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STRONGLY *E*-REFLEXIVE INVERSE SEMIGROUPS

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Let S be an inverse semigroup with semilattice of idempotents E. We denote by σ the minimum group congruence on S (**6**), and by τ the maximum idempotent-determined congruence on S (**2**). (Recall that the congruence η on S is called *idempotent-determined* if $(e, x) \in \eta$ and $e \in E$ imply that $x \in E$.) In general $\tau \subseteq \sigma$.

The semigroup S is said to be *E-unitary* if $E\sigma = E$, or alternatively if, given e in E and x in S, $ex \in E$ implies that $x \in E$; such semigroups were originally called *proper* (**15**). The semigroup S is said to be *E-reflexive* if, given x and y in S, $xy \in E$ implies that $yx \in E$. More restrictively, S is said to be *strongly E-reflexive* if, given $e \in E^1$ and x and y in S, $exy \in E$ implies that $eyx \in E$, where the element 1 is the identity of S^1.

Thus an E-unitary inverse semigroup is strongly E-reflexive, and a strongly E-reflexive inverse semigroup is E-reflexive. As shown below, neither of these statements can be reversed.

In (**4**), McAlister described the structure of an E-unitary inverse semigroup S in terms of the group S/σ acting on a certain partially ordered set. Now an inverse semigroup S is E-unitary if and only if $\tau = \sigma$ on S, and in (**11**) McAlister's construction was generalised to the case where a given idempotent-determined congruence η was specified on an arbitrary inverse semigroup S. There, S was shown to be embedded in an inverse semigroup $L = L(\eta)$ in a prescribed way, L being defined in terms of $T = S/\eta$ acting on a certain partially ordered set. We recall that if S is E-reflexive, then τ can be described concretely (**12**).

In Section 2 of this paper, the theory of (**11, 12**) is supplemented so as to describe how S may be recovered from a suitable inverse semigroup L, and the full theory is then examined in detail whenever T is a semilattice of groups. This restriction is shown to be equivalent to the supposition that S is a strongly E-reflexive inverse semigroup. In the penultimate section we sketch how the theory specialises in other cases, while the last section is devoted to illustrative examples.

Thus, a structure theorem for strongly E-reflexive inverse semigroups is obtained which generalises McAlister's structure theorem for E-unitary inverse semigroups. In fact, we show that an inverse semigroup is strongly E-reflexive if and only if it is a semilattice of E-unitary inverse semigroups. We also use the theory of (**11, 12**) to show, in the terminology of (**13**), that a strongly E-reflexive inverse semigroup is embedded in a strong semilattice of inverse semigroups, each of which is a semidirect product of a semilattice and a group.

In the main we adhere to the notation of (**1**), and we assume familiarity with the basic theory of inverse semigroups contained therein; the symbol '$\subset$' means 'properly contained in'.

I should like to record my thanks to the referee for his extremely helpful suggestions. In particular, Lemma 2 and Theorem 8 are due to him. The former helped to simplify the original exposition of Section 2, while the latter generalised the original result, which was the analogue for strongly E-reflexive semigroups, and enabled the exposition of these results to be clarified.

1.

Proposition 1. *An inverse semigroup S is strongly E-reflexive if and only if S/τ is a semilattice of groups.*

Proof. Suppose that S is strongly E-reflexive. Then S is E-reflexive, so that by (**12**, Theorem 1)

$$\tau = \{(a, b) \in S \times S | E \cdot a = E \cdot b\} \tag{1}$$

where, for example, $E \cdot a = \{x \in S | ax \in E\}$, E being the semilattice of idempotents of S. It is easily seen that $(ex, xe) \in \tau$, for each $x \in S$ and $e \in E$. Hence S/τ is a semilattice of groups.

Conversely, suppose that S/τ is a semilattice of groups. Then S/τ is E-reflexive, and since τ is an idempotent-determined congruence, it follows that S itself is E-reflexive. Reversing the argument of the first part of the proof shows that S is strongly E-reflexive.

Corollary 1. *An inverse semigroup S is strongly E-reflexive if and only if S/η is a semilattice of groups for some idempotent-determined congruence η on S.*

Proof. Suppose that η is an idempotent-determined semilattice of groups congruence on S. Then $\eta \subseteq \tau$, so that S/τ is also a semilattice of groups, being a homomorphic image of S/η.

The result now follows almost immediately from Proposition 1.

Corollary 2. *An inverse semigroup S is strongly E-reflexive if and only if the minimum semilattice of groups congruence ν is idempotent-determined.*

Proof. First we recall that ν is the congruence generated by the relation $\{(ex, xe) | e, x \in S, \ e = e^2\}$. Suppose η is an idempotent-determined congruence such that S/η is a semilattice of groups. Then ν is also idempotent-determined since $\nu \subseteq \eta$.

The result now follows from Corollary 1.

In analogy with (**10**, Theorem 2) we have the following result. As above, the minimum semilattice of groups congruence is denoted by ν.

Proposition 2. *Let S be an inverse semigroup. Then the congruence generated by $\nu \cap R$ is the minimum congruence κ such that S/κ is strongly E-reflexive.*

Proof. Let ξ be a congruence on S such that S/ξ is strongly E-reflexive, and let $(a, b) \in \nu \cap R$. Then $aa^{-1} = bb^{-1}$, so that $a\xi(a\xi)^{-1} = b\xi(b\xi)^{-1}$. Let $\bar{\nu}$ denote the minimum semilattice of groups congruence on S/ξ. Then ν induces a surjective homomorphism $\phi: S/\nu \to (S/\xi)/\bar{\nu}$ such that $\xi^{\natural}\bar{\nu}^{\natural} = \nu^{\natural}\phi$. Since $(a, b) \in \nu$, it follows that $(a\xi, b\xi) \in \bar{\nu}$. By Corollary 2 to Proposition 1, $\bar{\nu}$ is an idempotent-determined congruence and hence, by (**11**, Proposition 2.1), $a\xi = b\xi$. Thus $\nu \cap R \subseteq \xi$.

