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# A NOTE ON LIPSCHITZIAN MAPPINGS IN CONVEX METRIC SPACES

BY

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ABSTRACT. Local criteria are given which imply the lipschitzian character of mappings defined in complete and convex metric spaces.

It is our purpose in this note to respond to a recent question of F. Clarke [1] concerning the lipschitzian character of mappings defined on complete and metrically convex metric spaces.

For points $x$, $y$ of a metric space $(X, \rho)$ with $x \neq y$ we denote:

$$(x, y) = \{z \in X : x \neq z \neq y \text{ and } \rho(x, z) + \rho(z, y) = \rho(x, y)\}.$$

The space $X$ is said to be *metrically convex* if $(x, y) \neq \varnothing$ for each distinct pair $x, y \in X$. We begin with the following definitions:

Let $X$ and $Y$ be metric spaces and $M$ a positive constant.

(a) (Clarke [1]). The mapping $T: X \to Y$ is said to be *pointwise lipschitzian on* $X$ with constant $M$ if $T$ is continuous and for each $x \in X$,

$$\limsup_{\substack{y \to x \\ y \neq x}} \frac{\rho(Tx, Ty)}{\rho(x, y)} \leq M.$$

$T$ is called a *pointwise contraction* if $M \in [0, 1)$.

(b) The mapping $T: X \to Y$ is said to be *almost directionally lipschitzian on* $X$ with constant $M$ if for each $x, y \in X$ with $x \neq y$:

$$\inf_{z \in (x,y)} \frac{\rho(Tx, Tz)}{\rho(x, z)} \leq M.$$

(c) (Clarke [1]). The mapping $T: X \to X$ is said to be *weakly directionally lipschitzian on* $X$ with constant $M$ if $T$ is continuous and for each $x \in X$:

$$\liminf_{\substack{z \to x \\ z \in (x, Tx)}} \frac{\rho(Tx, Tz)}{\rho(x, z)} \leq M.$$

In [1] Clarke proves that every weak directional contraction defined on a complete metric space has a fixed point, and he raises the question: Is every

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pointwise contraction on a complete and convex metric space a global contraction? Since it is immediate that (a)$\Rightarrow$(b) in metrically convex spaces (but not conversely), the following result yields an affirmative answer to Clarke's question. Our proof is a modification of the central argument of [2] where similar questions are treated in Banach spaces.

THEOREM. Let $X$ and $Y$ be complete metric spaces with $X$ metrically convex, and suppose $T: X \to Y$ is almost directionally lipschitzian with constant $M$. Suppose in addition that $T$ has a closed graph. Then $T$ is a lipschitzian mapping with global Lipschitz constant $M$ on $X$.

REMARK. This theorem actually shows more than asked in Clarke's question since the assumption of closedness of $T$ is weaker than continuity, and moreover it represents a generalization of Theorem 1 of [2].

**Proof of the theorem.** Fix $x, y \in X$ with $x \neq y$, let $\hat{M} > M$, and let $\Omega_1$ denote the countable ordinals. Suppose for $\alpha$ less than some fixed $\gamma \in \Omega_1$ the set $\{x_\alpha\}$ has been defined so that:

(1) $x_0 = x$.
(2) If $x_\alpha = y$ for some $\alpha$ and $\alpha \leq \eta < \gamma$, then $x_\eta = y$.
(3) If $\alpha < \beta < \eta < \gamma$ and $x_\eta \neq y$, then $x_\beta \in (x_\alpha, x_\eta)$.
(4) If $\beta < \eta < \gamma$ and $x_\eta \neq y$, then $x_\eta \in (x_\beta, y)$.
(5) $T$ is $\hat{M}$-lipschitzian on the set $\{x_\eta : \eta < \gamma\}$.

In order to define $x_\gamma$ we distinguish two cases.

CASE 1. $\gamma = \mu + 1$ for some $\mu \in \Omega_1$. If $x_\mu = y$, then take $x_\gamma = y$. Otherwise use (b) to choose $x_\gamma \in (x_\mu, y)$ so that

$$\rho(Tx_\gamma, Tx_\mu) \leq \hat{M}\rho(x_\gamma, x_\mu).$$

We now show that (3)–(5) hold for $\eta = \gamma$: If $\alpha < \mu$, then (4) implies

$$\begin{aligned} \rho(x_\alpha, x_\gamma) &\leq \rho(x_\alpha, x_\mu) + \rho(x_\mu, x_\gamma) \\ &= \rho(x_\alpha, y) - \rho(x_\mu, y) + \rho(x_\mu, y) - \rho(x_\gamma, y) \\ &= \rho(x_\alpha, y) - \rho(x_\gamma, y) \\ &\leq \rho(x_\alpha, x_\gamma); \end{aligned}$$

hence $x_\mu \in (x_\alpha, x_\gamma)$. Thus if $\alpha < \beta \leq \mu < \gamma$, by (3):

$$\begin{aligned} \rho(x_\alpha, x_\gamma) &= \rho(x_\alpha, x_\mu) + \rho(x_\mu, x_\gamma) \\ &= \rho(x_\alpha, x_\beta) + \rho(x_\beta, x_\mu) + \rho(x_\mu, x_\gamma) \\ &\geq \rho(x_\alpha, x_\beta) + \rho(x_\beta, x_\gamma) \geq \rho(x_\alpha, x_\gamma); \end{aligned}$$

i.e., $x_\beta \in (x_\alpha, x_\gamma)$, proving (3) for $\eta=\gamma$. Also (4) holds for $\eta=\gamma$ since, for $\beta<\gamma$,

$$\begin{aligned}\rho(x_\beta, y) &= \rho(x_\beta, x_\mu)+\rho(x_\mu, x_\gamma)+\rho(x_\gamma, y)\\ &= \rho(x_\beta, x_\gamma)+\rho(x_\gamma, y).\end{aligned}$$

