f(n)$:

$$f(z) = y^{1/2} \sum_{n \neq 0} \lambda_f(n) K_{i\nu}(2\pi|n|y) e(nx)$$

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normalized by setting $\lambda_f(1) = 1$. Here $\lambda_f(n)$ is then the n th Hecke eigenvalue of f , K_{iv} is the modified Bessel function of second kind, $z = x + iy$ and $e(x) = e^{2\pi ix}$. The Maass cusp form f is said to be primitive if it is normalized cusp form which is a Hecke eigenform of all the Hecke operators.

The Ramanujan conjecture asserts that $|\lambda_f(n)| = \mathcal{O}(n^\epsilon)$ for any positive ϵ . In this direction, to the best of our knowledge, the best known bound for the Fourier coefficients was given by Kim and Sarnak [14], which is

$$|\lambda_f(n)| = \mathcal{O}\left(n^{7/64+\epsilon}\right) \tag{1.1}$$

for any positive ϵ .

Let $\rho : \mathbb{H} \rightarrow \mathbb{H}$ be the antiholomorphic involution given by $\rho(x + iy) = -x + iy$. We call a Maass cusp form f even if $f \circ \rho = f$ and odd if $f \circ \rho = -f$. We may attach an automorphic L -function associated with the primitive Maass cusp form f over $\Gamma_0(N)$ as

$$L(s, f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}, \quad \text{Re}(s) > 1. \tag{1.2}$$

It has an Euler product of the form (cf. [2, p. 208])

$$L(s, f) = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right)^{-1} \tag{1.3}$$

for $\text{Re}(s) > 1$, where χ_0 denotes the principle character modulo N . The L -function $L(s, f)$ can be analytically continued to the entire complex plane. The completed L -function associated with f can be defined as

$$\Lambda(s, f) = L_\infty(s, f)L(s, f),$$

where

$$L_\infty(s, f) := \pi^{-s} \Gamma\left(\frac{s + \epsilon + i\nu}{2}\right) \Gamma\left(\frac{s + \epsilon - i\nu}{2}\right). \tag{1.4}$$

Here ϵ takes the value 0 for f even and 1 for f odd. The completed L -function $\Lambda(s, f)$ satisfies the functional equation [2, p. 208]

$$\Lambda(s, f) = \epsilon_f N^{1/2-s} \Lambda(1 - s, f), \tag{1.5}$$

where ϵ_f is a complex number of modulus 1. The non-trivial zeros of $L(s, f)$ lie in the critical strip $0 < \text{Re}(s) < 1$ and, by the grand Riemann hypothesis, are conjectured to be on the critical line $\text{Re}(s) = 1/2$. The goal of this article is to obtain a transformation formula involving the non-trivial zeros of $L(s, f)$ and to find an equivalent criterion of the grand Riemann hypothesis for $L(s, f)$.

In the classical case, the problem was initiated by Ramanujan. He showed an elegant transformation formula to Hardy involving an infinite series of the Möbius function [16, p. 312], during his stay at Trinity. Later, the identity was corrected

by Hardy and Littlewood in [12, p. 156, § 2.5], which precisely states that for α and β being two positive real numbers with $\alpha\beta = \pi$, the following identity holds:

$$\sqrt{\alpha} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-\alpha^2/n^2} - \sqrt{\beta} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-\beta^2/n^2} = -\frac{1}{2\sqrt{\beta}} \sum_{\rho} \frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)} \beta^{\rho}, \tag{1.6}$$

provided the infinite series on the right-hand side of the above equation converges where the sum is running over the non-trivial zeros ρ of the Riemann zeta function with an assumption that the non-trivial zeros are simple. For more details, one can look into [4, p. 467–468].

The series on the right-hand side of (1.6) converges if we bracket the terms of the series in such a way that the terms for which

$$|\text{Im}(\rho) - \text{Im}(\rho')| < \exp\left(-c \frac{\text{Im}(\rho)}{\log(\text{Im}(\rho))}\right) + \exp\left(-c \frac{\text{Im}(\rho')}{\log(\text{Im}(\rho'))}\right)$$

are included in the same bracket (cf. [20, p. 220]) but the convergence of the series is still unknown unconditionally.

The identity (1.6) leads Hardy and Littlewood [12] to obtain an equivalent criterion of the Riemann hypothesis for $\zeta(s)$, which precisely states that for the function $P(y) := \sum_{n=1}^{\infty} \frac{(-y)^n}{n! \zeta(2n+1)}$, the bound $P(y) = \mathcal{O}(y^{-1/4+\delta})$ as $y \rightarrow \infty$ for any positive δ is equivalent to the Riemann hypothesis. This equivalent criterion for the Riemann hypothesis is sometimes known as Riesz-type criterion since Riesz [17] was the first mathematician to find a similar result around the same time.

Later, several analogues and generalizations have been investigated in different directions. For instance, a similar problem involving an extra complex variable was considered in [11]. The problem in the setting of Dirichlet L -function and Dedekind zeta function has been studied in [8] and [9] respectively. In [10], Dixit *et al.* have obtained an identity analogous to (1.6) for holomorphic Hecke eigenforms and found the similar criterion of the Riemann hypothesis for the associated L -function. Recently, in [1], one variable generalization of (1.6) have been studied in different direction.

We next provide the analogues of (1.6) for the non-holomorphic primitive Maass cusp form over the congruence subgroup $\Gamma_0(N) \subset \Gamma$. The reciprocal of $L(s, f)$ for $\text{Re}(s) > 1$ can be written as

$$\frac{1}{L(s, f)} = \sum_{n=1}^{\infty} \frac{\tilde{\lambda}_f(n)}{n^s}.$$

where

$$\tilde{\lambda}_f(n) = \begin{cases} \mu(d)\chi_0(D)\lambda_f(d) & \text{if } n = dD^2, (d, D) = 1 \text{ and } d, D \text{ squarefree} \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

The above expression of $\tilde{\lambda}_f(n)$ can be derived mainly from the Euler product of $L(s, f)^{-1}$, which follows from (1.3) that

$$\sum_{n=1}^{\infty} \frac{\tilde{\lambda}_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right). \tag{1.8}$$

The multiplicativity of $\tilde{\lambda}_f(n)$ and (1.8) together implies that for $p^m|n$ with $m \geq 3$, we have $\tilde{\lambda}_f(n) = 0$. Therefore, from the fact that both $\lambda_f(n)$ and $\chi_0(n)$ are multiplicative, we obtain (1.7).