On the other hand, let χ denote the congruence generated by $\nu \cap R$. Then $\chi \subseteq \nu$. Thus if we denote by ν' the minimum semilattice of groups congruence on S/χ, it follows that $(a\chi, b\chi) \in \nu'$ if and only if $(a, b) \in \nu$. We wish to prove that ν' is idempotent-determined (see Corollary 2 to Proposition 1).

Consider $a \in S$ such that $(a\chi, (aa^{-1})\chi) \in \nu'$. Then $(a, aa^{-1}) \in \nu$. Now $(a, aa^{-1}) \in R$ and $\nu \cap R \subseteq \chi$. Hence $(a, aa^{-1}) \in \chi$, so that $a\chi = (aa^{-1})\chi$, and the result follows.

A semilattice of groups is strongly E-reflexive, as is an E-unitary inverse semigroup. Clearly, however, a semilattice of groups need not be E-unitary. Further, consider the bisimple inverse ω-semigroup $S(G, \alpha)$, where the endomorphism α of G is *not* injective. Then $S(G, \alpha)$ is not E-unitary (**8**), so that $\tau \subset \sigma$ on $S(G, \alpha)$. However, $S(G, \alpha)$ is E-reflexive; this follows either by direct computation or else from the result due to Schein (private communication) that an inverse semigroup is E-reflexive whenever its semilattice of idempotents forms a chain. Suppose that $S(G, \alpha)$ were strongly E-reflexive. By Proposition 1, $S(G, \alpha)/\tau$ is a bisimple semilattice of groups, that is to say, a group. Hence $\sigma \subseteq \tau$, a contradiction.

The class of strongly E-reflexive inverse semigroups is closed under the taking of inverse subsemigroups and direct products.

Now an inverse semigroup T is *τ-reduced*, that is, of the form S/τ for some inverse semigroup S, if and only if the identity congruence is the maximum idempotent-determined congruence on T. Since any congruence on T is specified by its idempotent classes, which in the case of an idempotent-determined congruence are subsemilattices of the semilattice of idempotents, one sees from (1) above that the semilattices of groups T

arising in Proposition 1 are those for which

$$e < f \text{ implies that } E.\cdot e \neq E.\cdot f; \quad e, f \in E \tag{2}$$

where E is the semilattice of idempotents of T. It is easily seen that (2) is equivalent to the condition

$$e < f \text{ implies that } E.\cdot f \subset E.\cdot e; \quad e, f \in E. \tag{3}$$

Proposition 3. *Let T be a semilattice of groups with semilattice of idempotents E and structural homomorphisms $\{\phi_{f,e} | e, f \in E, f \geqslant e\}$. Then T satisfies* (3) *if and only if, given $f > e$ in E, there exists $g \in E$ with $g \leqslant f$, $g \not\leqslant e$ such that* Ker $\phi_{g,eg}$ *is a non-trivial subgroup of T.*

Proof. This is easily seen.

The final result of this section examines the gross structure of a strongly E-reflexive inverse semigroup S. The fine structure will be given in Section 3.

First we need a result due to Hardy and Tirasupa; for the sake of completeness we include a short proof.

Lemma 1. **(3)** *Let S be an inverse semigroup which is a semilattice W of inverse semigroups S_α, $\alpha \in W$, and let σ_α denote the minimum group congruence on S_α, $\alpha \in W$. Then $\cup \sigma_\alpha$ is a semilattice of groups congruence on S.*

Proof. The result follows easily from the characterisation of σ_α due to Vagner (**16**), see (**5**, Theorem 1), namely that $\sigma_\alpha = \{(x, y) \in S_\alpha \times S_\alpha | z \leqslant x, y$ for some $z \in S_\alpha\}$.

As is natural, a homomorphism on an inverse semigroup will be called an idempotent-determined homomorphism whenever the associated congruence is idempotent-determined.

Theorem 1. *An inverse semigroup S is strongly E-reflexive if and only if S is a semilattice of E-unitary inverse semigroups.*

Proof. Suppose that S is strongly E-reflexive. By Corollary 1 to Proposition 1, there exists an idempotent-determined semilattice of groups congruence η on S. Let θ denote the canonical homomorphism onto S/η, where S/η is the semilattice W of the groups G_α, $\alpha \in W$, say. Let ϕ denote the canonical homomorphism from S/η onto W, and let $\chi = \theta\phi$. Then χ is a homomorphism from S onto the semilattice W; let $S_\alpha = \alpha\chi^{-1}$ for each $\alpha \in W$.

Now $S_\alpha = G_\alpha\theta^{-1}$. Let $e, x \in S_\alpha$ where $e = e^2$ and $ex = (ex)^2$. Since $e\theta$ is the identity element of G_α, $x\theta = e\theta \cdot x\theta = (ex)\theta$. Since θ is an idempotent-determined homomorphism, it follows that $x = x^2$. Hence S_α is E-unitary, and the result follows.

Conversely, let S be a semilattice W of E-unitary inverse semigroups S_α, $\alpha \in W$. It follows from Lemma 1 that $\cup\sigma_\alpha$ is an idempotent-determined semilattice of groups congruence on S. By Corollary 1 to Proposition 1, therefore, S is strongly E-reflexive.

Remarks. (i) Following the first part of the proof of Theorem 1, we have shown that $\eta_\alpha \equiv \eta \cap (S_\alpha \times S_\alpha)$ is a group congruence which is also idempotent-determined. Hence η_α is the minimum group congruence on S_α $(\alpha \in W)$.

(ii) Suppose that a semilattice W indexes a set of mutually disjoint E-unitary inverse semigroups S_α, $\alpha \in W$, and suppose that we have connecting homomorphisms $\phi_{\alpha\beta}$: $S_\alpha \to S_\beta$ $(\alpha \geqslant \beta)$ where each $\phi_{\alpha\alpha}$ is the identity map and $\phi_{\alpha\gamma} = \phi_{\alpha\beta}\phi_{\beta\gamma}$ whenever $\alpha \geqslant \beta \geqslant \gamma$. Then under the obvious multiplication $S \equiv \cup\{S_\alpha | \alpha \in W\}$ becomes a strongly E-reflexive inverse semigroup. Such connecting homomorphisms can always be defined. In this case, following the notation of (**13**), we shall call S a *strong* semilattice of E-unitary inverse semigroups.

