Can. J. Math., Vol. XXIV, No. 4, 1972, pp. 734–736

A NOTE ON DIVISION ALGEBRAS

I. N. HERSTEIN AND A. RAMER

In this note we prove some results on the intersection properties of maximal subfields of division algebras which are finite dimensional over their centers. These results indicate that we can get very small intersections with any subalgebra if we use the appropriate maximal subfields. As a consequence of our first theorem, we obtain some theorems which are known and some which can be obtained from these known theorems (see, for instance, Theorem 3, Chapter VII in [**3**]). The proofs of these known results given here are very elementary and are quite different from the ones in the literature.

We begin with

THEOREM 1. *Let D be a division algebra finite dimensional over its center and let $D_0 \neq D$ be a sub-division algebra of D. Then there exists a maximal subfield K of D such that $K \cap D_0 = F$. In fact, if L is any maximal subfield of D which is a simple extension of F, then for some $x \in D$, $xLx^{-1} \cap D_0 = F$. Thus if L is a maximal separable subfield of D, $xLx^{-1} \cap D_0 = F$ for some appropriate $x \in D$.*

Proof. We may assume that F is an *infinite* field, otherwise D itself would be a field by Wedderburn's theorem. Let $L = F(t)$ be a maximal subfield of D which is a simple extension of F. By a well-known theorem (see [**1**] or [**4**]) there are only a finite number of fields L_i with

$$L \supset L_i \underset{\neq}{\supset} F.$$

Let $C_D(L_i) = \{x \in D | xu = ux \text{ for all } u \in L_i\}$. Since $L_i \not\subset F$, $C_D(L_i)$ is not D; $C_D(L_i)$ is a vector space over F (in fact, a sub-division algebra). Since F is an infinite field and D is a vector space over F, D *cannot* be the set-theoretic union of a finite number of proper subspaces. Hence $D_0 \cup_i C_D(L_i) \neq D$. Thus there is an element $x \in D$, $x \notin D_0$ which centralizes no L_i; hence x does *not* commute with any element $a \in L$ if $a \notin F$.

Suppose that the theorem is false, that is, that $yLy^{-1} \cap D_0 \neq F$ for every $y \in D$. Then, certainly, for the x above and any $\alpha \neq 0$ in F,

$$(x + \alpha)^{-1}L(x + \alpha) \cap D_0 \neq F,$$

which is to say $L \cap (x + \alpha)D_0(x_0 + \alpha)^{-1} \neq F$. Hence $L \cap (x + \alpha)D(x + \alpha)^{-1}$ is one of the L_i. Since F is infinite, by the pigeon-hole principle, there exist

Received August 27, 1971. The research of the first named author was supported by NSF Grant GP-29269.

three distinct elements, $\alpha_1, \alpha_2, \alpha_3 \in F$, none of them 0, such that

$$F \neq M = L \cap (x + \alpha_1)D_0(x + \alpha_1)^{-1} = L \cap (x + \alpha_2)D_0(x + \alpha_2)^{-1}$$
$$= L \cap (x + \alpha_3)D_0(x + \alpha_3)^{-1}.$$

Let $a \in M$, $a \notin F$. Hence, for some $d_1, d_2, d_3 \in D_0$ we have

$$(x + \alpha_i)d_i(x + \alpha_i)^{-1} = a,$$

that is,

$$(1) \qquad \begin{aligned} (x + \alpha_1)d_1 &= a(x + \alpha_1) \\ (x + \alpha_2)d_2 &= a(x + \alpha_2) \\ (x + \alpha_3)d_3 &= a(x + \alpha_3). \end{aligned}$$

From (1), by subtracting one of the relations from another we get:

$$(2) \qquad \begin{aligned} x(d_1 - d_2) + \alpha_1 d_1 - \alpha_2 d_2 &= a(\alpha_1 - \alpha_2) = a\beta, \quad \beta \neq 0 \in F \\ x(d_2 - d_3) + \alpha_2 d_2 - \alpha_3 d_3 &= a(\alpha_2 - \alpha_3) = a\gamma, \quad \gamma \neq 0 \in F. \end{aligned}$$

Playing the relations in (2) off against each other yields that

$$x(\gamma(d_1 - d_2) - \beta(d_2 - d_3)) + \gamma(\alpha_1 d_1 - \alpha_2 d_2) - \beta(\alpha_2 d_2 - \alpha_3 d_3) = 0.$$