We next define two functions

$$\mathcal{P}_{f_o}(y) := \sum_{n=1}^{\infty} \frac{\tilde{\lambda}_f(n)}{n^2} K_{i\nu} \left(\frac{2\pi y}{n} \right) \tag{1.9}$$

and

$$\mathcal{P}_{f_e}(y) := \sum_{n=1}^{\infty} \frac{\tilde{\lambda}_f(n)}{n} \left[2K_{i\nu} \left(\frac{2\pi y}{n} \right) - \frac{\Gamma(i\nu)}{(\pi y/n)^{i\nu}} - \frac{\Gamma(-i\nu)}{(\pi y/n)^{-i\nu}} \right]. \tag{1.10}$$

In the following theorem, we obtain transformation formulas involving the non-trivial zeros of $L(s, f)$.

THEOREM 1.1. *Let f be a normalized primitive Maass cusp form over the congruence subgroup $\Gamma_0(N)$ with eigenvalue $1/4 + \nu^2$. Let $L(s, f)$ be its associated L -function as defined in (1.2). Let ϵ_f be the complex number of modulus 1 appeared in (1.5). If α and β are any positive numbers with $\alpha\beta = 1/N$. If the multiplicity of a non-trivial zero ρ of $L(s, f)$ is n_ρ , Then*

(a) for f odd, we have

$$\begin{aligned} &\alpha^{3/2}\mathcal{P}_{f_o}(\alpha) - \epsilon_f\beta^{3/2}\mathcal{P}_{f_o}(\beta) \\ &= -\frac{\epsilon_f}{4\pi} \sum_{\rho} \frac{1}{(n_\rho - 1)!} \lim_{s \rightarrow \rho} \frac{d^{n_\rho-1}}{ds^{n_\rho-1}} \left\{ \frac{(s - \rho)^{n_\rho} L_\infty(1 - s, f) \beta^{s-\frac{1}{2}}}{L(s, f)} \right\}, \end{aligned} \tag{1.11}$$

(b) for f even, we have

$$\begin{aligned} \sqrt{\alpha}\mathcal{P}_{f_e}(\alpha) - \epsilon_f\sqrt{\beta}\mathcal{P}_{f_e}(\beta) &= \frac{\epsilon_f\pi^{i\nu}\Gamma(-i\nu)}{L(-i\nu, f)}\beta^{\frac{1}{2}+i\nu} + \frac{\epsilon_f\pi^{-i\nu}\Gamma(i\nu)}{L(i\nu, f)}\beta^{\frac{1}{2}-i\nu} \\ &\quad - \epsilon_f \sum_{\rho} \frac{1}{(n_\rho - 1)!} \lim_{s \rightarrow \rho} \frac{d^{n_\rho-1}}{ds^{n_\rho-1}} \\ &\quad \times \left\{ \frac{(s - \rho)^{n_\rho} L_\infty(1 - s, f) \beta^{s-\frac{1}{2}}}{L(s, f)} \right\}, \end{aligned} \tag{1.12}$$

provided the series on the right-hand of (1.11) and (1.12) converge where ρ runs through the non-trivial zeros of $L(s, f)$. In particular, if all non-trivial zeros of $L(s, f)$ are simple,

(a) for f odd, we have

$$\alpha^{3/2}\mathcal{P}_{f_o}(\alpha) - \epsilon_f\beta^{3/2}\mathcal{P}_{f_o}(\beta) = -\frac{\epsilon_f}{4\pi} \sum_{\rho} \frac{L_\infty(1 - \rho, f)}{L'(\rho, f)}\beta^{\rho-\frac{1}{2}}, \tag{1.13}$$

(b) for f even, we have

$$\begin{aligned} \sqrt{\alpha}\mathcal{P}_{f_e}(\alpha) - \epsilon_f\sqrt{\beta}\mathcal{P}_{f_e}(\beta) &= \frac{\epsilon_f\pi^{i\nu}\Gamma(-i\nu)}{L(-i\nu, f)}\beta^{\frac{1}{2}+i\nu} + \frac{\epsilon_f\pi^{-i\nu}\Gamma(i\nu)}{L(i\nu, f)}\beta^{\frac{1}{2}-i\nu} \\ &\quad - \epsilon_f\sum_{\rho} \frac{L_{\infty}(1-\rho, f)}{L'(\rho, f)}\beta^{\rho-\frac{1}{2}}. \end{aligned} \tag{1.14}$$

The next theorem provides an equivalent criterion of the grand Riemann hypothesis for $L(s, f)$ when f is odd, motivated from the above theorem.

THEOREM 1.2. *If f is an odd normalized primitive Maass cusp form over the congruence subgroup $\Gamma_0(N)$ with eigenvalue $1/4 + \nu^2$. Then the bound $\mathcal{P}_{f_o}(y) = \mathcal{O}(y^{-3/2+\delta})$ as $y \rightarrow \infty$ for every positive δ is equivalent to the grand Riemann hypothesis for $L(s, f)$.*

Now we consider the remaining case when f is even. In obtaining the criterion of the grand Riemann hypothesis for $L(s, f)$ for f even, we need to consider the derivative of the function $\mathcal{P}_{f_e}(y)$, unlike to the case when f is odd. The reason behind the above fact is technical, which is mainly due to the poles coming from the gamma factors involved in the functional equation (1.5) of $L(s, f)$. Let $\mathcal{Q}_{f_e}(y)$ be the derivative of the function $\mathcal{P}_{f_e}(y)$. Thus it follows from the derivative of the K-Bessel function [5, p. 36, Formula 1.14.1.1] that

$$\begin{aligned} \mathcal{Q}_{f_e}(y) &= \pi\sum_{n=1}^{\infty} \frac{\tilde{\lambda}_f(n)}{n^2} \left[2K_{1+i\nu}\left(\frac{2\pi y}{n}\right) + 2K_{1-i\nu}\left(\frac{2\pi y}{n}\right) \right. \\ &\quad \left. - \frac{\Gamma(1+i\nu)}{(\pi y/n)^{1+i\nu}} - \frac{\Gamma(1-i\nu)}{(\pi y/n)^{1-i\nu}} \right]. \end{aligned} \tag{1.15}$$