As will be seen in Sections 3 and 5, not every strongly E-reflexive inverse semigroup is of this form, although any strongly E-reflexive inverse semigroup can be embedded in a strongly E-reflexive inverse semigroup which is a strong semilattice of inverse semigroups, each of which is a semidirect product of a semilattice and a group.

2.

The aim of this section is to summarise and to supplement the theory of (**11**, **12**).

Let X be a down-directed partially ordered set, and let Y be a subsemilattice and order-ideal of X. Let T be an inverse semigroup acting on X, *on the left*, via a given homomorphism $\phi: T \to I_X$, where each $\alpha \in \operatorname{Im}\phi$ has the property that its domain $\Delta\alpha$ and its range $\nabla\alpha$ are order-ideals of X and α is an order-isomorphism. Suppose further that $X = TY$ and that $\Delta t \neq \square$ for each $t \in T$ (as usual, the map $t\phi$ is denoted by t). Define $L = L(T, X, Y)$ to be

$$\{(a, t) | t \in T,\ a \in Y \cap \Delta t^{-1},\ t^{-1}a \in Y\}$$

under the multiplication

$$(a, t)(b, s) = (t(t^{-1}a \wedge b), ts).$$

Then L is an inverse semigroup with semilattice of idempotents $E = \{(a, t) \in L | t = t^2\}$; given (a, e) and (b, f) in E, $(a, e)(b, f) = (a \wedge b, ef)$ and $(a, t) \in L$ has inverse $(t^{-1}a, t^{-1})$. Moreover, $(a, t)(a, t)^{-1} = (a, tt^{-1})$. Let π_1: $L \to Y$ and π_2: $L \to T$ denote the projection maps.

It is easily seen that $\pi_1|E$ is a homomorphism which maps onto Y. Moreover, π_2 is an idempotent-determined homomorphism which is also surjective. This follows because, for each $t \in T$, the hypotheses about X, Y and Δt^{-1} imply that $t^{-1}(Y \cap \Delta t^{-1}) \cap Y \neq \square$. In short, we shall refer to (T, X, Y) as an *L-triple.*

The L-triple (T, X, Y) is called a *strict L-triple* if T has the further property that

for each $a \in Y$, there is a least idempotent $e(a)$ in T such that $a \in \Delta e(a)$ (that is, such that $(a, e(a)) \in E$) and moreover, for a, b in Y, $e(a \wedge b) = e(a) \cdot e(b)$. (4)

Given a strict L-triple (T, X, Y), define $L_m = L_m(T, X, Y)$ to be $\{(a, t) \in L | tt^{-1} = e(a)\}$, and let $E_m = \{(a, e(a)) | a \in Y\}$. By (4), E_m is a non-empty subsemilattice of E and $\pi_1|E_m$ maps onto Y.

Lemma 2. *Let (T, X, Y) be a strict L-triple and let $t \in T$. If $a \in Y \cap \Delta t$ and $ta \in Y$, then $e(ta) = te(a)t^{-1}$.*

Proof. Clearly $ta \in \Delta te(a)t^{-1}$, so that $e(ta) \leq te(a)t^{-1}$. Replacing a by ta and t by t^{-1}, we deduce that $e(a) = e(t^{-1}ta) \leq t^{-1}e(ta)t$. Hence $e(ta) \leq te(a)t^{-1} \leq tt^{-1}e(ta)tt^{-1} \leq e(ta)$, and the result follows.

Theorem 2. *L_m is an inverse subsemigroup of L with E_m as its semilattice of idempotents.*

Proof. Let (a, t), $(b, s) \in L_m$. Then $(a, t)(b, s) = (t(t^{-1}a \wedge b), ts)$ where, by Lemma 2, $e(t(t^{-1}a \wedge b)) = te(t^{-1}a \wedge b)t^{-1} = te(t^{-1}a)e(b)t^{-1} = tt^{-1}e(a)te(b)t^{-1} = tt^{-1}tt^{-1}tss^{-1}t^{-1} = ts(ts)^{-1}$. Hence L_m is closed under multiplication.

Similarly, $e(t^{-1}a) = t^{-1}e(a)t = t^{-1}tt^{-1}t = t^{-1}t$, so that $(a, t)^{-1} = (t^{-1}a, t^{-1}) \in L_m$, and the result follows.

The next theorem brings out the connection between L_m and the theory of (**11**, **12**).

Theorem 3. *$L_m = \{(a, t) \in L | (a, s) \in L$ and $s \leq t$ imply $s = t\}$.*

Proof. Let $(a, t) \in L_m$ and suppose we have $(a, s) \in L$ with $s \leq t$. Then $(a, ss^{-1}) = (a, s)(a, s)^{-1} \in L$ with $ss^{-1} \leq tt^{-1} = e(a)$. By definition of $e(a)$ we deduce that $ss^{-1} = tt^{-1}$. Since $s \leq t$, $s = ss^{-1}t = tt^{-1}t = t$.

Conversely, suppose that $(a, t) \in L$ has the property that if $(a, s) \in L$ and $s \leq t$ then $s = t$, and suppose that $(a, e) \in E$. Then $(a, et) = (a, e)(a, t) \in L$ where $et \leq t$. Hence $et = t$, so that $tt^{-1} \leq e$. Since $(a, tt^{-1}) \in E$, it follows that $tt^{-1} = e(a)$. Hence $(a, t) \in L_m$.

Remark. Simple examples (with T a chain) show that L-triples exist in which $E_m = \square$ or in which E_m is non-empty but not a subsemilattice of E.

Even where we have a strict L-triple (T, X, Y), another simple example with T a two element chain shows that $\pi_2(L_m)$ need not equal T.

In view of the preceding remark, a strict L-triple is said to be *fully strict* whenever $\pi_2(L_m) = T$.

We point out that if (T, X, Y) is a strict L-triple and if $T_m = \pi_2(L_m)$, $X_m = T_m Y$, then (T_m, X_m, Y) is a fully strict L-triple and $L_m(T, X, Y) = L_m(T_m, X_m, Y)$.

