LARGE AUTOMORPHISM GROUPS OF COMPACT KLEIN SURFACES WITH BOUNDARY, I

by COY L. MAY

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0. Introduction. Let X be a Klein surface [**1**], that is, X is a surface with boundary ∂X together with a dianalytic structure on X. A homeomorphism $f: X \rightarrow X$ of X onto itself that is dianalytic will be called an *automorphism* of X.

In a recent paper [**8**] we showed that a compact Klein surface of (algebraic) genus $g \geqq 2$, with non-empty boundary, cannot have more than $12(g-1)$ automorphisms. We also showed that the bound $12(g-1)$ is attained, by exhibiting some surfaces of low genus ($g = 2, 3, 5$) together with their automorphism groups. The corresponding bound for Riemann surfaces is quite well known; Hurwitz showed that a compact Riemann surface of genus $g \geqq 2$ has at most $84(g-1)$ (orientation-preserving) automorphisms.

Here we study those groups that act as a group of $12(g-1)$ automorphisms of a compact Klein surface with boundary of genus $g \geqq 2$. Our main result is a characterization of these groups in terms of their presentations. We call these finite groups *M^*-groups.* It is easy to find examples of M^*-groups. In fact, by using known results about normal subgroups of the modular group, we are able to find an infinite family of M^*-groups. Consequently the bound $12(g-1)$ is attained for infinitely many values of the genus g.

In the final section of the paper we show how to obtain other infinite families for which the bound $12(g-1)$ is achieved, without use of results about the modular group. To obtain an infinite family we only need a single Klein surface with boundary that has $12(g-1)$ automorphisms. Some of the infinite families we get in this manner consist of orientable surfaces; others consist of non-orientable surfaces.

1. Compact Klein surfaces and NEC groups. Let X be a compact Klein surface, and let E be the field of all meromorphic functions on X. E is an algebraic function field in one variable over $\mathbf{R}$ [**1**, p. 102] and as such has an *algebraic genus g*. This is the non-negative integer that makes the algebraic version of the Riemann–Roch Theorem work [**2**, p. 22]. We will refer to g simply as the *genus* of the compact Klein surface X. In case X is a Riemann surface, g is equal to the topological genus of X.

Now let (X_c, π, σ) be the *complex double* of X [**1**, pp. 37–41], that is, X_c is a compact Riemann surface, $\pi : X_c \rightarrow X$ is an unramified 2-sheeted covering of X, and σ is the unique antianalytic involution of X_c such that $\pi = \pi \circ \sigma$. If F is the field of meromorphic functions on X_c, then $F = E(i)$ and by a well-known classical result [**2**, p. 99], the genus of X is equal to the genus of X_c.

If X and Y are Klein surfaces, then a *morphism* [**1**, p. 17] is a continuous map $f: X \rightarrow Y$, with $f(\partial X) \subset \partial Y$ and the following local behavior. For every point $p \in X$ there exist dianalytic charts (U, z) and (V, w) at p and $f(p)$ respectively such that $f(U) \subset V$ and $f|_U = w^{-1} \circ \varphi \circ F \circ z$, where F is an analytic function on $z(U)$ and φ is the folding map, $\varphi(x+yi) = x + |y| i$.

A

To study automorphism groups of Klein surfaces with boundary, we use the theory of non-euclidean crystallographic (NEC) groups, as developed by Wilkie [**11**] and Macbeath [**7**]. Singerman [**10**] used NEC groups to obtain results about automorphism groups of non-orientable Klein surfaces without boundary.

Let $H = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$ denote the open upper half-plane and Ω the group of automorphisms of H. The elements of Ω are transformations of one of two types:

$$T(z) = \frac{az+b}{cz+d}, \quad ad-bc = 1, \quad a, b, c, d \quad \text{real}; \tag{1.1}$$

$$T(z) = \frac{a\bar{z}+b}{c\bar{z}+d}, \quad ad-bc = -1, \quad a, b, c, d \quad \text{real}. \tag{1.2}$$

Those of type (1.1) preserve orientation and form a subgroup Ω^+ of index 2 in Ω. The only transformations of type (1.2) that have fixed points in H are *reflections* (in case $a+d=0$). Reflections are of order two and have a circle of fixed points in H.