THEOREM 1.3. *Let f be an even normalized primitive Maass cusp form over the congruence subgroup $\Gamma_0(N)$ with eigenvalue $1/4 + \nu^2$. Then*

- (a) *The bound $\mathcal{Q}_{f_e}(y) = \mathcal{O}(y^{-3/2+\delta})$ for any $\delta > 0$ implies the grand Riemann hypothesis for $L(s, f)$.*
- (b) *If the grand Riemann hypothesis for $L(s, f)$ is true, then as $y \rightarrow \infty$,*

$$\mathcal{Q}_{f_e}(y) = -\pi\sum_{n=1}^{\lfloor y^{1-\epsilon} \rfloor - 1} \frac{\tilde{\lambda}_f(n)}{n^2} \left\{ \frac{\Gamma(1+i\nu)}{(\pi y/n)^{1+i\nu}} + \frac{\Gamma(1-i\nu)}{(\pi y/n)^{1-i\nu}} \right\} + \mathcal{O}\left(y^{-3/2+\delta}\right) \tag{1.16}$$

and

$$\mathcal{P}_{f_e}(y) = -\sum_{n=1}^{\lfloor y^{1-\epsilon} \rfloor - 1} \frac{\tilde{\lambda}_f(n)}{n} \left\{ \frac{\Gamma(i\nu)}{(\pi y/n)^{i\nu}} + \frac{\Gamma(-i\nu)}{(\pi y/n)^{-i\nu}} \right\} + \mathcal{O}\left(y^{-1/2+\delta}\right).$$

The article is organized as follows. The next section is devoted to obtaining the transformation formulas of $L(s, f)$. We prove the criterion for the grand Riemann hypothesis for $L(s, f)$ for both odd and even case in § 3 and § 4, respectively.

2. Transformation formula involving the non-trivial zeros of $L(s, f)$

In this section, we obtain an analogous result of (1.6) in the setting of primitive Maass cusp form. In the following lemma, we provide a lower bound for $L(s, f)$ which will be used to prove theorem 1.1.

LEMMA 2.1. *For a sequence of positive numbers T as $T \rightarrow \infty$ through values such that $|T - \gamma| > \exp(-A_1\gamma/\log \gamma)$ for every ordinate γ of a zero of $L(s, f)$, where A_1 is sufficiently small positive constant, the bound*

$$|L(\sigma + iT, f)| \geq e^{-A_2T}$$

holds for $\sigma \in [-1/2, 3/2]$, where $0 < A_2 < \frac{\pi}{4}$.

Proof. The key ingredients to prove the above lemma are followed from [13, p. 102, Formula 5.24] and [18, Lemma 3.6]. One can therefore apply a similar argument as in [20, p. 219, § 9.8] or [9, p. 6] to complete the proof. \square

We next provide the proof of theorem 1.1 only for the case when f odd since the proof for f even goes along the similar direction.

2.1. Proof of theorem 1.1

The inverse Mellin transform of the modified Bessel function $K_\mu(x)$ [19, p. 253, Formula 10.32.13] is given by

$$K_\mu(x) = \frac{1}{2\pi i} \int_{(c)} 2^{s-2} \Gamma\left(\frac{s-\mu}{2}\right) \Gamma\left(\frac{s+\mu}{2}\right) x^{-s} ds, \tag{2.1}$$

which is valid for any $c > \pm \text{Re}(\mu)$. Here and throughout the article, $\int_{(c)}$ denotes the integral $\int_{c-i\infty}^{c+i\infty}$. Replacing s by $s + 1$ in (2.1) and letting $x = 2\pi\alpha/n$ and $\mu = i\nu$, we have

$$\frac{4\pi\alpha}{n} K_{i\nu}\left(\frac{2\pi\alpha}{n}\right) = \frac{1}{2\pi i} \int_{(c)} L_\infty(s, f) \left(\frac{\alpha}{n}\right)^{-s} ds, \tag{2.2}$$

for $-\frac{1}{2} < c < -\frac{7}{64}$, where $L_\infty(s, f)$ is defined in (1.4) for f odd. We next insert (2.2) into (1.9) to obtain

$$4\pi\alpha \mathcal{P}_{f_o}(\alpha) = \sum_{n=1}^{\infty} \frac{\tilde{\lambda}_f(n)}{n} \frac{1}{2\pi i} \int_{(c)} L_\infty(s, f) \left(\frac{\alpha}{n}\right)^{-s} ds = \frac{1}{2\pi i} \int_{(c)} \frac{L_\infty(s, f)}{L(1-s, f)} \alpha^{-s} ds, \tag{2.3}$$

where in the last step we have interchanged the order of summation and integration. The functional equation of $L(s, f)$, as in (1.5) thus yields

$$4\pi\alpha \mathcal{P}_{f_o}(\alpha) = \frac{\epsilon_f \sqrt{N}}{2\pi i} \int_{(c)} \frac{L_\infty(1-s, f)}{L(s, f)} (N\alpha)^{-s} ds. \tag{2.4}$$

We next shift the line of integration from $\text{Re}(s) = c$ to $\text{Re}(s) = \lambda$ where $1 < \lambda < \frac{3}{2}$. Consider the positively oriented contour formed by the line segments

$[\lambda - iT, \lambda + iT]$, $[\lambda + iT, c + iT]$, $[c + iT, c - iT]$ and $[c - it, \lambda - iT]$, where T is any positive real number. Clearly, the poles of the Gamma factors in $L_\infty(1 - s, f)$ are on the right side of the line $\text{Re}(s) = \lambda$. Thus, the only poles inside the contour are arising from the non-trivial zeros ρ of $L(s, f)$. Therefore, by invoking the Cauchy residue theorem, we arrive at

$$\int_{c-iT}^{c+iT} \frac{L_\infty(1-s, f)}{L(s, f)} (N\alpha)^{-s} ds = \left[\int_{c-iT}^{\lambda-iT} + \int_{\lambda-iT}^{\lambda+iT} + \int_{\lambda+iT}^{c+iT} \right] \times \frac{L_\infty(1-s, f)}{L(s, f)} (N\alpha)^{-s} ds - 2\pi i \sum_{|\text{Im}(\rho)| < T} R_\rho, \tag{2.5}$$

where R_ρ denotes a residual term at a non-trivial zero ρ of $L(s, f)$. If we assume the multiplicity of a non-trivial zero ρ of $L(s, f)$ as n_ρ , then the residual term R_ρ can be calculated as

$$\begin{aligned} R_\rho &= \frac{1}{(n_\rho - 1)!} \lim_{s \rightarrow \rho} \frac{d^{n_\rho-1}}{ds^{n_\rho-1}} \left\{ \frac{(s - \rho)^{n_\rho} L_\infty(1 - s, f) (N\alpha)^{-s}}{L(s, f)} \right\} \\ &= \frac{1}{(n_\rho - 1)!} \lim_{s \rightarrow \rho} \frac{d^{n_\rho-1}}{ds^{n_\rho-1}} \left\{ \frac{(s - \rho)^{n_\rho} L_\infty(1 - s, f) \beta^s}{L(s, f)} \right\}. \end{aligned}$$