As is essentially shown in **(11)**, the above theory can be turned around. The details are summarised in the next result.

Theorem 4. *Let S be an inverse semigroup with semilattice of idempotents E, and let η be an idempotent-determined congruence on S. Then there exists a fully strict L-triple (T, X, Y), where $T = S/\eta$ and Y is isomorphic to E, and an isomorphism from S onto L_m; π_2 induces the congruence η on S.*

If T is a group, then an L-triple is a fully strict L-triple, (4) being satisfied automatically; in this case (T, X, Y) is called a *P-triple*. The above theory then specialises as follows:

Let X be a down-directed partially ordered set, and let Y be a subsemilattice and order-ideal of X. Let C be a group, with identity element 1, acting on X on the left by order-automorphisms. Suppose further that $X = GY$. Define $P = P(G, X, Y)$ to be $\{(a, g) \in Y \times G \mid g^{-1}a \in Y\}$ under the multiplication $(a, g)(b, h) = (a \wedge gb, gh)$. Then P is an E-unitary inverse semigroup with semilattice of idempotents $E = \{(a, 1) \mid a \in Y\}$ and maximal group homomorphic image G, π_2: $P \to G$ being the canonical homomorphism. (Note that $P_m = P$, this property being in fact characteristic of the E-unitary case.) Conversely, given an E-unitary inverse semigroup S, there exists a P-triple (G, X, Y), with $G = S/\sigma$ and Y isomorphic to E, and an isomorphism from S onto $P(G, X, Y)$.

This theory is due to McAlister **(4, 9)**.

3.

We now apply the theory of the first two sections to obtain the fine structure of strongly E-reflexive inverse semigroups.

All order-ideals of partially ordered sets are tacitly assumed to be non-empty.

Theorem 5. (i) *Let (T, X, Y) be a strict L-triple where T is a semilattice of groups. Then L_m is a strongly E-reflexive inverse semigroup.*

(ii) *Conversely, let S be a strongly E-reflexive inverse semigroup. Then there exists a fully strict L-triple (T, X, Y) where T is a semilattice of groups and S is isomorphic to L_m.*

Proof. (i) As noted in Section 2, $\pi_2|L_m$ is an idempotent-determined homomorphism into the semilattice of groups T. Its image is therefore a semilattice of groups, and the result follows from Corollary 1 to Proposition 1.

(ii) This follows from Proposition 1 and Theorem 4.

Remarks. Let (T, X, Y) be an L-triple where T is a semilattice of groups. An easy analogue of the proof of part (i) of Theorem 5 shows that $L(T, X, Y)$ is a strongly E-reflexive inverse semigroup. In particular, if T is a semilattice then $L(T, X, Y)$ is just a semilattice.

If S is a semilattice of groups with semilattice of idempotents E, then (S, E, E) is a fully strict L-triple with each $s \in S$ acting as the identity map on $\Delta s = \{e \in E | e \leq s^{-1}s\}$; see (**11**, page 21).

Let (T, X, Y) be an L-triple where T is the semilattice W of groups $G_\alpha, \alpha \in W$. For each $\alpha \in W$, we denote by ϵ_α the identity element of G_α. Let $J = \{\Delta\epsilon_\alpha | \alpha \in W\}$; then J is a semilattice under intersection and the map ψ: $W \to J$ defined by the rule ψ: $\alpha \to \Delta\epsilon_\alpha$ is a surjective homomorphism. By a slight abuse of notation, we define, for each $\alpha \in W$, $Y_\alpha = Y \cap \Delta\epsilon_\alpha$ and $X_\alpha = G_\alpha Y_\alpha$; note that each $g \in G_\alpha$ is an order-automorphism of $\Delta\epsilon_\alpha$.

We remark that if (T, X, Y) is fully strict then ψ is injective. To see this, suppose there exist $\alpha, \beta \in W$ with $\alpha\psi = \beta\psi$. Since $\pi_2(L_m) = T$, there exist $a, b \in Y$ such that $\epsilon_\alpha = e(a)$ and $\epsilon_\beta = e(b)$. Then $a \in \alpha\psi = (\alpha\beta)\psi$, so that $\epsilon_\alpha \leq \epsilon_{\alpha\beta}$, that is to say, $\alpha \leq \beta$. Similarly $\beta \leq \alpha$, and we deduce that $\alpha = \beta$. By Example 5.1 the converse does not hold.

Returning to our original situation where (T, X, Y) is not necessarily fully strict, we now exhibit $L(T, X, Y)$ as a semilattice of E-unitary inverse semigroups.

Lemma 3. *For each $\alpha \in W$, $(G_\alpha, X_\alpha, Y_\alpha)$ is a P-triple.*

Proof. Let $\alpha \in W$. It is clear that G_α acts on X_α by order-automorphisms, and that Y_α is a subsemilattice and order-ideal of X_α. Let $a, b \in Y_\alpha$ and let $g, h \in G_\alpha$. Then there exists $z \in X$ such that $z \leq ga$ and $z \leq hb$. Then $z \in \Delta\epsilon_\alpha$ and $g^{-1}z \leq a$. Hence $g^{-1}z \in Y$, and it follows that $g^{-1}z \in Y_\alpha$. Thus $z \in X_\alpha$, and it follows that X_α is down-directed, whence the result.

Theorem 6. *$L(T, X, Y)$ is the semilattice W of the E-unitary inverse semigroups $P(G_\alpha, X_\alpha, Y_\alpha)$, $\alpha \in W$.*

Proof. By the proof of Theorem 1, $L = L(T, X, Y)$ is the semilattice W of the E-unitary inverse semigroups $\{(a, t) \in L | t \in G_\alpha\}$, $\alpha \in W$—one uses the second projection map π_2 as the candidate for θ there. Let $\alpha \in W$. Then $(a, t) \in L$ with $t \in G_\alpha$ if and only if $a \in Y_\alpha$ and $t^{-1}a \in Y_\alpha$. Since t acts on $\Delta\epsilon_\alpha$ as an order-automorphism, the result is almost immediate from Lemma 3.