An *NEC group* is a subgroup of Ω that acts discontinuously on H. An NEC group contained in Ω^+ is called a Fuchsian group. If Γ is an NEC group containing orientation-reversing elements, then Γ has a subgroup $\Gamma^+ = \Gamma \cap \Omega^+$ of index two.

Let Γ be an NEC group such that the quotient space H/Γ is compact. A fundamental region for Γ is defined in the same way as for Fuchsian groups. We will denote the non-euclidean area of a fundamental region F by $\mu(F)$. Many of the basic results about fundamental regions for Fuchsian groups continue to hold for fundamental regions for NEC groups. In particular, if Γ' is a subgroup of finite index in Γ and F, F' are fundamental regions for Γ, Γ' respectively, then

$$[\Gamma : \Gamma'] = \mu(F')/\mu(F).$$

An NEC group Γ will be called a *surface group* if the quotient space $X = H/\Gamma$ is compact and the quotient map $p : H \to X$ is unramified. Γ will be called a *bordered surface group* if further $\partial X \neq \emptyset$. Let $x \in H$. It is easy to see that $p(x) \in \partial X$ if and only if there exists a reflection $r \in \Gamma$ such that $r(x) = x$. Thus bordered surface groups contain reflections, but no other elements of finite order.

Now we recall two important results about the relationship between Klein surfaces and NEC groups.

THEOREM A. *Let Γ be an NEC group. Then the quotient space H/Γ has a unique dianalytic structure such that the quotient map $p : H \to H/\Gamma$ is a morphism of Klein surfaces.*

Proof. Since Γ acts discontinuously on H, this follows immediately from a result of Alling and Greenleaf [**1**, p. 52].

THEOREM B. *Let X be a compact Klein surface with boundary of genus $g \geqq 2$. Then X can be represented in the form H/Γ, where Γ is a bordered surface group. If F is a fundamental region for Γ, then*

$$\mu(F) = 2\pi(g-1).$$

For a proof, see [**9**]. A good reference for the corresponding result about Riemann surfaces is [**6**, p. 66].

2. Induced mappings on the quotient space. This section was inspired by the treatment of the Riemann surface case in [**6**, pp. 59–63]. Also see [**5**].

For any Klein surface Y, let $\mathrm{Aut}(Y)$ denote the group of automorphisms of Y. If Y is orientable, let $\mathrm{Aut}^+(Y)$ denote the subgroup of orientation-preserving automorphisms.

Let Γ be a bordered surface group. Let $X = H/\Gamma$ be the quotient space, $p: H \to X$ the quotient map. X is a compact Klein surface with boundary, by Theorem A.

A transformation $f \in \Omega$ is said to be Γ-*admissible* in case the following condition holds:

$$p(x) = p(y) \quad \text{if and only if} \quad p(f(x)) = p(f(y)).$$

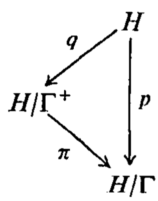
If f is Γ-admissible, then we can define a map $f^*: X \to X$ by

$$f^*(p(x)) = p(f(x)).$$

It is easy to check that f^* is an automorphism of X.

Let $A(H, \Gamma)$ denote the group of Γ-admissible automorphisms of H. Clearly $\Gamma \subset A(H, \Gamma)$. Let $N(\Gamma)$ denote the normalizer of Γ in Ω. It is not hard to see that $N(\Gamma) \subset A(H, \Gamma)$ (see [**6**, p. 60]).