We can bind $\Gamma(s)$ for $s = \sigma + it$, in any vertical strip using Stirling’s formula, which is given by (cf. [7, p. 224])

$$|\Gamma(s)| = (2\pi)^{1/2} |t|^{\sigma - \frac{1}{2}} e^{-\frac{1}{2}\pi|t|} \left(1 + \mathcal{O}\left(\frac{1}{|t|}\right) \right). \tag{2.6}$$

Now, lemma 2.1 and (2.6) together implies that as $T \rightarrow \infty$,

$$\frac{L_\infty(1-s, f)}{L(s, f)} (N\alpha)^{-s} = \mathcal{O}\left(e^{(A_2 - \frac{\pi}{2})|T|}\right),$$

where $A_2 < \pi/4$. Therefore, the horizontal integrals in (2.5) vanishes as $T \rightarrow \infty$. We next concentrate on the vertical integral in (2.5) and denote the vertical integral as \mathcal{V} . Substituting s by $1 - s$ and applying the relation $\alpha\beta = 1/N$ respectively, the vertical integral \mathcal{V} reduces to

$$\mathcal{V} := \int_{\lambda-iT}^{\lambda+iT} \frac{L_\infty(1-s, f)}{L(s, f)} (N\alpha)^{-s} ds = \beta \int_{1-\lambda-iT}^{1-\lambda+iT} \frac{L_\infty(s, f)}{L(1-s, f)} \beta^{-s} ds.$$

Clearly $-\frac{1}{2} < 1 - \lambda < 0$. Therefore as $T \rightarrow \infty$, we can apply (2.3) to evaluate the vertical integral \mathcal{V} as

$$\mathcal{V} = 2\pi i \cdot 4\pi\beta^2 \mathcal{P}_{f_o}(\beta). \tag{2.7}$$

Combining (2.4), (2.5) and (2.7) together, we have

$$\frac{4\pi\alpha}{\epsilon_f \sqrt{N}} \mathcal{P}_{f_o}(\alpha) = 4\pi\beta^2 \mathcal{P}_{f_o}(\beta) - \sum_{\rho} R_\rho.$$

Finally, we multiply both sides of the above equation by $\frac{\epsilon_f \sqrt{N\alpha}}{4\pi}$ and invoke the relation $\alpha\beta = 1/N$ to conclude the first part (1.11) of our theorem.

For the second part, when f is even, one can first apply Cauchy residue theorem in (2.1) by shifting the line of integration from c to λ such that $-1 < \lambda < -\frac{1}{2}$ and substitute $x = \frac{2\pi\alpha}{n}$, $\mu = i\nu$ to obtain

$$\begin{aligned} & 2K_{i\nu} \left(\frac{2\pi\alpha}{n} \right) - \left(\frac{\pi\alpha}{n} \right)^{-i\nu} \Gamma(i\nu) - \left(\frac{\pi\alpha}{n} \right)^{i\nu} \Gamma(-i\nu) \\ &= \frac{1}{4\pi i} \int_{(\lambda)} \Gamma \left(\frac{s+i\nu}{2} \right) \Gamma \left(\frac{s-i\nu}{2} \right) \left(\frac{\pi\alpha}{n} \right)^{-s} ds. \end{aligned} \tag{2.8}$$

Next, one can argue along a similar direction as was shown for part (a) to complete the proof of (1.12).

In particular, if all the non-trivial zeros of $L(s, f)$ are simple, the term R_ρ in (2.5) can be evaluated as

$$R_\rho = \lim_{s \rightarrow \rho} (s - \rho) \frac{L_\infty(1-s, f)}{L(s, f)} (N\alpha)^{-s} = \frac{L_\infty(1-\rho, f)}{L'(\rho, f)} (N\alpha)^{-\rho} = \frac{L_\infty(1-\rho, f)}{L'(\rho, f)} \beta^\rho, \tag{2.9}$$

where in the last step we applied the relation $\alpha\beta = 1/N$. This completes the proof of the theorem.

3. Equivalent criterion for grand Riemann hypothesis when f is odd

We first provide a heuristic stemming from theorem 1.1, which motivates us to get an equivalent criterion of the grand Riemann hypothesis for $L(s, f)$. Invoking lemma 3.1 in (1.9), we obtain

$$\alpha^{3/2} \mathcal{P}_{f_o}(\alpha) \rightarrow 0, \tag{3.1}$$

as $\alpha \rightarrow 0$. We now assume the grand Riemann hypothesis for $L(s, f)$ and the convergence of the series $\sum_\rho \frac{L_\infty(1-\rho, f)}{L'(\rho, f)} \beta^{\rho-\frac{1}{2}}$. Then (1.13) and (3.1) together conclude

$$\mathcal{P}_{f_o}(\beta) = \mathcal{O} \left(\beta^{-3/2} \right),$$

as $\beta \rightarrow \infty$.

The heuristic assumes the convergence of the series $\sum_\rho \frac{L_\infty(1-\rho, f)}{L'(\rho, f)} \beta^{\rho-\frac{1}{2}}$. It has been shown in theorem 1.2 that the assumption of the grand Riemann hypothesis is enough to obtain the bound $\mathcal{P}_{f_o}(\beta) = \mathcal{O}(\beta^{-3/2+\delta})$, for any $\delta > 0$.

In the following lemma, we provide a bound for the function $\mathcal{P}_{f_o}(y)$ as $y \rightarrow 0$.