Corollary. *If $X = Y$ is a semilattice, then $L(T, X, X)$ is the semilattice W of the inverse semigroups $P(G_\alpha, \Delta\epsilon_\alpha, \Delta\epsilon_\alpha)$, $\alpha \in W$, each of which is a semidirect product of the semilattice $\Delta\epsilon_\alpha$ by the group G_α, $\alpha \in W$.*

Proof. For each $\alpha \in W$, $Y_\alpha = Y \cap \Delta\epsilon_\alpha = X \cap \Delta\epsilon_\alpha = \Delta\epsilon_\alpha$. The result follows from Theorem 6.

Suppose further that (T, X, Y) is a strict L-triple. We wish to exhibit L_m as a semilattice of E-unitary inverse semigroups. To smooth the argument, we in fact assume that (T, X, Y) is a fully strict L-triple. By the remarks preceding Theorem 4, this involves no real loss in generality.

So let (T, X, Y) be a fully strict L-triple, with T the semilattice W of groups G_α, $\alpha \in W$.

For each $\alpha \in W$, let $V_\alpha = \{a \in Y \mid e(a) = \epsilon_\alpha\}$ and let $U_\alpha = G_\alpha V_\alpha$. Each V_α is non-empty; note that $Y_\alpha = \cup\{V_\beta \mid \beta \leq \alpha\}$.

Lemma 4. *For each $\alpha \in W$, $(G_\alpha, U_\alpha, V_\alpha)$ is a P-triple.*

Proof. Let $\alpha \in W$, and let $a, b \in V_\alpha$. Then $e(a \wedge b) = e(a) \cdot e(b) = \epsilon_\alpha$, and it follows that V_α is a subsemilattice of Y.

Let $g \in G_\alpha$, and suppose that $ga \leqslant b$. Then $ga \in Y_\alpha$. By Lemma 2, $e(ga) = ge(a)g^{-1} = g\epsilon_\alpha g^{-1} = \epsilon_\alpha$, so that $ga \in V_\alpha$. Hence V_α is an order-ideal of U_α.

Let $c, d \in U_\alpha$. Then $c = gx$ and $d = hy$ for some $g, h \in G_\alpha$ and $x, y \in V_\alpha$. Since (T, X, Y) is a fully strict L-triple, there exists $u \in Y$ such that $(u, g) \in L_m$. Hence, by Theorem 2, $(g(g^{-1}u \wedge x), g) = (u, g)(x, \epsilon_\alpha) \in L_m$, so that $u' \equiv g(g^{-1}u \wedge x) \in V_\alpha$. Now $u' \leqslant gx$. Similarly, there exists $v' \in V_\alpha$ such that $v' \leqslant hy$. Then $u' \wedge v' \in V_\alpha$ and is a lower bound for the set $\{c, d\}$. Hence U_α is down-directed, and the result follows.

Theorem 7. *L_m is the semilattice W of the E-unitary inverse semigroups $P(G_\alpha, U_\alpha, V_\alpha)$, $\alpha \in W$.*

Proof. The proof is analogous to that of Theorem 6 and follows almost immediately from Lemma 4. One notes in particular that if $a \in Y$ with $e(a) = \epsilon_\alpha$, and if $g \in G_\alpha$ with $g^{-1}a \in Y$, then as seen in the proof of Lemma 4, $e(g^{-1}a) = \epsilon_\alpha$.

Remark. Let S be a strongly E-reflexive inverse semigroup, and let η be an idempotent-determined semilattice of groups congruence on S. Then S can be expressed as a semilattice W of E-unitary inverse semigroups S_α, $\alpha \in W$, as in Theorem 1. Moreover, by Theorem 4, there exists a fully strict L-triple (T, X, Y) with $T = S/\eta$ such that S is isomorphic to L_m. Since the second projection map π_2 induces the congruence η on S, it is clear from Theorem 7 that L_m is the semilattice W of E-unitary inverse semigroups $P(G_\alpha, U_\alpha, V_\alpha)$, $\alpha \in W$, and that the isomorphism between S and L_m induces an isomorphism between each S_α and $P(G_\alpha, U_\alpha, V_\alpha)$.

We now broach the question: when are the semigroups $L(T, X, Y)$ and L_m, of Theorems 6 and 7 respectively, strong semilattices of E-unitary inverse semigroups?

First of all, following (14), we say that an order-ideal I of a partially ordered set X is a *p-ideal* if, for each $a \in X$, the set $\{x \in I | x \leqslant a\}$ is a principal order-ideal of X; that is, if the set $\{x \in I | x \leqslant a)$ has a greatest element.

The next theorem, which was pointed out to me by the referee, shows that our question admits a wholly general treatment.

Theorem 8. *Let S be an inverse semigroup which is the semilattice W of inverse semigroups S_α, $\alpha \in W$, each with semilattice of idempotents E_α. Let $E = \cup\{E_\alpha | \alpha \in W\}$ denote the semilattice of idempotents of S, and for each $\alpha \in W$, let $F_\alpha = \cup\{E_\beta | \beta \leqslant \alpha\}$; each F_α is an ideal of E. Then the following are equivalent*:

(i) *S is a strong semilattice of the S_α*;

(ii) *E is a strong semilattice of the E_α*;

(iii) *each F_α is a p-ideal of E*;

(iv) *if $\alpha \geqslant \beta$, $e \in F_\alpha$ then the set $\{f \in F_\beta | f \leqslant e\}$ has a maximum element*;

(v) *if $\alpha \geqslant \beta$, $e \in E_\alpha$ then the set $\{f \in E_\beta | f \leqslant e\}$ has a maximum element e_β, where if $\alpha \geqslant \beta \geqslant \gamma$, $(e_\beta)_\gamma = e_\gamma$.*

Proof. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) If $\alpha, \beta \in W$ with $\alpha \geqslant \beta$, let $\theta_{\alpha\beta}: E_\alpha \to E_\beta$ denote the linking homomorphism. Let $\gamma \in W$, and let $e \in E_\gamma$. Then $e\theta_{\gamma,\alpha\gamma} \in F_\alpha$ and $e\theta_{\gamma,\alpha\gamma} \leqslant e$. On the other hand, suppose $f \in F_\alpha$ and $f \leqslant e$. Then $f \in E_\beta$ where $\beta \leqslant \alpha$ and $\beta \leqslant \gamma$. Hence $\beta \leqslant \alpha\gamma$, and it follows almost immediately that $f \leqslant e\theta_{\gamma,\alpha\gamma}$. Thus $e\theta_{\gamma,\alpha\gamma}$ is the maximum element in the set $\{f \in F_\alpha | f \leqslant e\}$, and the result follows.