Now the quotient space H/Γ^+ is a compact Riemann surface. Let $q: H \to H/\Gamma^+$ be the quotient map. Then the group of order two $C_2 = \Gamma/\Gamma^+$ acts on H/Γ^+ to exhibit H/Γ^+ as an unramified double cover of $X = H/\Gamma$. If $\pi: H/\Gamma^+ \to H/\Gamma$ denotes the quotient map, the following diagram commutes.



There is one point in the fiber $\pi^{-1}(x)$ if $x \in \partial X$; otherwise there are two. Let $f \in \Gamma - \Gamma^+$. Since $f \in N(\Gamma^+) \subset A(H, \Gamma^+)$, we can define an automorphism σ of H/Γ^+ by

$$\sigma(q(x)) = q(f(x)).$$

σ is an orientation-reversing automorphism of order 2 and $\pi \circ \sigma = \pi$. Thus we see that $(H/\Gamma^+, \pi, \sigma)$ is the complex double of $X = H/\Gamma$. Now let X_c denote H/Γ^+. We will use the complex double to establish the algebraic relationship between Γ and $A(H, \Gamma)$ within Ω. We have a group homomorphism $p^*: A(H, \Gamma) \to \mathrm{Aut}(X)$ defined by $p^*(f) = f^*$. We have a similar homomorphism $q^*: A(H, \Gamma^+) \to \mathrm{Aut}(X_c)$.

PROPOSITION 1. *Γ = kernel p^* and $A(H, \Gamma) = N(\Gamma)$.*

Proof. Clearly $\Gamma \subset \text{kernel}\, p^*$. Let $f \in \text{kernel}\, p^*$ such that f is analytic. Then $p = p \circ f$, i.e. $\pi \circ q = \pi \circ q \circ f$. Since (X_c, π, σ) is the complex double of X, for each $x \in H$ either

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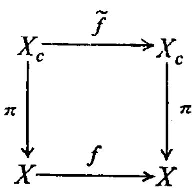
$q(x) = q \circ f(x)$ or $\sigma \circ q(x) = q \circ f(x)$. Suppose there exists a point $y \in H$ such that $q(y) \neq q \circ f(y)$. Then we can find an open set U about y such that $\sigma \circ q = q \circ f$ on U. Hence f is antianalytic on U, a contradiction. Therefore $q = q \circ f$, and $f \in \text{kernel}\, q^*$. Since f is analytic, we have $f \in \Gamma^+$ [**6**, p. 60].

If $h \in \text{kernel}\, p^*$ and h is antianalytic, then choose $k \in \Gamma - \Gamma^+$. $k \circ h$ is analytic and $k \circ h \in \text{kernel}\, p^*$. Hence $k \circ h \in \Gamma^+$ and $h \in \Gamma$. Therefore $\Gamma = \text{kernel}\, p^*$.

Now Γ is normal in $A(H, \Gamma)$. Thus $A(H, \Gamma) \subset N(\Gamma)$. We have already noted that $N(\Gamma) \subset A(H, \Gamma)$.

PROPOSITION 2. $p^* : A(H, \Gamma) \to \text{Aut}(X)$ *is surjective.*

Proof. Let $f \in \text{Aut}(X)$. There exists a unique map $\tilde{f} \in \text{Aut}^+(X_c)$ such that the following diagram commutes.



Moreover, $\sigma \circ \tilde{f} \circ \sigma = \tilde{f}$ and $\tilde{f} = \tilde{h}$ implies that $f = h$ [**1**, p. 79].

There is an analytic map $\theta \in A(H, \Gamma^+)$ such that $q^*(\theta) = \tilde{f} \in \text{Aut}^+(X_c)$ [**6**, p. 63]. Now $\tilde{f} \circ q = q \circ \theta$. It is easy to check that θ is Γ-admissible, so that we have an automorphism $p^*(\theta) = \theta^* \in \text{Aut}(X)$. As before there is a unique map $\tilde{\theta}^* \in \text{Aut}^+(X_c)$ such that $\pi \circ \tilde{\theta}^* = \theta^* \circ \pi$. But if $q(x) \in X_c$, $\pi \circ \tilde{f}(q(x)) = \pi \circ q \circ \theta(x) = p \circ \theta(x) = \theta^* \circ p(x) = \theta^* \circ \pi(q(x))$. Hence $\tilde{f} = \tilde{\theta}^*$ and $f = \theta^*$, that is, $p^*(\theta) = f$. Thus p^* is surjective.