LEMMA 3.1. *Let $-\frac{1}{2} < c < -\frac{7}{64}$. We have*

$$\mathcal{P}_{f_o}(y) = \mathcal{O} \left(y^{-1-c} \right),$$

as $y \rightarrow 0$.

Proof. We first apply both (2.2) and (2.6) to obtain that for $-\frac{1}{2} < c < -\frac{7}{64}$,

$$\frac{1}{n} K_{i\nu} \left(\frac{2\pi y}{n} \right) = \frac{1}{4\pi y} \frac{1}{2\pi i} \int_{(c)} L_\infty(s, f) \left(\frac{y}{n} \right)^{-s} ds \ll \frac{y^{-c-1}}{n^{-c}}. \tag{3.2}$$

Therefore, (1.9) and (3.2) imply that

$$\mathcal{P}_{f_o}(y) \ll y^{-c-1} \sum_{n=1}^\infty \frac{\tilde{\lambda}_f(n)}{n^{1-c}} \ll y^{-c-1},$$

where in the last step we used the convergence of the series $\sum_{n=1}^\infty \frac{\tilde{\lambda}_f(n)}{n^{1-c}}$ for $-\frac{1}{2} < c < -\frac{7}{64}$. □

We next obtain the Mellin transform of the function $\mathcal{P}_{f_o}(y)$, which plays an important role to prove theorem 1.2, when f is odd.

LEMMA 3.2. *For any complex number s with $0 < \text{Re}(s) < \frac{1}{2}$, we have*

$$\int_0^\infty y^{-s} \mathcal{P}_{f_o}(y) dy = \frac{\pi^{s-1} \Gamma\left(\frac{1-s+i\nu}{2}\right) \Gamma\left(\frac{1-s-i\nu}{2}\right)}{4 L(s+1, f)}.$$

Proof. Let

$$\phi(s) := \int_0^\infty y^{-s} \mathcal{P}_{f_o}(y) dy.$$

We first make the change of variable y to x/n and then multiply both sides by $\lambda_f(n)/n$ to write the above integral as

$$\frac{\lambda_f(n)}{n^{s+1}} \phi(s) = \int_0^\infty x^{-s} \frac{\lambda_f(n)}{n^2} \mathcal{P}_{f_o} \left(\frac{x}{n} \right) dx.$$

Summing over n from 1 to ∞ , we interchange the order of summation and integration by applying the Weierstrass M-test and the Lebesgue dominated convergence theorem to obtain

$$L(s+1, f) \phi(s) = \int_0^\infty x^{-s} \sum_{n=1}^\infty \frac{\lambda_f(n)}{n^2} \mathcal{P}_{f_o} \left(\frac{x}{n} \right) dx. \tag{3.3}$$

It follows from the integral representation of $\mathcal{P}_{f_o}(y)$ as in (2.3) that for any $c \in (-\frac{1}{2}, 0)$, we have

$$\begin{aligned} \sum_{n=1}^\infty \frac{\lambda_f(n)}{n^2} \mathcal{P}_{f_o} \left(\frac{x}{n} \right) &= \sum_{n=1}^\infty \frac{\lambda_f(n)}{n^2} \frac{n}{4\pi x} \frac{1}{2\pi i} \int_{(c)} \frac{L_\infty(s, f)}{L(1-s, f)} \left(\frac{x}{n} \right)^{-s} ds \\ &= \frac{1}{4\pi i x} \int_{(c)} L_\infty(s, f) x^{-s} ds, \end{aligned}$$

where in the last step we interchanged the order of summation and integration and used the series definition of $L(s, f)$. Therefore, substituting α by nx in expression

(2.2), the above series reduces to

$$\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^2} \mathcal{P}_{f_o} \left(\frac{x}{n} \right) = K_{i\nu}(2\pi x). \tag{3.4}$$

Finally, we insert (3.4) into (3.3) and apply the Mellin transform of K -Bessel function for $0 < \text{Re}(s) < \frac{1}{2}$ to conclude that

$$L(s + 1, f)\phi(s) = \int_0^{\infty} x^{-s} K_{i\nu}(2\pi x) = \frac{\pi^{s-1}}{4} \Gamma \left(\frac{1 - s + i\nu}{2} \right) \Gamma \left(\frac{1 - s - i\nu}{2} \right).$$

This completes the proof of our lemma. □

We are now ready to prove an equivalent criterion of the grand Riemann hypothesis for $L(s, f)$ when f is odd.

3.1. Proof of theorem 1.2

We first prove the sufficient condition of the grand Riemann hypothesis for $L(s, f)$ and for that we assume that the bound $\mathcal{P}_{f_o}(y) = \mathcal{O}(y^{-\frac{3}{2}+\delta})$ holds for any $\delta > 0$, as $y \rightarrow \infty$. It is already known from lemma 3.2 that the identity

$$L(s + 1, f) \int_0^{\infty} y^{-s} \mathcal{P}_{f_o}(y) dy = \frac{\pi^{s-1}}{4} \Gamma \left(\frac{1 - s + i\nu}{2} \right) \Gamma \left(\frac{1 - s - i\nu}{2} \right). \tag{3.5}$$

holds in $0 < \text{Re}(s) < \frac{1}{2}$. We claim that the identity is also valid inside the region $-\frac{1}{2} < \text{Re}(s) \leq 0$ under the assumption of the above bound of $\mathcal{P}_{f_o}(y)$.

To prove the claim, we first split the integral in the above identity into two parts, one from 0 to 1 and another from 1 to ∞ . Lemma 3.1 yields the convergence of the first integral inside the region $-\frac{1}{2} < \text{Re}(s) \leq 0$. It follows from the bound $\mathcal{P}_{f_o}(y) = \mathcal{O}(y^{-\frac{3}{2}+\delta})$ that the second integral also converges inside the same region. Therefore, the result [19, theorem 3.2, p. 30] implies that the integral $\int_0^{\infty} y^{-s} \mathcal{P}_{f_o}(y) dy$ represents an analytic function in $-\frac{1}{2} < \text{Re}(s) \leq 0$. The gamma factors on the right-hand side of (3.5) are also analytic inside $-\frac{1}{2} < \text{Re}(s) \leq 0$ and $L(s + 1, f)$ is entire. Therefore, the principle of analytic continuation yields our claim.