If $s = \gamma(d_1 - d_2) - \beta(d_2 - d_3) \neq 0$, being in D_0, $s^{-1} \in D_0$ and so

$$x = [\beta(\alpha_2 d_2 - \alpha_3 d_3) - \gamma(\alpha_1 d_1 - \alpha_2 d_2)]s^{-1}$$

would be in D_0, contrary to our choice of x. Hence

$$(3) \qquad \gamma(d_1 - d_2) = \beta(d_2 - d_3).$$

This, in turn, forces

$$(4) \qquad \gamma(\alpha_1 d_1 - \alpha_2 d_2) = \beta(\alpha_2 d_2 - \alpha_3 d_3).$$

Eliminating d_3 between (3) and (4) we end up with

$$(5) \qquad \gamma(\alpha_3 - \alpha_1)d_1 = (\alpha + \beta)(\alpha_3 - \alpha_2)d_2.$$

However, recalling that $\gamma = \alpha_2 - \alpha_3 \neq 0$, $\beta = \alpha_1 - \alpha_2 \neq 0$, (5) readily gives us that $d_1 = d_2$.

Returning to (2) with the result $d_1 = d_2$ in hand, we get $(\alpha_1 - \alpha_2)d_1 = (\alpha_1 - \alpha_2)a$, and so $a = d_1$. But then (1) tells us that $(x + \alpha_1)a = a(x + \alpha_1)$ and so $xa = ax$. This contradicts the fact that x does not commute with any element in L outside of F, and $a \in L$, $a \notin F$. The theorem is thus proved.

The theorem has some immediate implications, namely:

COROLLARY 1. *Let D be a division algebra finite dimensional over its center F and let L be a maximal subfield of D. If K is a maximal separable subfield of D then $L \cap xKx^{-1} = F$ for some $x \in D$.*

COROLLARY 2. *If D is a division algebra finite dimensional over its center F and if $K = F(a)$ is a maximal subfield which is a simple extension of F, then for some $x \in D$, a and xax^{-1} generate D over F.*

Proof. By the theorem there is an $x \in D$ such that $xKx^{-1} \cap K = F$. Thus if D_0 is the subalgebra of D generated by a and xax^{-1} over F, then K and xKx^{-1} are both maximal subfields of D_0. Since the center of D_0 is contained in every maximal subfield of D_0 and $K \cap xKx^{-1} = F$, F must be the center of D_0. But then, since K is a maximal subfield of D_0, $[D_0:F] = [K:F]^2 = [D:F]$ (where $[\quad :F]$ denotes the degree over F). Hence $D = D_0$.

COROLLARY 3. *If D is a division algebra finite dimensional over its center F then there exists an element $a \in D$ such that a and xax^{-1} generate D over F, for some $x \in D$.*

Proof. D has a maximal subfield K which is separable over F. Thus $K = F(a)$ for some $a \in K$. Apply Corollary 2 to this.

We now prove

THEOREM 2. *Let D be a division algebra finite dimensional over its center F, and let $M \supset F$ be any subfield of D. Then $M = K_1 \cap K_2$ for some maximal subfields K_1, K_2 of D. In fact, if K is any maximal subfield of D which contains M then for some maximal subfield L of D, $K \cap L = M$.*

Proof. Let $D_1 = C_D(M) = \{x \in D | xm = mx \text{ for all } m \in M\}$; D_1 is a division algebra and, by the double centralizer theorem [**2**], M is the center of D_1. Applying Corollary 1, if K is a maximal subfield of D_1 then for some other maximal subfield L of D_1, $K \cap L = M$.

Now, any maximal subfield of D_1 is a maximal subfield of D, for if $T \supset K \supset M$, where T is a field and K is a maximal subfield of D_1, then clearly T centralizes M, so $T \subset C_D(M) = D_1$. Hence $T = K$. This finishes the proof.

REFERENCES

1. E. Artin, *Galois theory* (Notre Dame Mathematical Lectures No. 2, 1946).
2. I. N. Herstein, *Non-commutative rings* (Carus Monograph No. 15, 1968).
3. Nathan Jacobson, *Structure of rings* (A.M.S. Colloquium Series, No. XXXVII, 1956, 1964).
4. Irving Kaplansky, *Fields and rings* (Chicago Lectures in Mathematics, 1969).

University of Chicago,
Chicago, Illinois;
Weizmann Institute of Science,
Rehovot, Israel

https://doi.org/10.4153/CJM-1972-069-4 Published online by Cambridge University Press