Propositions 1 and 2 have the following important consequence.

PROPOSITION 3. *Let Γ be a bordered surface group. G is a group of automorphisms of the Klein surface H/Γ if and only if $G \cong \Delta/\Gamma$, where Δ is an NEC group such that $\Gamma \subset \Delta \subset N(\Gamma)$.*

The corresponding result about Riemann surfaces is similar and well known.

3. Large groups of automorphisms. The closed disc D is a compact Klein surface of genus 0; D has (up to isomorphism) a unique dianalytic structure [**1**, p. 60].

Let G be a group of automorphisms of a compact Klein surface X of genus $g \geqq 2$ with non-empty boundary. Then the quotient space $\Phi = X/G$ has a unique dianalytic structure such that the quotient map $\pi : X \to \Phi$ is a morphism of Klein surfaces. Moreover, π is a ramified r-sheeted covering of Φ, where $r = |G|$ (see [**8**] and [**1**, p. 52]).

Using a form of the Hurwitz ramification formula, we showed that $|G| \leqq 12(g-1)$. Also, $|G| = 12(g-1)$ if and only if the quotient space Φ is the disc D, and the quotient map $\pi : X \to D$ is ramified above precisely 4 points of the boundary ∂D, the ramification indices k_i in the fibers above these 4 points being $k_1 = k_2 = k_3 = 2$, $k_4 = 3$ [**8**].

Let F be a four-sided non-euclidean polygon with angles $\frac{1}{2}\pi$, $\frac{1}{2}\pi$, $\frac{1}{2}\pi$, and $\frac{1}{3}\pi$. Let A be the NEC group generated by the reflections R_1, R_2, R_3, R_4 in the sides of the polygon F.

Then A has the following presentation [3, p 55]:

$$\left.\begin{array}{l}\text{generators } R_1, R_2, R_3, R_4,\\ \text{and relations}\\ (R_1)^2=(R_2)^2=(R_3)^2=(R_4)^2=(R_1R_2)^2=(R_2R_3)^2=(R_3R_4)^2=(R_4R_1)^3=1.\end{array}\right\} \quad (2)$$

The polygon F is a fundamental region for the group A, of course, and by the Gauss–Bonnet Theorem [6, p. 21],

$$\mu(F)=2\pi-\tfrac{1}{2}\pi-\tfrac{1}{2}\pi-\tfrac{1}{2}\pi-\tfrac{1}{3}\pi=\tfrac{1}{6}\pi.$$

Now suppose that a bordered surface group Γ is a normal subgroup of finite index in A. Then A/Γ is a group of automorphisms of the compact Klein surface $Y=H/\Gamma$. If Y is of genus g, then

$$|A/\Gamma|=[A:\Gamma]=\frac{2\pi(g-1)}{\pi/6}=12(g-1).$$

Thus we obtain a large automorphism group whenever we find a finite group G and a homomorphism $\varphi: A\to G$ onto G such that kernel φ is a bordered surface group.

Before proceeding we need a definition.

Definition. A finite group G will be called an *M^*-group* in case G is generated by three distinct elements t, u, v of order two which satisfy the relations

$$(tu)^2=(tv)^3=1. \quad (3)$$

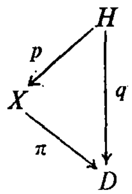
Now we are ready to prove one of our main results.