Now the right-hand side of (3.5) has no zeros in $-\frac{1}{2} < \text{Re}(s) \leq 0$. On the other hand, the integral on the left-hand side is analytic inside the same region. Therefore, $L(s + 1, f)$ does not vanish in $-\frac{1}{2} < \text{Re}(s) \leq 0$ which concludes that the grand Riemann hypothesis is true.

We next prove the converse part and for that we first assume that the grand Riemann hypothesis is true. Under the assumption of the grand Riemann hypothesis for $L(s, f)$, it follows from [13, proposition 5.14, p. 113] that as $x \rightarrow \infty$,

$$\sum_{n \leq x} \tilde{\lambda}_f(n) = \mathcal{O} \left(x^{\frac{1}{2}+\delta} \right),$$

where δ is any positive real number. Let

$$M(\ell, n) := \sum_{m=\ell}^n \frac{\tilde{\lambda}_f(m)}{m}.$$

Applying the Euler’s partial summation formula, the above function can be bounded as

$$M(\ell, n) = \mathcal{O}\left(\ell^{-1/2+\delta}\right). \tag{3.6}$$

Let $\ell = \lfloor y^{1-\delta} \rfloor$. We split the sum $\mathcal{P}_{f_o}(y)$ into two parts as

$$\mathcal{P}_{f_o}(y) = S_1(f, y) + S_2(f, y), \tag{3.7}$$

where

$$S_1(f, y) := \sum_{n=1}^{\ell-1} \frac{\tilde{\lambda}_f(n)}{n^2} K_{i\nu}\left(\frac{2\pi y}{n}\right),$$

and

$$S_2(f, y) := \sum_{n=\ell}^{\infty} \frac{\tilde{\lambda}_f(n)}{n^2} K_{i\nu}\left(\frac{2\pi y}{n}\right).$$

We first estimate $S_2(f, y)$. We have

$$\begin{aligned} \sum_{n=\ell}^N \frac{\tilde{\lambda}_f(n)}{n^2} K_{i\nu}\left(\frac{2\pi y}{n}\right) &= \sum_{n=\ell}^{N-1} M(\ell, n) \left(\frac{K_{i\nu}\left(\frac{2\pi y}{n}\right)}{n} - \frac{K_{i\nu}\left(\frac{2\pi y}{n+1}\right)}{n+1} \right) \\ &\quad + M(\ell, N) \frac{K_{i\nu}\left(\frac{2\pi y}{N}\right)}{N}, \end{aligned}$$

where the last term vanishes as $N \rightarrow \infty$. Therefore, the above equation implies

$$S_2(f, y) = \sum_{n=\ell}^{\infty} M(\ell, n) \left(\frac{K_{i\nu}\left(\frac{2\pi y}{n}\right)}{n} - \frac{K_{i\nu}\left(\frac{2\pi y}{n+1}\right)}{n+1} \right). \tag{3.8}$$

We next estimate the above sum by utilizing the mean value theorem and for that we define

$$g(x) := \frac{K_{i\nu}\left(\frac{2\pi y}{x}\right)}{x}. \tag{3.9}$$

Invoking (2.2) into (3.9), we obtain

$$g(x) = \frac{1}{4\pi y} \frac{1}{2\pi i} \int_{(c)} \Gamma\left(\frac{s+1+i\nu}{2}\right) \Gamma\left(\frac{s+1-i\nu}{2}\right) (\pi y)^{-s} ds,$$

where $-\frac{1}{2} < c < 0$. We differentiate $g(x)$ with respect to x and apply Stirling’s formula (2.6) on the gamma factors to derive that

$$g'(x) = \frac{1}{4\pi y} \int_{(c)} \Gamma\left(\frac{s+1+i\nu}{2}\right) \Gamma\left(\frac{s+1-i\nu}{2}\right) s (\pi y)^{-s} x^{s-1} ds \ll \frac{x^{c-1}}{y^{c+1}}. \tag{3.10}$$

It follows from the mean value theorem that there exists $\lambda_n \in (n, n + 1)$ such that $g(n) - g(n + 1) = -g'(\lambda_n)$. Therefore, the sum $S_2(f, y)$ in (3.8) can be written as

$$S_2(f, y) = \sum_{n=\ell}^{\infty} M(\ell, n)(g(n) - g(n + 1)) = \sum_{n=\ell}^{\infty} M(\ell, n)(-g'(\lambda_n)).$$

Inserting the bounds from (3.6) and (3.10) into the above equation, we arrive at

$$S_2(f, y) \ll \frac{\ell^{-\frac{1}{2}+\delta}}{y^{c+1}} \sum_{n=\ell}^{\infty} n^{c-1} \ll \frac{\ell^{c-\frac{1}{2}+\delta}}{y^{c+1}} \ll y^{-\frac{3}{2}+\delta}, \tag{3.11}$$

where in the penultimate step we have used the fact that $\sum_{n>x} n^{-s} = \mathcal{O}(x^{1-s})$ (cf. [3, p. 55, theorem 3.2(c)]) and in the last step, we put $\ell = \lfloor y^{1-\delta} \rfloor$.

Next, we will concentrate on $S_1(f, y)$. Utilizing the asymptotics of $K_{i\nu}(z)$ as $z \rightarrow \infty$ [15, Formula 10.40.2, p. 255], we can bound $S_1(f, y)$ as

$$S_1(f, y) \ll \frac{e^{-\frac{2\pi y}{\ell}}}{\sqrt{y}} \sum_{n=1}^{\ell-1} \frac{\tilde{\lambda}_f(n)}{n^{3/2}}.$$

Therefore, the bound of $\tilde{\lambda}_f(n)$, followed from (1.1) and the value $\ell = \lfloor y^{1-\delta} \rfloor$, yields

$$S_1(f, y) \ll \frac{e^{-\frac{2\pi y}{\ell}}}{\sqrt{y}} \ell^{-\frac{25}{64}+\delta} \ll y^{-\frac{57}{64}+\delta} e^{-2\pi y^\delta}, \tag{3.12}$$

for any $\delta > 0$. Combining the estimates of $S_1(f, y)$ and $S_2(f, y)$ as in (3.12) and (3.11) respectively into (3.7), we can conclude that as $y \rightarrow \infty$, $\mathcal{P}_{f_\rho}(y) = \mathcal{O}(y^{-\frac{3}{2}+\delta})$ for any $\delta > 0$. This completes the proof of theorem 1.2(a).