(iii) $\Rightarrow$ (iv) This is immediate.

(iv) $\Rightarrow$ (v) Let $\alpha \geqslant \beta$ and let $e \in E_\alpha$. Then $e \in F_\alpha$. Denote by e_β the maximum element of the set $\{f \in F_\beta | f \leqslant e\}$. Pick $g \in E_\beta$. Then $eg \in E_\beta \subseteq F_\beta$ and $eg \leqslant e$. Hence $eg \leqslant e_\beta$, and it follows that $e_\beta \in E_\beta$. Thus e_β is the maximum element in $\{f \in E_\beta | f \leqslant e\}$.

Let $\beta \geqslant \gamma$. By the above, $(e_\beta)_\gamma \in E_\gamma$ and $(e_\beta)_\gamma \leqslant e_\beta \leqslant e$, so that $(e_\beta)_\gamma \leqslant e_\gamma$. On the other hand $e_\gamma \in E_\gamma \subseteq F_\beta$ and $e_\gamma \leqslant e$, so that $e_\gamma \leqslant e_\beta$, and it follows that $e_\gamma \leqslant (e_\beta)_\gamma$. Hence $e_\gamma = (e_\beta)_\gamma$.

(v) $\Rightarrow$ (i) We claim that

$$\text{given } e \in E_\alpha,\ x \in S_\alpha, \text{ and } \beta \leqslant \alpha,\ xe_\beta x^{-1} = (xex^{-1})_\beta. \tag{5}$$

Certainly $xe_\beta x^{-1} \in E_\beta$ and $xe_\beta x^{-1} \leqslant xex^{-1}$. On the other hand, if $f \in E_\beta$ and $f \leqslant xex^{-1}$, then $f \leqslant xx^{-1}$ and $x^{-1}fx \leqslant e$. But $x^{-1}fx \in E_\beta$, so that $x^{-1}fx \leqslant e_\beta$. Hence $f = xx^{-1}fxx^{-1} \leqslant xe_\beta x^{-1}$, and (5) follows.

Given $\alpha \geq \beta$ and $x \in S_\alpha$, define the map $\phi_{\alpha\beta}: S_\alpha \to S_\beta$ by the rule $\phi_{\alpha\beta}: x \mapsto (xx^{-1})_\beta x$.

Let $y \in S_\alpha$. Then, using (5), we see that $(xy)\phi_{\alpha\beta} = x(yy^{-1})_\beta y$, while $x\phi_{\alpha\beta} \cdot y\phi_{\alpha\beta} = x(x^{-1}x)_\beta(yy^{-1})_\beta y$, so that $x\phi_{\alpha\beta} \cdot y\phi_{\alpha\beta} \leq (xy)\phi_{\alpha\beta}$. However, $x^{-1}x \cdot (yy^{-1})_\beta \leq (x^{-1}x)_\beta$, and from this it follows easily that $(xy)\phi_{\alpha\beta} \leq x\phi_{\alpha\beta} \cdot y\phi_{\alpha\beta}$. Hence $\phi_{\alpha\beta}$ is a homomorphism.

Clearly $\phi_{\alpha\alpha}$ is the identity map, and it is easily seen that if $\beta \geq \gamma$, then $\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}$.

Finally, let $\delta \in W$ and let $z \in S_\delta$. Then, setting $\nu = \alpha\delta$,

$$\begin{aligned} xz &= xx^{-1}xzz^{-1}z \\ &= x(x^{-1}x)_\nu(zz^{-1})_\nu z, \text{ since } x^{-1}xzz^{-1} \leq (x^{-1}x)_\nu(zz^{-1})_\nu \\ &= x(x^{-1}x)_\nu x^{-1}x(zz^{-1})_\nu z \\ &= (xx^{-1})_\nu x(zz^{-1})_\nu z, \text{ by (5)} \\ &= x\phi_{\alpha\nu} \cdot z\phi_{\delta\nu}. \end{aligned}$$

Thus (i) holds with the $\phi_{\alpha\beta}$ as linking homomorphisms.

We can use Theorem 1 to reformulate condition (iii) of Theorem 8 as follows:

Theorem 9. *Let S be an inverse semigroup with semilattice of idempotents E. Then S is a strong semilattice of E-unitary inverse semigroups if and only if there exists an idempotent-determined semilattice of groups congruence η on S such that $(Ee)\eta$ is a p-ideal in E, for all $e \in E$.*

Proof. This is immediate.

Remark. Examples of the kind given in Section 5 below show that a given strongly E-reflexive inverse semigroup may be expressed as a strong semilattice of E-unitary inverse semigroups, as in Theorem 1, by means of ν (the minimum semilattice of groups congruence) but not by means of τ, and vice versa. Thus there does not exist a test idempotent-determined congruence which would enable us to decide whether a strongly E-reflexive inverse semigroup, however presented, is or is not a strong semilattice of E-unitary inverse semigroups. However, it follows from Theorem 8 or 9 that if S is a strongly E-reflexive inverse semigroup whose semilattice of idempotents is inversely well-ordered, then S is a strong semilattice of E-unitary inverse semigroups.

Before applying Theorem 8 we prove a preliminary lemma.