Theorem 1. *G is a group of $12(g-1)$ automorphisms of a compact Klein surface of genus $g\geqq 2$ with non-empty boundary if and only if G is an M^*-group.*

Proof. Sufficiency. Suppose G is an M^*-group with generators t, u, v of order two that satisfy the relations (3). Define a homomorphism $\varphi: A\to G$ onto G by $\varphi(R_1)=t$, $\varphi(R_2)=u$, $\varphi(R_3)=1$, $\varphi(R_4)=v$. Let $\Gamma=\text{kernel}\,\varphi$. Then $R_3\in\Gamma$, and using a result of Macbeath [7, p. 1198], it is easy to see that any element of finite order in Γ is conjugate to R_3 and thus is a reflection. Therefore Γ is a bordered surface group. Now G is a group of automorphisms of the Klein surface $Y=H/\Gamma$, and if Y is of genus g, $|G|=12(g-1)$.

Necessity. Suppose G is a group of $12(g-1)$ automorphisms of a compact Klein surface X of genus $g\geqq 2$ with non-empty boundary. By Theorem B, we can represent X in the form $X=H/\Gamma$, where Γ is a bordered surface group. Let $p: H\to X$ be the quotient map. Since $|G|=12(g-1)$ the quotient space X/G is the disc D and the quotient map $\pi: X\to D$ is ramified above exactly 4 points of ∂D, the ramification indices being 2, 2, 2, 3.

By Proposition 3, there exists an NEC group B, where $\Gamma\subset B\subset N(\Gamma)$, and a homomorphism $\varphi: B\to G$ onto G such that kernel $\varphi=\Gamma$. Let $Y=H/B$ and let $q: H\to Y$ be the quotient map. It is easy to check that the map $f: Y\to D$ defined by $f(q(x))=\pi(p(x))$ is a dianalytic isomorphism. Thus the quotient space H/B is the disc D, and the following diagram commutes.



The map p is unramified. Therefore q is ramified above precisely the same points of D that π is, and the ramification indices above these points must also be the same. Then it follows from Wilkie's results [**11**, p. 96] that, after a simple elimination of redundant generators, the group B has the presentation (2).

Now Γ is a bordered surface group, so that Γ contains a reflection. Since Γ is normal in B, it follows that one of the R_i is in Γ [**7**, p. 1198]. Suppose $R_1 \in \Gamma = \text{kernel}\,\varphi$. Since $R_4^2 = (R_4R_1)^3 = 1$, $R_4 \in \Gamma$ as well. Then R_4R_1 is an analytic element of finite order in the surface group Γ, a contradiction. Hence $R_1 \notin \Gamma$. Similarly $R_4 \notin \Gamma$.

Suppose $R_3 \in \Gamma$. Let $t = \varphi(R_1)$, $u = \varphi(R_2)$, and $v = \varphi(R_4)$. Then t, u, v generate G and satisfy the relations (3). Hence G is an M^*-group.

If $R_2 \in \Gamma$, then let $t = \varphi(R_4)$, $u = \varphi(R_3)$, and $v = \varphi(R_1)$. Again we see that G is an M^*-group. This concludes the proof.

The proof indicates the relationship between [**8**] and the work of Wilkie [**11**].

4. M*-groups. Let Γ be the *extended modular group*, the subgroup of Ω consisting of all transformations of types (1.1) and (1.2) where a, b, c, d are integers. Then Γ^+ is the usual *modular group*. The group Γ is generated by the transformations T, U, V where $T(z) = 1/\bar{z}$, $U(z) = -\bar{z}$, and $V(z) = -\bar{z}/(\bar{z}+1)$. They satisfy the relations

$$T^2 = U^2 = V^2 = (TU)^2 = (TV)^3 = 1,$$

and these relations actually define Γ [**3**, pp. 85–86]. Immediately we see that M^*-groups are finite quotient groups of Γ. On the other hand, suppose K is any normal subgroup of Γ of finite index larger than 6. It can be checked that T, U, V, TU, and TV are not in K. Therefore Γ/K is an M^*-group.