4. Equivalent criterion for grand Riemann hypothesis when f is even

In this section, we provide a criterion of the grand Riemann hypothesis for $L(s, f)$ when f is even, which is motivated from theorem 1.1(b). Under the assumption of the grand Riemann hypothesis for $L(s, f)$ and the convergence of the series $\sum_{\rho} \frac{L_{\infty}(1-\rho, f)}{L'(\rho, f)} \beta^{\rho-\frac{1}{2}}$, it follows from a similar argument in the beginning of § 3 that as $\alpha \rightarrow 0$, or equivalently, as $\beta \rightarrow \infty$,

$$\mathcal{P}_{f_e}(\beta) = \frac{\pi^{i\nu}\Gamma(-i\nu)}{L(-i\nu, f)}\beta^{i\nu} + \frac{\pi^{-i\nu}\Gamma(i\nu)}{L(i\nu, f)}\beta^{-i\nu} + \mathcal{O}(\beta^{-1/2})$$

The heuristic is true under the assumption of the convergence of the series $\sum_{\rho} \frac{L_{\infty}(1-\rho, f)}{L'(\rho, f)} \beta^{\rho-\frac{1}{2}}$ but without this assumption, it is shown in theorem 1.2 that the estimate of $\mathcal{P}_{f_e}(\beta)$ provides the main term of order $\beta^{|i\nu|}$ and the error term of the order $\beta^{-1/2+\delta}$ for any $\delta > 0$. As mentioned earlier in section § 1, we need to consider $\mathcal{Q}_{f_e}(\beta)$ to find the criterion of the grand Riemann hypothesis for $L(s, f)$.

In order to prove the sufficient condition of the grand Riemann hypothesis, the following two lemmas are crucial.

LEMMA 4.1. Let $-\frac{3}{2} < c < -\frac{1}{2}$. Then for any $y > 0$,

$$\mathcal{Q}_{f_e}(y) = \mathcal{O}(y^{-c-1}).$$

Proof. We first consider the function

$$\Omega_\nu(y, x) := 2K_{1+i\nu}\left(\frac{2\pi y}{x}\right) + 2K_{1-i\nu}\left(\frac{2\pi y}{x}\right) - \frac{\Gamma(1+i\nu)}{(\pi y/x)^{1+i\nu}} - \frac{\Gamma(1-i\nu)}{(\pi y/x)^{1-i\nu}}. \tag{4.1}$$

The above function can be written as

$$\begin{aligned} \Omega_\nu(y, x) &= -\frac{x}{\pi} \frac{d}{dy} \left[2K_{i\nu}\left(\frac{2\pi y}{x}\right) - \frac{\Gamma(i\nu)}{(\pi y/x)^{i\nu}} - \frac{\Gamma(-i\nu)}{(\pi y/x)^{-i\nu}} \right] \\ &= -\frac{x}{\pi} \frac{d}{dy} \frac{1}{4\pi i} \int_{(c)} \Gamma\left(\frac{s+i\nu}{2}\right) \Gamma\left(\frac{s-i\nu}{2}\right) \left(\frac{\pi y}{x}\right)^{-s} ds \\ &= \frac{1}{4\pi i} \int_{(c)} s \Gamma\left(\frac{s+i\nu}{2}\right) \Gamma\left(\frac{s-i\nu}{2}\right) \left(\frac{\pi y}{x}\right)^{-s-1} ds, \end{aligned} \tag{4.2}$$

where in the penultimate step we have applied (2.8). Invoking Stirling formula for gamma functions in the above integral, we obtain

$$\Omega_\nu(y, x) = \mathcal{O}\left(\left(\frac{x}{y}\right)^{c+1}\right). \tag{4.3}$$

Therefore, the definition of $\mathcal{Q}_{f_e}(y)$ together with (4.3) yields

$$\mathcal{Q}_{f_e}(y) \ll y^{-1-c} \sum_{n=1}^{\infty} \frac{\tilde{\lambda}_f(n)}{n^{1-c}} \ll y^{-c-1},$$

where in the last step we used the fact that the series $\sum_{n=1}^{\infty} \frac{\tilde{\lambda}_f(n)}{n^{1-c}}$ converges as $-\frac{3}{2} < c < -\frac{1}{2}$. This proves the lemma. \square

In the following lemma, we provide the Mellin transform of $\mathcal{Q}_{f_e}(y)$, which plays an important role to prove the sufficient condition of the grand Riemann hypothesis for $L(s, f)$, when f is even.

LEMMA 4.2. For any complex number s with $\frac{1}{2} < \text{Re}(s) < \frac{3}{2}$, we have

$$\int_0^\infty y^{-s} \mathcal{Q}_{f_e}(y) dy = \frac{\pi^s s}{2} \frac{\Gamma\left(\frac{-s+i\nu}{2}\right) \Gamma\left(\frac{-s-i\nu}{2}\right)}{L(s+1, f)}. \tag{4.4}$$

Proof. Proceeding similarly as was done in the proof of lemma 3.2, we can obtain

$$\int_0^\infty y^{-s-1} \mathcal{P}_{f_e}(y) dy = \frac{\pi^s}{2} \frac{\Gamma\left(\frac{-s+i\nu}{2}\right) \Gamma\left(\frac{-s-i\nu}{2}\right)}{L(s+1, f)}.$$

We finally perform the integration by parts and utilize the fact that $\mathcal{Q}_{f_e}(y) = \frac{d}{dy} \mathcal{P}_{f_e}(y)$ to conclude our lemma. \square

4.1. Proof of theorem 1.3

We first prove part (a) and for that we assume the bound $\mathcal{Q}_{f_e}(y) = \mathcal{O}(y^{-3/2+\delta})$ for any $\delta > 0$. Multiplying both sides of (4.4) with $(\frac{s^2+\nu^2}{4})$, the identity

$$L(s + 1, f) \left(\frac{s^2 + \nu^2}{4} \right) \int_0^\infty y^{-s} \mathcal{Q}_{f_e}(y) dy = \frac{\pi^s s}{2} \Gamma \left(1 + \frac{-s + i\nu}{2} \right) \Gamma \left(1 + \frac{-s - i\nu}{2} \right) \tag{4.5}$$

holds for $\frac{1}{2} < \text{Re}(s) < \frac{3}{2}$. Invoking lemma 4.1 and the bound of $\mathcal{Q}_{f_e}(y)$, it can be shown by a similar argument of the proof of theorem 1.2 that the identity is also valid inside the region $-\frac{1}{2} < \text{Re}(s) < \frac{3}{2}$.