Lemma 5. *Let (T, X, Y) be an L-triple, where T is the semilattice W of groups G_α, $\alpha \in W$. Then the following are equivalent:*

(i) *Each element of $\{Y_\alpha | \alpha \in W\}$ is a p-ideal of Y;*

(ii) *Each element of $J = \{\Delta\epsilon_\alpha | \alpha \in W\}$ is a p-ideal of X.*

Proof. (i) $\Rightarrow$ (ii). Let $a \in X$, $\alpha \in W$. Then there exists $\beta \in W$ such that $a = gb$ for some $g \in G_\beta$, $b \in Y_\beta$. By hypothesis there exists a maximum element $c \in Y_\alpha$ such that $c \leq b$. Then $gc \leq a$, and moreover, since $c \in \Delta\epsilon_{\alpha\beta}$, $gc = g\epsilon_{\alpha\beta}c \in \Delta\epsilon_{\alpha\beta} \subseteq \Delta\epsilon_\alpha$.

Conversely, consider $x \in \Delta\epsilon_\alpha$ such that $x \leq a$. Then $g^{-1}x \leq b$, so that $g^{-1}x \in Y$. Moreover, $x \in \Delta\epsilon_\beta$ so that $x \in \Delta\epsilon_{\alpha\beta}$. Hence $g^{-1}x = g^{-1}\epsilon_{\alpha\beta}x \in \Delta\epsilon_{\alpha\beta} \subseteq \Delta\epsilon_\alpha$. Thus $g^{-1}x \leq c$, so that $x \leq gc$. The result follows.

(ii) $\Rightarrow$ (i). This is trivial.

The next result is now almost immediate from Theorem 8 and Lemma 5.

Theorem 10. (A) *Let (T, X, Y) be an L-triple where T is the semilattice W of groups G_α, $\alpha \in W$. Then the following are equivalent:*

(i) *$L(T, X, Y)$ is a strong semilattice of the E-unitary inverse semigroups $P(G_\alpha, X_\alpha, Y_\alpha)$, $\alpha \in W$;*

(ii) *Given α and β in W with $\alpha \geq \beta$ and $a \in Y_\alpha$, there is a largest element $b \in Y_\beta$ such that $b \leq a$;*

(iii) *Each element of $J = \{\Delta\epsilon_\alpha \mid \alpha \in W\}$ is a p-ideal of X.*

(B) *Suppose further that (T, X, Y) is a fully strict L-triple. Then the following are equivalent to each other and to any one of* (i), (ii), (iii):

(iv) *L_m is a strong semilattice of the E-unitary inverse semigroups $P(G_\alpha, U_\alpha, V_\alpha)$, $\alpha \in W$;*

(v) *Given α, β and γ in W with $\alpha \geq \beta \geq \gamma$ and $a \in V_\alpha$, there is a largest element $a_\beta \in V_\beta$ such that $a_\beta \leq a$, where $(a_\beta)_\gamma = a_\gamma$.*

Proof. In view of the remarks before Lemma 4 and after Theorem 7, under the hypotheses in (B), the equivalence of (ii), (iii), (iv) and (v) follows straightaway from Theorem 8 and Lemma 5.

In the terminology of Theorem 8, in (A), $S = L(T, X, Y)$, $S_\alpha = P(G_\alpha, X_\alpha, Y_\alpha)$, $E_\alpha = Y_\alpha \times \{\epsilon_\alpha\}$, and $F_\alpha = \cup\{Y_\beta \times \{\epsilon_\beta\} \mid \beta \leq \alpha\}$. Thus the application of Theorem 8 to this situation is not entirely straightforward. However, an easy argument establishes that (ii) above is the analogue of condition (iv) (and of condition (v)!) of Theorem 8, and that, because of Lemma 5, (iii) above is the analogue of condition (iii) of Theorem 8.

Following the hypotheses of Theorem 10 (A), let $\bar{X}$ be the set of order-ideals A of X such that $A \subseteq gY_\alpha$ for some $g \in G_\alpha$, $\alpha \in W$, the order on $\bar{X}$ being that of inclusion. For each $\alpha \in W$, let $\bar{\Delta}\epsilon_\alpha = \{A \in \bar{X} \mid A \subseteq \Delta\epsilon_\alpha\}$. As shown in (**11**), $(T, \bar{X}, \bar{X})$ is an L-triple, $g \in G_\alpha$ having domain $\bar{\Delta}\epsilon_\alpha$ for each $\alpha \in W$. Note that each $\bar{\Delta}\epsilon_\alpha$ is a p-ideal of $\bar{X}$; if $B \in \bar{X}$, $B \cap \Delta\epsilon_\alpha$ is the largest element C of $\bar{\Delta}\epsilon_\alpha$ such that $C \subseteq B$. Combining these remarks with the Corollary to Theorem 6 and with Theorem 10 (A) we have one half of the following result, the other half of which is immediate.

Theorem 11. *An inverse semigroup is strongly E-reflexive if and only if* *it can be embedded in a strong semilattice of inverse semigroups, each of which is a semidirect product of a semilattice and a group.*

Recall that homomorphisms between two E-unitary inverse semigroups presented in the form $P(G, X, Y)$ have been described explicitly (**4**).

4

We remark here that the theory of Section 2 can be brought full circle in the following sense.

Let (T, X, Y) be a strict L-triple, where T is now an arbitrary inverse semigroup, and as before, let $T_m = \pi_2(L_m)$, $X_m = T_m Y$. Now the congruence $\pi_2 \circ \pi_2^{-1}|L_m \times L_m$ is an idempotent-determined congruence on L_m, so that L_m is embedded in a semigroup $L^{(1)}$ as in Theorem 4. Further, as mentioned prior to Theorem 4, (T_m, X_m, Y) is a fully strict L-triple, so that one can form $L^{(2)} = L(T_m, X_m, Y)$. Finally, let $L^{(3)} = \{(a, t) \in Y \times T_m | (a, t) \in L(T, X, Y)\}$. Then it can be shown that $L^{(1)} \approx L^{(2)} = L^{(3)}$.