For each integer $q \geqq 2$, let Γ_q be the *principal congruence subgroup of level* q, consisting of all transformations (1.1) of Γ^+ such that $a \equiv d \equiv \pm 1 \pmod{q}$, and $b \equiv c \equiv 0 \pmod{q}$. Let $\mu = [\Gamma^+ : \Gamma_q]$ be the index of Γ_q in Γ^+. If $q = 2$, then $\mu = 6$. If $q > 2$, then

$$\mu = \tfrac{1}{2}q^3 \prod_{p|q} \left(1 - \frac{1}{p^2}\right),$$

where the product is over the prime divisors of q [**4**, pp. 8–10]. Now note that Γ_q is normal in the big group Γ. $[\Gamma : \Gamma_q] = 2\mu \geqq 12$. Thus we have the following important result.

THEOREM 2. *For each integer $q \geqq 2$, Γ/Γ_q is an M^*-group.*

COROLLARY. *There are infinitely many values of the genus g for which there is a compact Klein surface with boundary with $12(g-1)$ automorphisms.*

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It is possible to find one topological type of Klein surface with boundary on which Γ/Γ_q acts. Consider the quotient space H/Γ_q. For more details on the following, see the treatment in [4, pp. 6–16]. Let M denote the compactification of H/Γ_q. M is a compact Riemann surface of genus h, where $h = 0$ if $q = 2$, and

$$h = 1 + \frac{q^2(q-6)}{24} \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right) \quad \text{if} \quad q > 2.$$

The quotient group Γ/Γ_q acts as a group of automorphisms of M. Let $\lambda = \mu/q$. There are exactly λ points of the quotient space M that are images of points equivalent under Γ^+ (or under Γ) to $i\infty$. These λ points form an orbit of the quotient group Γ/Γ_q. Now remove an open disc centered about each of these λ points, and let X denote the resulting surface. X inherits an analytic structure from M, and Γ/Γ_q acts as a group of automorphisms of X. Topologically X is a sphere with h handles and λ holes. Then X is of genus $g = 2h + \lambda - 1$ [1]. We need to calculate the order of Γ/Γ_q in terms of the genus g. From [4, p. 14] we have the following relation.

$$12(h-1) = \mu(1-6/q) = \mu - 6\lambda.$$

Therefore $|\Gamma/\Gamma_q| = 2\mu = 2[12(h-1)+6\lambda] = 12[2(h-1)+\lambda] = 12[(2h+\lambda-1)-1] = 12(g-1)$. Thus Γ/Γ_q acts as an automorphism group of X of maximum possible order.

The values of 2μ, g, h, λ are tabulated below for some small values of q.

q	2μ	g	h	λ	q	2μ	g	h	λ
2	12	2	0	3	7	336	29	3	24
3	24	3	0	4	8	384	33	5	24
4	48	5	0	6	9	648	55	10	36
5	120	11	0	12	10	720	61	13	36
6	144	13	1	12	11	1320	111	26	60

Before proceeding, we give a table of M^*-groups of low order that we have found. These groups are quite familiar, and it is easy enough to check that they are M^*-groups. S_n denotes the symmetric group on n letters, A_n the alternating group on n letters, and C_n the cyclic group of order n.

M^*-groups of low order

Group	Order	Genus
$C_2 \times S_3$	12	2
S_4	24	3
$S_3 \times S_3$	36	4
$C_2 \times S_4$	48	5
A_5	60	6
$C_2 \times A_5$	120	11
$S_3 \times S_4$	144	13
$C_2 \times C_2 \times A_5$	240	21

5. Infinite families. In this section we exhibit other infinite families of surfaces for which the bound $12(g-1)$ is attained. The technique used is essentially the same as that employed in [**5**] and [**10**].

Suppose Γ is a bordered surface group such that the quotient space H/Γ is of genus g and has $12(g-1)$ automorphisms. Then, as in the proof of Theorem 1, there exists an NEC group B with presentation (2), such that Γ is a normal subgroup of B and $[B:\Gamma]=12(g-1)$.