Now, the right-hand side of (4.5) has no zeros in $-\frac{1}{2} < \text{Re}(s) < 0$. On the other hand, integral is also analytic inside the same region. Therefore, $L(s + 1, f)$ has no zeros in $-\frac{1}{2} < \text{Re}(s) < 0$. This completes the proof of the grand Riemann hypothesis.

We next prove part (b) and for that we assume the grand Riemann hypothesis for $L(s, f)$. The argument to prove the estimates of $\mathcal{P}_{f_e}(y)$ and $\mathcal{Q}_{f_e}(y)$ are similar, we here provide the proof for $\mathcal{Q}_{f_e}(y)$.

It follows from [13, proposition 5.14, p. 113] that the grand Riemann hypothesis for $L(s, f)$ implies

$$\sum_{n \leq x} \tilde{\lambda}_f(n) = \mathcal{O} \left(x^{\frac{1}{2}+\delta} \right),$$

as $x \rightarrow \infty$, where δ is any positive real number. Let

$$M(\ell, n) := \sum_{m=\ell}^n \frac{\tilde{\lambda}_f(m)}{m^2}.$$

Applying the Euler’s partial summation formula, the above function can be bounded as

$$M(\ell, n) = \mathcal{O} \left(\ell^{-3/2+\delta} \right). \tag{4.6}$$

Inserting $\Omega_\nu(y, x)$ from (4.1) into the definition of $\mathcal{Q}_{f_e}(y)$ in (1.15), we have

$$\mathcal{Q}_{f_e}(y) = \pi \sum_{n=1}^\infty \frac{\tilde{\lambda}_f(n)}{n^2} \Omega_\nu(y, n).$$

Let $\ell = \lfloor y^{1-\delta} \rfloor$. We next split the above sum into two parts as

$$\begin{aligned} \mathcal{Q}_{f_e}(y) &= \sum_{n=1}^{\ell-1} \frac{\tilde{\lambda}_f(n)}{n^2} \Omega_\nu(y, n) + \sum_{n=\ell}^\infty \frac{\tilde{\lambda}_f(n)}{n^2} \Omega_\nu(y, n) \\ &=: T_1(f, y) + T_2(f, y). \end{aligned} \tag{4.7}$$

We first evaluate $T_2(f, y)$. We have

$$\sum_{n=\ell}^N \frac{\tilde{\lambda}_f(n)}{n^2} \Omega_\nu(y, n) = \sum_{n=\ell}^{N-1} M(\ell, n) (\Omega_\nu(y, n) - \Omega_\nu(y, n + 1)) - M(\ell, N) \Omega_\nu(y, N).$$

Note that the last term in the above equation vanishes as $N \rightarrow \infty$, therefore, we obtain

$$T_2(f, y) = \sum_{n=\ell}^{\infty} M(\ell, n) (\Omega_\nu(y, n) - \Omega_\nu(y, n + 1)). \tag{4.8}$$

The mean value theorem implies that there exists $\lambda_n \in (n, n + 1)$ such that

$$\Omega_\nu(y, n) - \Omega_\nu(y, n + 1) = -\frac{d}{dx} \Omega_\nu(y, x) \Big|_{x=\lambda_n}. \tag{4.9}$$

We next find the estimate for the expression on the right-hand side of the above equation. Differentiating both sides of (4.2) with respect to x , we arrive at

$$\frac{d}{dx} \Omega_\nu(y, x) = -\frac{1}{4\pi^2 iy} \int_{(c)} s(s+1) \Gamma\left(\frac{s+i\nu}{2}\right) \Gamma\left(\frac{s-i\nu}{2}\right) \left(\frac{x}{\pi y}\right)^s ds \ll y^{-c-1} x^c. \tag{4.10}$$

The combination of (4.6), (4.8), (4.9) and (4.10) together with $\ell = \lfloor y^{1-\epsilon} \rfloor$, yields

$$T_2(f, y) \ll \ell^{-\frac{3}{2}+\epsilon} y^{-c-1} \sum_{n=\ell}^{\infty} n^c \ll y^{-\frac{3}{2}+\epsilon}, \tag{4.11}$$

where in the last step we used the fact $\sum_{n>x} n^{-s} = \mathcal{O}(x^{1-s})$ (cf. [3, p. 55, theorem 3.2(c)]).

We now estimate $T_1(f, y)$. Invoking the estimate of $K_z(x)$ from [15, p. 255, Formula 10.40.2], we arrive at

$$\begin{aligned} T_1(f, y) &= -\sum_{n=1}^{\ell-1} \frac{\tilde{\lambda}_f(n)}{n^2} \left\{ \left(\frac{\pi y}{n}\right)^{i\nu-1} \Gamma(1-i\nu) + \left(\frac{\pi y}{n}\right)^{-i\nu-1} \Gamma(1+i\nu) \right\} \\ &\quad + \mathcal{O}\left(\frac{e^{-\frac{2\pi y}{\ell}}}{\sqrt{y}} \sum_{n=1}^{\ell-1} \frac{\tilde{\lambda}_f(n)}{n^{3/2}}\right). \end{aligned}$$

Therefore, the bound of $\tilde{\lambda}_f(n)$, followed from (1.1) and the value $\ell = \lfloor y^{1-\delta} \rfloor$, yields

$$\begin{aligned} T_1(f, y) &= -\sum_{n=1}^{\ell-1} \frac{\tilde{\lambda}_f(n)}{n^2} \left\{ \left(\frac{\pi y}{n}\right)^{i\nu-1} \Gamma(1-i\nu) + \left(\frac{\pi y}{n}\right)^{-i\nu-1} \Gamma(1+i\nu) \right\} \\ &\quad + \mathcal{O}\left(y^{-\frac{57}{64}+\delta} e^{-2\pi y^\delta}\right), \end{aligned} \tag{4.12}$$

for any $\delta > 0$. Thus combining (4.7), (4.11) and (4.12), we conclude (1.16). This completes the proof of our theorem.

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