To see that (5) holds for $\eta=\gamma$, suppose $\beta<\gamma$. Then, as seen above, $x_\beta \in (x_\alpha, x_\gamma)$ and

$$\begin{aligned}\rho(Tx_\beta, Tx_\gamma) &\le \rho(Tx_\beta, Tx_\mu)+\rho(Tx_\mu, Tx_\gamma)\\ &\le \hat{M}\rho(x_\beta, x_\mu)+\hat{M}\rho(x_\mu, x_\gamma)\\ &= \hat{M}\rho(x_\beta, x_\gamma).\end{aligned}$$

CASE 2. Now suppose $\gamma \in \Omega_1$ is a limit ordinal. In this case there exists a sequence $\{\gamma_n\}_{n=1}^\infty \subset \Omega_1$ with $\gamma_n \uparrow \gamma$. By (1) and (3) (with $\alpha=0$):

$$\rho(x, x_{\gamma_{n+1}}) = \rho(x, x_{\gamma_n})+\rho(x_{\gamma_n}, x_{\gamma_{n+1}});$$

thus the sequence $\{\rho(x, x_{\gamma_n})\}$ is non-decreasing. Since $\rho(x, x_{\gamma_n}) = \rho(x, y)-\rho(x_{\gamma_n}, y)$ (by (4)), it follows that $\{\rho(x, x_{\gamma_n})\}$ converges. Moreover

$$\rho(x, x_{\gamma_n}) = \sum_{k=0}^{n-1} \rho(x_{\gamma_k}, x_{\gamma_{k+1}}) \quad \text{where} \quad x_{\gamma_0}=x,$$

from which $\sum_{k=0}^\infty \rho(x_{\gamma_k}, x_{\gamma_{k+1}})<\infty$. Therefore $\{x_{\gamma_n}\}$ is a Cauchy sequence in $X$ and by completeness there exists $w \in X$ such that $x_{\gamma_n} \to w$ as $n \to \infty$. In this case, define $x_\gamma = w$. Note that if, for any $\alpha<\gamma$, $x_\alpha = y$, then $\{x_{\gamma_n}\}$ is eventually the constant sequence $\{y\}$ and in this case (2)–(5) obviously hold for $\eta=\gamma$. Thus we suppose for all $\alpha<\gamma$ it is the case that $x_\alpha \ne y$, and we show that in this case (3)–(5) hold for $\eta=\gamma$.

By (5), $\rho(Tx_{\gamma_n}, Tx_{\gamma_m}) \le \hat{M}\rho(x_{\gamma_n}, x_{\gamma_m})$ for all $m$, $n$; thus $\{Tx_{\gamma_n}\}$ is a Cauchy sequence in $Y$. Since $Y$ is complete and $T$ has a closed graph, $Tx_\gamma = \lim_n Tx_{\gamma_n}$. If $\alpha<\beta<\gamma$ and $n$ is chosen so large that $\gamma_n \ge \beta$, then by (3),

$$\rho(x_\alpha, x_{\gamma_n}) = \rho(x_\alpha, x_\beta)+\rho(x_\beta, x_{\gamma_n})$$

and letting $n \to \infty$ we obtain

$$\rho(x_\alpha, x_\gamma) = \rho(x_\alpha, x_\beta)+\rho(x_\beta, x_\gamma),$$

proving $x_\beta \in (x_\alpha, x_\gamma)$. Also by (4),

$$\rho(x_\beta, y) = \rho(x_\beta, x_{\gamma_n})+\rho(x_{\gamma_n}, y),$$

and again passing to the limit: $x_\gamma \in (x_\beta, y)$. Therefore (3) and (4) hold for $\eta=\gamma$. By (5),

$$\rho(Tx_\beta, Tx_{\gamma_n}) \le \hat{M}\rho(x_\beta, x_{\gamma_n});$$

hence

$$\rho(Tx_\beta, Tx_\gamma) \le \hat{M}\rho(x_\beta, x_\gamma).$$

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Therefore a set $\{x_\gamma : \gamma \in \Omega_1\}$ may be defined in $X$ so that (1)–(5) are satisfied. If $x_\gamma \neq y$ for all $\gamma \in \Omega_1$ then (3) implies $\{\rho(x, x_\gamma) : \gamma \in \Omega_1\}$ is an uncountable discrete set of real numbers—a contradiction. Thus for some $\gamma \in \Omega_1$, $x_\gamma = y$, and by (5): $\rho(Tx, Ty) \le \hat{M}\rho(x, y)$. Since $\hat{M} > M$ is arbitrary, $\rho(Tx, Ty) \le M\rho(x, y)$, completing the proof.

## References

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