The full L-theory can be viewed as a structure theorem machine. In this paper, we have fed into the machine the stipulation that T lies in the class C of inverse semigroups consisting of semilattices of groups. The main properties of C used in Propositions 1 and 2 were that C is a subclass of the class of E-reflexive inverse semigroups, that C is closed under homomorphic images, and that there is a minimum congruence ρ on an arbitrary inverse semigroup S such that $S/\rho \in C$. Using (1) in the proof of Proposition 1, the exact defining conditions were found on an inverse semigroup S so that $S/\tau \in C$; the first property of C mentioned above entails that S itself is E-reflexive. The class C' of groups also has these properties, and the results in McAlister's theory (**4**) can be viewed as arising from the stipulation that T lies in C'. Thus Propositions 1 and 2 can be extended in an obvious way for suitable subclasses of the class of E-reflexive inverse semigroups.

Another class of semigroups which recommends itself for consideration is the class of bisimple inverse ω-semigroups $S(G, \alpha)$. As mentioned before, such semigroups are E-reflexive. Recall (**8**) that if α is injective $S(G, \alpha)$ is E-unitary, while if α is not injective any non-group congruence separates idempotents, so that $S(G, \alpha)$ is then τ-reduced. Stipulating that $T = S(G, \alpha)$ in the former case yields E-unitary inverse semigroups for L_m, since the composition of idempotent-determined homomorphisms is again idempotent-determined. Something new is achieved in the latter case, namely a description of all inverse semigroups S for which $S/\tau = S(G, \alpha)$. In practice the two cases may as well be treated together, so that for a given bisimple inverse ω-semigroup B we find a description of all inverse semigroups S for which $S/\eta = B$, where η is an idempotent-determined congruence on S. The theory, which has features in common with that of (7), is left to the reader.

5.

This section is devoted to illustrative examples. The first is typical of the class of examples in which T is a chain.

Example 5.1. Let X be the real numbers R under the usual ordering, let $Y = X$, let J be the set of those order-ideals of R which are either of the form $\{x \in R | x \leqslant t\}$ or of the form $\{x \in R | x < t\}$ for some $t \in R$, and let T be the chain consisting of the identity maps on the elements of J. Then (T, X, Y) is a strict L-triple which is not fully strict; if $t \in T$, the identity map on $\{x \in R | x < t\}$ is not of the form $e(a)$ for any $a \in R$.

The next two examples are more complicated. The first shows that a strongly E-reflexive inverse semigroup need not be a strong semilattice of E-unitary inverse semigroups, while the second shows that the technicality in conditions (v) of Theorem 8 and (v) of Theorem 9 is indeed essential.

Example 5.2. Let $X = Y$ be the semilattice $\{a, b, c, d, e, f\}$ where $a = b \wedge c$, $b \vee c = d = e \wedge f$, b and c are incomparable, and e and f are also incomparable. Let $W = \{\alpha, \beta\}$ where $\alpha > \beta$, and let T be the semilattice W of the groups G_α = Klein 4-group and G_α = cyclic 2-group given by a surjective homomorphism from G_α to G_β. Set $J = \{\{a, b, c\}, X\}$. Then there is an obvious injective homomorphism $\phi: T \to I_X$ such that (T, X, X) is a fully strict L-triple, where each element of G_α is an order-automorphism of X, and each element of G_β is an order-automorphism of $\{a, b, c\}$. Moreover, $L_m = (\{a, b, c\} \times G_\beta) \cup (\{d, e, f\} \times G_\alpha)$.

Suppose that L_m is the semilattice Λ of E-unitary inverse semigroups $S_\lambda, \lambda \in \Lambda$, where $(e, x) \in S_\lambda$ say, x being any element of G_α such that $x^{-1}e = f$. Then $(f, x) = (e, x)^{-1} \in S_\lambda$, so that $(f, \epsilon_\alpha) = (f, x)(f, x)^{-1} \in S_\lambda$ and $(e, \epsilon_\alpha) = (e, x)(e, x)^{-1} \in S_\lambda$. Hence $(d, \epsilon_\alpha) = (e, \epsilon_\alpha)(f, \epsilon_\alpha)$ is in S_λ, and it follows that $\{d, e, f\} \times G_\alpha \subseteq S_\lambda$. Similarly $\{a, b, c\} \times G_\beta \subseteq S_\mu$ for some $\mu \in \Lambda$.

By Proposition 3 (with $f = g = \epsilon_\alpha$ and $e = \epsilon_\beta$ there), T is τ-reduced. Since T is an idempotent-determined homomorphic image of L_m, it follows that L_m is not E-unitary. Hence $\mu \neq \lambda$, so that $\Lambda = \{\alpha, \beta\}$ with $S_\mu = S_\beta = P(G_\beta, U_\beta, V_\beta)$ and $S_\lambda = S_\alpha = P(G_\alpha, U_\alpha, V_\alpha)$. Now $d \in Y_\alpha$ and there is no largest element in Y_β below d. It follows from Theorem 10 (B) that L_m is not a strong semilattice of E-unitary inverse semigroups.

Example 5.3. Let $X = Y$ be the semilattice $\{a, b, c, d\}$ where $a = b \wedge c$, $d = b \vee c$, and b and c are incomparable. Let $X_1 = X$, $X_2 = \{a, b, c\}$, and $X_3 = \{a, b\}$, and let $J = \{X_1, X_2, X_3\}$. Let T be the chain $\{\epsilon_\alpha, \epsilon_\beta, \epsilon_\gamma\}$ where $\epsilon_\alpha[\epsilon_\beta, \epsilon_\gamma]$ is the identity map on $X_1[X_2, X_3]$. Then (T, X, X) is a fully strict

L-triple, and $V_\gamma = \{a, b\}$, $V_\beta = \{c\}$, and $V_\alpha = \{d\}$. Now $d_\gamma = b$, while $(d_\beta)_\gamma = c_\gamma = a$.

The final example shows that Theorem 10 (B) is not sufficiently general for our purposes, but that Theorem 10 (A) is needed.

Example 5.4. Let E be the chain $\{a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots\cdots < b_{n+1} < b_n < \cdots < b_2 < b_1\}$, and consider the fully strict L-triple (E, E, E) described in the remarks following Theorem 5. Then the L-triple $(E, \bar{E}, \bar{E})$ is not even strict, as can be seen by considering the element $\{a_i | i = 1, 2, 3, \ldots\}$ of $\bar{E}$.

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