The commutator subgroup Γ' is a characteristic subgroup of Γ. The subgroup Γ^m of Γ generated by the m^{th} powers of elements of Γ is also a characteristic subgroup. Γ^m contains orientation-reversing elements if and only if m is odd.

For odd m, let $\Delta_m=\Gamma'\Gamma^m$. Δ_m is a characteristic subgroup of Γ. Γ is finitely generated by Wilkie's results [**11**]. The quotient group Γ/Δ_m is a finitely generated abelian group in which every element has finite order. Thus Γ/Δ_m is a finite abelian group, so that Δ_m is of finite index in Γ. Hence Δ_m is a surface group. But Γ contains reflections, and since m is odd, so does Δ_m. Thus Δ_m is a bordered surface group. Since Δ_m is characteristic in Γ, Δ_m is normal in B. Now, as we have seen, H/Δ_m is a compact Klein surface with boundary of genus g' that has $12(g'-1)$ automorphisms. When the topological type of H/Γ is known, we can calculate the genus of each Klein surface H/Δ_m. For example, we have the following.

LEMMA 1. *Suppose H/Γ topologically is a sphere with k holes. Then the genus g' of H/Δ_m is*

$$g'=(k-2)m^{k-1}+1.$$

Further, each surface H/Δ_m is orientable.

Proof. Γ has the following canonical presentation [**11**, p. 96]:

generators $c_1,\dots,c_k,e_1,\dots,e_k$,
and relations $c_i^2=e_ic_ie_i^{-1}c_i=1\quad(i=1,\dots,k),\quad e_1e_2\dots e_k=1$.

Now $c_i^m=c_i\in\Delta_m$ for each i. Therefore Γ/Δ_m has the presentation:

generators $\bar{e}_1,\dots,\bar{e}_k$,
and relations $m\bar{e}_i=0\quad(i=1,\dots,k),\quad \bar{e}_1+\bar{e}_2+\dots+\bar{e}_k=0$.

Clearly $\Gamma/\Delta_m\cong(C_m)^{k-1}$, so that $[\Gamma:\Delta_m]=m^{k-1}$. The genus of the Klein surface H/Γ is $(k-1)$. Both Γ and Δ_m are bordered surface groups. Therefore

$$m^{k-1}=[\Gamma:\Delta_m]=\frac{2\pi(g'-1)}{2\pi[(k-1)-1]}.$$

Then $g'=(k-2)m^{k-1}+1$.

Let $X=H/\Gamma$ and $Y=H/\Delta_m$, and let $p:H\to X$ and $q:H\to Y$ be the quotient maps. Then $G=\Gamma/\Delta_m$ is a group of automorphisms of Y. As in the proof of Theorem 1, the quotient space Y/G is dianalytically isomorphic to X. If $\pi:Y\to X$ denotes the quotient map, then $p=\pi\circ q$. π is an unramified m^{k-1}-sheeted covering of X, since $|G|=m^{k-1}$ [**8**]. Suppose $\pi(q(x))=p(x)\in\partial X$. Then there is a reflection $r\in\Gamma$ with $r(x)=x$. But $r\in\Delta_m$ as well, since m

is odd, so that $q(x) \in \partial Y$. Thus $\pi^{-1}(\partial X) = \partial Y$, that is, Y is not folded along ∂X. Hence we can lift the orientation of X to define an orientation of Y.

THEOREM 3. *There is a compact orientable Klein surface of genus g with non-empty boundary that has* $12(g-1)$ *automorphisms for each of the following values of g:*

$$g = m^2+1, \quad g = 2m^3+1, \quad g = 4m^5+1, \quad g = 10m^{11}+1,$$

where m is any positive odd integer.

Proof. Lemma 1 applies to 4 examples of the previous section (see the first table with $q = 2, 3, 4, 5$).

To find infinite families of non-orientable surfaces, we first need examples of non-orientable surfaces with $12(g-1)$ automorphisms.

EXAMPLE. Let X be a sphere with 12 holes, with the holes centered around the vertices of an inscribed regular icosahedron. X is an orientable Klein surface of genus 11. X has a group of automorphisms isomorphic to the complete symmetry group (including reflections) of the regular icosahedron, which is $C_2 \times A_5$. Let $\tau : X \to X$ be the antipodal map. The quotient space $W = X/\tau$ is a real projective plane with 6 holes, a non-orientable Klein surface of genus 6. W has an automorphism group isomorphic to A_5. Thus W has $60 = 12(6-1)$ automorphisms.

A similar example in **[8]** gives a non-orientable Klein surface of genus 3, topologically a real projective plane with 3 holes, that has 24 automorphisms.

LEMMA 2. *Suppose* H/Γ *topologically is a real projective plane with k holes. Then the genus* g' *of* H/Δ_m *is*

$$g' = (k-1)m^k+1.$$

Further, each surface H/Δ_m *is non-orientable.*

Proof. Γ has the following canonical presentation **[11**, p. 101]:

generators $c_1, \ldots, c_k, \quad e_1, \ldots, e_k, \quad d,$
and relations $c_i^2 = 1 \quad (i = 1, \ldots, k), \quad e_i c_i e_i^{-1} c_i = 1 \quad (i = 1, \ldots, k), \quad d^2 e_1 \ldots e_k = 1.$

Then Γ/Δ_m has the following presentation:

generators $\bar{e}_1, \ldots, \bar{e}_k, \quad \bar{d},$
and relations $2\bar{d}+\bar{e}_1+\ldots+\bar{e}_k = 0, \quad m\bar{d} = 0, \quad m\bar{e}_i = 0 \quad (i = 1, \ldots, k).$

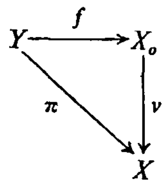
Thus $\Gamma/\Delta_m \cong (C_m)^k$ and $[\Gamma : \Delta_m] = m^k$. The genus of the Klein surface H/Γ is k. Therefore

$$m^k = [\Gamma : \Delta_m] = \frac{2\pi(g'-1)}{2\pi(k-1)}$$

and $g' = (k-1)m^k+1$.

Now, as in the proof of Lemma 1, let $X = H/\Gamma$, $Y = H/\Delta_m$, and $G = \Gamma/\Delta_m$. The quotient space Y/G is the Klein surface X. If $\pi : Y \to X$ denotes the quotient map, then π is an unramified m^k-sheeted covering of X and $\pi^{-1}(\partial X) = \partial Y$. Suppose Y is orientable. Then let

(X_o, ν, τ) be the orienting double of X [**1**, pp. 42–43], that is, X_o is a compact orientable Klein surface with boundary, $\nu: X_o \to X$ is an unramified 2-sheeted covering of X, and τ is the unique antianalytic involution of X_o such that $\nu \circ \tau = \nu$. Since $\pi^{-1}(\partial X) = \partial Y$, there exists a unique analytic map $f\colon Y \to X_o$ such that the following diagram commutes [**1**, p. 42].



Then f is an r-sheeted covering of X_o for some integer r, and $2r = m^k$. But m^k is odd, a contradiction. Therefore Y is non-orientable.

THEOREM 4. *There is a compact non-orientable Klein surface of genus g with non-empty boundary that has* $12(g-1)$ *automorphisms for each of the following values of g:*

$$g = 2m^3 + 1, \quad g = 5m^6 + 1,$$

where m is any odd positive integer.

We are working (with Newcomb Greenleaf) on other techniques for finding infinite families of surfaces for which the bound $12(g-1)$ is attained. A sequel is planned.

Thanks are due the referee for calling our attention to the relationship of the extended modular group to M^*-groups.

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THE UNIVERSITY OF TEXAS AT AUSTIN
AUSTIN, TEXAS 78712, U.S.A.

Present address: UNIVERSITY OF KENTUCKY
LEXINGTON, KENTUCKY 40506, U.S.A.

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